THE FUNDAMENTALS OF LINEAR ALGEBRA

Zachary Lihn

Columbia Undergraduate Math Society

July 11, 2023
Part I

WHAT’S LINEAR ALGEBRA?
WHAT’S LINEAR ALGEBRA?

- "Mathematics is the art of reducing any problem to linear algebra." - William Stein
- Linear algebra is one of the only mathematical theories that we understand almost completely
- All of math uses linear algebra as a source of examples, proof techniques, etc. Most problems are solved by reducing to linear algebra (e.g. linear approximation via derivatives)
**SUMMER PLAN**

- Review of the fundamentals (today): vector spaces, linear maps, bases, products, dual spaces
- Tensors: a unified language for multilinear maps
- Exterior Products: A powerful construction, using tensors, that allows intrinsic definitions of trace, determinant, and rank. Motivated geometrically by "area."
- Matrix operators: Power series of matrices (e.g. matrix exponential). Derivatives of these power series.
- Canonical forms: Diagonalizability and Jordan canonical form. Allows "reading off" the geometry of a matrix from the canonical form. Powerful for theory and applications.
- Scalar products: Generalizes the dot product in $\mathbb{R}^n$.
- Bilinear forms: Bilinear maps $V \times V \to \mathbb{R}$ (or $\mathbb{C}$).
Part II

VECTOR SPACES
**Abstract Vector Spaces**

Let $k$ be a **field** (e.g. $\mathbb{R}$ or $\mathbb{C}$).

**Definition 0.1**

A set $V$ is a **vector space over** $k$ if

1. $V$ is an abelian group under operation $+$: there is a zero element $0$, addition $u + v$ for $u, v \in V$ is defined, there are inverses $-u$ with $u + (-u) = 0$, and $u + v = v + u$ (commutativity).

2. **Scalar multiplication is defined**: for $\lambda \in k$, $v \in V$, we have $\lambda v \in V$.

3. For all $u, v \in V$ and $\lambda, \mu \in k$,

   \[(\lambda + \mu)v = \lambda v + \mu v, \quad \lambda(v + u) = \lambda v + \lambda u, \quad 1v = v, \quad 0v = 0\]

   *(distributivity)*

Note that the definition is abstract: we don’t explain how addition and multiplication work, we just say what properties they must satisfy.
EXAMPLES OF VECTOR SPACES

- $\mathbb{R}^n, \mathbb{C}^n, \mathbb{Q}^n$ are vector spaces over $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ respectively.
- If $v \in \mathbb{R}^n$, the set $\{u \in \mathbb{R}^n : u \cdot v = 0\}$ is a vector space, the orthogonal complement ($\cdot$ is the usual dot product).
- The set of all real-valued continuous functions on $[0, 1]$, denoted $C([0, 1])$, is a vector space.
- So is $\{f \in C([0, 1]) : f(0) = f(1) = 0\}$.
- Polynomials of degree $\leq d$ over a field $k$ form a finite-dimensional vector space over $k$. 
LINEAR INDEPENDENCE AND BASES

Definition 0.2
A tuple of vectors \((v_1, \ldots, v_n)\) is **linearly dependent** if there exist \(\lambda_1, \ldots, \lambda_n \in k\), not all equal to zero, such that

\[
\lambda_1 v_1 + \cdots + \lambda_n v_n = 0.
\]

If \((v_1, \ldots, v_n)\) are not linearly dependent, we say they are **linearly independent**.

Definition 0.3
A vector space is **n-dimensional** if there exists a linearly independent set of \(n\) vectors, but no linearly independent set of \(n+1\) vectors. It is **infinite-dimensional** if there exist \(n\) linearly independent vectors for any \(n\).

- \(\mathbb{R}^n\) is \(n\)-dimensional, but \(C([a, b])\) is infinite-dimensional.

Definition 0.4 (Equivalent)

The **dimension** of \(V\) is the longest length of any chain of strictly increasing vector subspaces

\[
V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n
\]
Bases

Definition 0.5

An (ordered) basis in a vector space $V$ is a tuple $(e_1, \ldots, e_n)$ of linearly independent vectors such that any vector $v \in V$, can be expressed as $v = \sum_{k=1}^{n} v_k e_k$ for some $v_k \in k$ (i.e. $(e_1, \ldots, e_n)$ span $V$).

The numbers $v_k$ are the components of $v$ with respect to $(e_1, \ldots, e_n)$.

▶ This definition only works (as written) for finite-dimensional vector spaces. For infinite dimensions, you will have an infinite basis, but only finite linear combinations are allowed.

▶ A basis is extra data associated to $V$! There are many choices of basis for a given vector space.

▶ Equivalent definition: A basis is a choice of isomorphism from $V$ to the "standard" vector space $k^n$, where $n = \dim V$. If $\varphi : V \rightarrow k^n$ is the isomorphism, the basis vectors are given by $e_i = \varphi^{-1}((0, \ldots, 0, 1, 0, \ldots, 0))$ with the 1 in the $i$th place.

Theorem 1

In a finite-dimensional vector space, all bases have equally many vectors (i.e. dimension is a well-defined integer).

Theorem 2

Every vector space has a basis.

This turns out to be equivalent to the axiom of choice! Think about what a basis of $\mathbb{R}$ over $\mathbb{Q}$ would look like.
Part III

LINEAR MAPS
LINEAR MAPS BETWEEN VECTOR SPACES

Definition 0.6

A function $A : V \to W$ between vector spaces $V$, $W$ is **linear** if for all $\lambda \in k$ and $u, v \in V$, $A(u + \lambda v) = A(u) + \lambda A(v)$.

- If $A$ is a linear map $V \to V$ and $(e_j)_j$ is a basis then there exist $A_{jk} \in k$ ($j, k = 1, \ldots, n$) such that if $v = \sum_{j=1}^n v_j e_j$, then $(Av)_j = \sum_{k=1}^n A_{jk} v_k$.
- By linearity, $Av = A(\sum_{k=1}^n v_k e_k) = \sum_{k=1}^n v_k A(e_k)$ so $A$ is determined by where it sends $e_k$.
- In basis $(e_j)_j$, we can write $A e_k$ as $A e_k = \sum_{j=1}^n A_{jk} e_j$.
- So $Av = \sum_{k=1}^n v_k \sum_{j=1}^n A_{jk} e_j$.
- The matrix $(A_{jk})_{j,k=1,...,n}$ determines $A$ in the given basis $(e_1, \ldots, e_n)$.

The composition of two linear maps $A : V \to W$ and $B : W \to Z$ is again a linear map $BA : V \to Z$. 
EXAMPLES OF LINEAR MAPS

1. The identity $I : V \rightarrow V$ given by $Iv = v$ is linear. Its matrix in any basis is the Kronecker delta $\delta_{ij} = 1$ if $i = j$, 0 otherwise.

2. Let $C^1([a, b])$ be the space of continuously differentiable real-valued functions of $[a, b]$. Then the derivative $d/dx : C^1([a, b]) \rightarrow C([a, b])$ is a linear map.

3. Solving a differential equation $d/dxu(x) = f$ consists of finding the preimage of $f$ under this linear map. This is the view of PDE from functional analysis.

4. Similarly, integration $f \mapsto \int_a^b f$ defines a linear function $C([a, b]) \rightarrow \mathbb{R}$. 
Let $\text{Hom}(V, W)$ be the set of all linear maps $V \rightarrow W$. Then $\text{Hom}(V, W)$ is a vector space: If $\lambda \in k$, $v \in V$, and $A, B \in \text{Hom}(V, W)$, define

$$(\lambda A)v = \lambda(Av)$$

$$(A + B)v = Av + Bv$$

- You can check that $\text{Hom}(V, W)$ is a vector space.
- We can also define $\text{End}(V) := \text{Hom}(V, V)$.
- Fact: $\text{End}(k) \cong k$.
- Later, we will see that $\text{Hom}(V, W) \cong V^* \otimes W$, where $V^*$ is the dual space.
Part IV

ISOMORPHISMS
**Definition 0.7**

Two vector spaces are **isomorphic** if there exists a bijective linear map between them.

- If $A : V \to W$ is an isomorphism and $(e_1, \ldots, e_n)$ is a basis for $V$, then $(Ae_1, \ldots, Ae_n)$ is a basis for $W$.

Coupled with the fact that a basis for $V$ is equivalent to an isomorphism $V \to k^n$, we get:

**Theorem 3**

Any vector space $V$ of dimension $n$ is isomorphic to the space $k^n$ of $n$-tuples.

Note that this isomorphism depends on the choice of basis! It is not *canonical*:

**Definition 0.8**

A linear map $V \to W$ is **canonically defined** or **canonical** if it's definition does not depend on the basis chosen.

$V$ and $W$ are **canonically isomorphic** if there is a canonically defined isomorphism between the two.
We can *construct* an isomorphism by choosing a basis, but to show its canonical we must show that it gives the same values for any other choice of basis.

As a general philosophy, canonically isomorphic vector spaces can be "identified" for all intents and purposes, while noncanonically isomorphic ones cannot be. Be careful though!

1. $V$ is canonically isomorphic to itself via the identity map. More generally, any $\lambda I$ for $0 \neq \lambda \in k$, gives a canonical isomorphism.

2. If $V$ is 1-dimensional, the isomorphism $\text{End}(V) \to k$ is constructed by noting every element of $\text{End}(V)$ is multiplication by a scalar. But this is not canonical: it requires a choice of vector $0 \neq v \in V$ that is mapped on $1 \in k$. 

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**Canonical Isomorphisms**

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Part V

MAKING NEW VECTOR SPACES FROM OLD ONES
Let $I$ be a (possibly infinite) set indexing a collection of vector spaces $\{V_i\}_{i \in I}$ defined over the same field $k$.

**Definition 0.9**

The **direct sum** $\bigoplus_{i \in I} V_i$ is the set of tuples $(v_i)_{i \in I}$ with $v_i = 0$ for all but finitely many $i$.

The **direct product** $\prod_{i \in I} V_i$ is the set of all tuples $(v_i)_{i \in I}$.

- We can define addition and scalar multiplication componentwise for both: $\lambda(v_i)_{i \in I} = (\lambda v_i)_{i \in I}$ and $(v_i)_{i \in I} + (w_i)_{i \in I} = (v_i + w_i)_{i \in I}$. This makes them into vector spaces.

- Note that $\bigoplus_{i \in I} V_i \subset \prod_{i \in I} V_i$ and the two are the same if $I$ is finite.

**Exercise:** Show that $\mathbb{R}^n \oplus \mathbb{R}^m$ is isomorphic to $\mathbb{R}^{n+m}$, but not canonically.
The Rank-Nullity Theorem

Theorem 4

Let \( A : V \to W \) be a linear transformation between finite-dimensional vector spaces. Then

\[
\dim \text{im} A + \dim \ker A = \dim V
\]

If you have taken abstract algebra, this theorem is just a consequence of the first isomorphism theorem: \( V / \ker A \cong \text{im} A \).
Part VI

THE DUAL SPACE AND HYPERPLANES
**Dual Space**

**Definition 0.10**

If $V$ is a vector space, the dual vector space $V^*$ is defined to be $\text{Hom}(V, k)$. In other words, it is the set of all linear maps $V \to k$ (called **linear functionals**).

1. Integration on $C([a, b])$ defines a linear functional. So does $\frac{d}{dx}|_{x=a}$ on the space of differentiable functions.
2. If $\mathbb{R}^2$ has coordinates $v = (x, y)$, then linear functionals are $f(v) = x - y$, $g(v) = 2x$. 
**Dual Basis**

We now show that there exists an isomorphism $V \rightarrow V^*$

- Choose a basis $(e_1, \ldots, e_n)$ of $V$. We claim that the tuple $(e^1, \ldots, e^n)$ is a basis of $V^*$, called the **dual basis**, where we characterize $e^i$ by

$$e^i(e_j) = \delta_{ij}$$

- They span the space: We have for $f \in V^*$ and $v \in V$ with $v = \sum_{k=1}^{n} v_k e_k$,

$$f(v) = f(\sum_{k=1}^{n} v_k e_k) = \sum_{k=1}^{n} v_k f(e_k) = \sum_{k=1}^{n} e^k(v) f(e_k)$$

so $f = \sum_{k=1}^{n} e^k f(e_k)$.

- They are linearly independent: If $\sum_{k=1}^{n} \lambda_k e^k = 0$, then acting on $e_j$ we get

$$0 = (\sum_{k=1}^{n} \lambda_k e^k)(e_j) = \lambda_j$$

so all the $\lambda_j$ are zero.
How does this look explicitly?

- Pick a basis and view elements of $V$ as column vectors. Let’s do this for $\mathbb{R}^3$ with the standard basis, so

  $$ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

- Then the dual basis $e^1, e^2, e^3$ are can be viewed as row vectors:

  $$ e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} $$

  since the usual matrix multiplication gives $e^i \cdot e_j = \delta_{ij}$

- So, after picking a basis, the isomorphism $V \rightarrow V^*$ is given by $v \mapsto v^T$, the transpose!

- This isomorphism is **not** canonical.
A Bit of Geometry

We are familiar with planes, such as the set \( \{x = 0\} \subset \mathbb{R}^3 \) or \( \{x+2y-z=0\} \subset \mathbb{R}^3 \).

Definition 0.11

The **hyperplane annihilated by** \( f \in V^* \) is the set \( \{x \in V : f(v) = 0\} = \ker f \).

Theorem 5

The hyperplane annihilated by nonzero \( f \in V^* \) has dimension \( n - 1 \). (\( n = \dim V \)).

Proof.

- The image of \( f \) either has dimension 0 or 1. Since it is nonzero, \( \dim \text{im} f = 1 \).
- From the rank-nullity theorem, \( \dim V = \dim \text{im} f + \dim \ker f = 1 + \dim \ker f \). So \( \dim \ker f = n - 1 \), the dimension of the hyperplane.

As an example, the linear functional corresponding to \( [a_1, a_2, a_3] \) in the standard basis is the hyperplane \( \{a_1 x + a_2 y + a_3 z = 0\} \).

This allows us to generalize our intuition to higher dimensions!

**Exercise** Let \( f_1, \ldots, f_m \in V^* \). Show that \( \{v \in V : f_i(v), i = 1, \ldots, m\} \) is a linear subspace of \( V \).

Show that if \( f_1, \ldots, f_m \) are linearly independent, then the dimension of that subspace is \( n - m \) (where \( n = \dim V \)).
EXERCISES

1. Show that $\mathbb{R}^n \oplus \mathbb{R}^m$ is isomorphic to $\mathbb{R}^{n+m}$, but not canonically.

2. If $V$ is one-dimensional, show that $\text{End}(V)$ is isomorphic to $k$, but not canonically.

3. Go on wikipedia and look up the universal properties of $\oplus$ and $\times$. Verify that they are true.

4. Construct a **canonical** isomorphism between $V$ and its double dual $V^{**}$, for $V$ finite-dimensional.

5. Let $V$ be the vector space of polynomials in $x$ of degree $\leq d$ with coefficients in $\mathbb{R}$. Let $(1, x, x^2, x^3, \ldots, x^3)$ be the basis of $V$. For notational convenience, set $e_i = x^i$. Express the corresponding dual basis $e'$ in terms of the (higher) derivative operator $\frac{d^i}{dx^i}|_{x=0}$.

6. Let $f_1, \ldots, f_m \in V^*$. Show that $\{v \in V : f_i(v), i = 1, \ldots, m\}$ is a linear subspace of $V$. Show that if $f_1, \ldots, f_m$ are linearly independent, then the dimension of that subspace is $n - m$ (where $n = \dim V$).