In this talk, we explained a joint work with Zhiwei Yun [3].

Let \( k = \mathbb{F}_q \) be a finite field of characteristic \( p > 0 \). Let \( X \) be a geometrically connected smooth proper curve over \( k \). Let \( \nu : X' \to X \) is a finite étale cover of degree 2 such that \( X' \) is also geometrically connected. Let \( F = k(X) \) and \( F' = k(X') \) be their function fields.

Let \( G = \text{PGL}_2 \) and \( T = (\text{Res}_{F'/F} \mathbb{G}_m)/\mathbb{G}_m \) the non-split torus associated to the double cover \( X' \) of \( X \). Let \( \text{Bun}_2 \) be the stack of rank two vector bundles on \( X \). The Picard stack \( \text{Pic}^1 \text{Bun}_2 \) acts on \( \text{Bun}_2 \) by tensoring a line bundle. Then \( \text{Bun}_G = \text{Bun}_2 / \text{Pic}^1 \text{Bun}_2 \) is the moduli stack of \( G \)-torsors over \( X \).

0.1. **The Heegner–Drinfeld cycle.** Let \( r \) be an even integer. Let \( \mu \in \{ \pm \}^r \) be an \( r \)-tuple of signs such that exactly half of them are equal to +. The Hecke stack \( \text{Hk}_2^\mu \) is the stack whose \( S \)-points is the groupoid of the data \((E_0, \ldots, E_r, x_1, \ldots, x_r, f_1, \ldots, f_r)\) where \( E_i \)'s are vector bundles of rank two over \( X \times S \), \( x_i \)'s are \( S \)-points of \( X \), and \( f_i \) is a minimal upper (i.e., increasing) modification if \( \mu_i = + \), and minimal lower (i.e., decreasing) modification if \( \mu_i = - \), and the modification takes place along the graph of \( E_0 \xrightarrow{f_1} \ldots \xrightarrow{f_r} E_r \).

The Picard stack \( \text{Pic}^1 \text{Hk}_2^\mu \) acts on \( \text{Hk}_2^\mu \) by simultaneously tensoring a line bundle. Define \( \text{Hk}_G^\mu = \text{Hk}_2^\mu / \text{Pic}^1 \text{Bun}_2 \). Assigning \( E_i \) to the data above descends to a morphism \( p^\mu_i : \text{Hk}_G^\mu \to \text{Bun}_G \).

The moduli stack \( \text{Sht}_G^\mu \) of Drinfeld \( G \)-Shtukas with \( r \)-modifications of type \( \mu \) for the group \( G \) is defined by the following cartesian diagram

\[
\begin{array}{ccc}
\text{Sht}_G^\mu & \longrightarrow & \text{Hk}_G^\mu \\
\downarrow & & \downarrow \\
\text{Bun}_G & \longrightarrow & \text{Bun}_G \times \text{Bun}_G
\end{array}
\]

The stack \( \text{Sht}_G^\mu \) is a Deligne-Mumford stack over \( X^r \) and the natural morphism

\[ \pi_G^\mu : \text{Sht}_G^\mu \longrightarrow X^r \]

is smooth of relative dimension \( r \), and locally of finite type. We remark that \( \text{Sht}_G^\mu \) as a stack over \( X^r \) is canonically independent of the choice of \( \mu \). The stack \( \text{Sht}_T^r \) of \( T \)-Shtukas is defined analogously, with the \( E_i \) replaced by line bundles on \( X' \), and the points \( x_i \) on \( X' \). Then we have a map

\[ \pi_T^r : \text{Sht}_T^r \longrightarrow X'^r \]

which is a torsor under the finite Picard stack \( \text{Pic}(X') / \text{Pic}(k) \). In particular, \( \text{Sht}_T^\mu \) is a proper smooth Deligne-Mumford stack over \( \text{Spec} \, k \).

There is a natural finite morphism of stacks over \( X^r \)

\[ \text{Sht}_G^\mu \longrightarrow \text{Sht}_G^\mu \times_{X^r} X'^r . \]

It induces a finite morphism

\[ \theta^\mu : \text{Sht}_T^\mu \longrightarrow \text{Sht}_G^\mu \times_{X^r} X'^r . \]

This defines a class in the Chow group of proper cycles of dimension \( r \) with \( \mathbb{Q} \)-coefficient

\[ \theta^\mu_\ast [\text{Sht}_T^\mu] \in \text{Ch}_{r\cdot} (\text{Sht}_G^\mu \otimes \mathbb{Q}) . \]
In analogy to the classical Heegner cycles \([1]\) in the number field case, we will call \(\theta^\pi_{\text{Eis}}[\text{Sh}_G]\) the
Heegner–Drinfeld cycle in our setting.

0.2. The spectral decomposition of the cycle space. We denote the set of closed points (places) of \(X\) by \(|X|\). For \(x \in |X|\), let \(\mathcal{O}_x\) be the completed local ring of \(X\) at \(x\) and let \(F_x\) be its fraction field. Let \(A = \prod_{x \in |X|} F_x\) be the ring of adeles, and \(\mathbb{O} = \prod_{x \in |X|} \mathcal{O}_x\) the ring of integers inside \(A\). Let \(K = \prod_{x \in |X|} K_x\) where \(K_x = G(\mathcal{O}_x)\). The (spherical) Hecke algebra \(\mathcal{H}\) is the \(\mathbb{Q}\)-algebra of bi-
\(K\)-invariant functions \(C^\infty((G(\mathbb{A})/K, \mathbb{Q})\) with the product given by convolution.

Let \(A = C^\infty(G(\mathbb{A})/G(\mathbb{A}), \mathbb{Q})\) be the space of everywhere unramified \(\mathbb{Q}\)-valued automorphic functions for \(G\). Then \(A\) is an \(\mathcal{H}\)-module. By an everywhere unramified cuspidal automorphic representation \(\pi\) of \(G(\mathbb{A})\) we mean an \(\mathcal{H}\)-submodule \(\mathcal{A}_\pi \subset A\) that is irreducible over \(\mathbb{Q}\). For every such \(\pi\), \(\text{End}_{\mathcal{H}}(\mathcal{A}_\pi)\) is a number field \(E_\pi\), which we call the coefficient field of \(\pi\). Then by the commutativity of \(\mathcal{H}\), \(\mathcal{A}_\pi\) is a one-dimensional \(E_\pi\)-vector space.

The Hecke algebra \(\mathcal{H}\) acts on the Chow group \(\text{Ch}_r(\text{Sh}_G)\) via Hecke correspondences. Let \(\widetilde{W} \subset \text{Ch}_r(\text{Sh}_G)\) be the sub \(\mathcal{H}\)-module generated by the Heegner–Drinfeld cycle \(\theta^\pi_{\text{Eis}}[\text{Sh}_G]\). There is a bilinear and symmetric intersection pairing
\[
\langle \cdot, \cdot \rangle_{\text{Sh}_G} : \widetilde{W} \times \widetilde{W} \longrightarrow \mathbb{Q}.
\] (0.2)

Let \(\widetilde{W}_0\) be the kernel of the pairing. The quotient \(W := \widetilde{W}/\widetilde{W}_0\) is then equipped with a non-degenerate pairing induced from \(\langle \cdot, \cdot \rangle_{\text{Sh}_G}\)
\[
\langle \cdot, \cdot \rangle : W \times W \longrightarrow \mathbb{Q}.
\]

The Hecke algebra \(\mathcal{H}\) acts on \(W\).

Let \(\pi\) be an everywhere unramified cuspidal automorphic representation of \(G\) with coefficient field \(E_\pi\), and let \(\lambda_\pi : \mathcal{H} \rightarrow E_\pi\) be the associated character, whose kernel \(\mathfrak{m}_\pi\) is a maximal ideal of \(\mathcal{H}\). Let
\[
W_\pi = \text{Ann}(\mathfrak{m}_\pi) \subset W
\] (0.3)
be the \(\lambda_\pi\)-eigenspace of \(W\). This is an \(E_\pi\)-vector space. Let \(I_{\text{Eis}} \subset \mathcal{H}\) be the Eisenstein ideal (cf. \([3]\)). Informally speaking, this is the annihilator of the Eisenstein spectrum in the space of automorphic functions \(A\). Define
\[
W_{\text{Eis}} = \text{Ann}(I_{\text{Eis}}).
\]

Theorem 0.1. We have an orthogonal decomposition of \(\mathcal{H}\)-modules
\[
W = W_{\text{Eis}} \oplus \left( \bigoplus_{\pi} W_\pi \right),
\] (0.4)
where \(\pi\) runs over the finite set of everywhere unramified cuspidal automorphic representation of \(G\), and \(W_\pi\) is an \(E_\pi\)-vector space of dimension at most one.

The \(\mathbb{Q}\)-bilinear pairing \(\langle \cdot, \cdot \rangle\) on \(W_\pi\) can be lifted to an \(E_\pi\)-bilinear symmetric pairing
\[
\langle \cdot, \cdot \rangle_\pi : W_\pi \times W_\pi \longrightarrow E_\pi
\] (0.5)
where for \(w, w' \in W_\pi\), \((w, w')_\pi\) is the unique element in \(E_\pi\) such that \(\text{Tr}_{E_\pi/\mathbb{Q}}(e \cdot (w, w')_\pi) = (ew, w')\) for all \(e \in E_\pi\).

0.3. Taylor expansion of \(L\)-functions. Let \(\pi\) be an everywhere unramified cuspidal automorphic representation of \(G\) with coefficient field \(E_\pi\). The standard \(L\)-function \(L(\pi, s)\) is a polynomial of degree \(4(g-1)\) in \(q^{-s-1/2}\) with coefficients in \(E_\pi\), where \(g\) is the genus of \(X\). Let \(\pi_{F'}\) be the base change to \(F'\), and let \(L(\pi_{F'}, s)\) be its standard \(L\)-function. This \(L\)-function is a product of two \(L\)-functions associated to cuspidal automorphic representations of \(G\) over \(F\):
\[
L(\pi_{F'}, s) = L(\pi, s) L(\pi \oplus \eta_{F'/F}, s),
\]
where
\[
\eta_{F'/F} : \mathbb{F}^\times \backslash \mathbb{A}^\times / \mathbb{Q}^\times \longrightarrow \{ \pm 1 \}.
\]
is the character corresponding to the étale double cover $X'$ via class field theory. The function $L(\pi_{F'}, s)$ satisfies a functional equation

$$L(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)L(\pi_{F'}, 1 - s),$$

where $\epsilon(\pi_{F'}, s) = q^{-8(g-1)(s-1/2)}$. Let $L(\pi, \text{Ad}, s)$ be the adjoint $L$-function of $\pi$. Denote

$$\mathcal{L}(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \text{Ad}, 1)},$$

where the the square root is understood as $\epsilon(\pi_{F'}, s)^{-1/2} := q^{4(g-1)(s-1/2)}$. In particular, we have a functional equation:

$$\mathcal{L}(\pi_{F'}, s) = \mathcal{L}(\pi_{F'}, 1 - s).$$

Consider the Taylor expansion at the central point $s = 1/2$:

$$\mathcal{L}(\pi_{F'}, s) = \sum_{r \geq 0} \mathcal{L}^{(r)}(\pi_{F'}, 1/2) \frac{(s - 1/2)^r}{r!},$$

i.e.,

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) = \frac{d^r}{ds^r} \bigg|_{s=0} \left( \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \text{Ad}, 1)} \right).$$

If $r$ is odd, by the functional equation we have

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) = 0.$$

Since $\mathcal{L}(\pi_{F'}, s) \in E_\pi[q^{-s-1/2}, q^{s-1/2}]$, we see that

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) \in E_\pi \cdot (\log q)^r. \quad (0.7)$$

Then our main result in [3] relates the $r$-th Taylor coefficient to the self-intersection number of the $\pi$-component of the Heegner–Drinfeld cycle $\theta_{\mu}^\prime [\text{Sht}_{\pi}^T]$ on the stack $\text{Sht}_{\pi}^T$.

**Theorem 0.2.** Let $\pi$ be an everywhere unramified cuspidal automorphic representation of $G$ with coefficient field $E_\pi$. Let $[\text{Sht}_{\pi}^T]_{\pi} \subseteq W_\pi$ be the projection of the image of $\theta_{\mu}^\prime [\text{Sht}_{\pi}^T] \subseteq W$ in $W$ to the direct summand $W_\pi$ under the decomposition $[3,4]$. Then we have an equality in $E_\pi$

$$\frac{1}{2(\log q)^r} |\omega_X| \mathcal{L}^{(r)}(\pi_{F'}, 1/2) = \left( [\text{Sht}_{\pi}^T]_{\pi}, [\text{Sht}_{\pi}^T]_{\pi} \right)_\pi,$$

where $\omega_X$ is the canonical divisor, and $|\omega_X| = q^{-2g+2}$.

**Remark 0.3.** When $r = 0$, this formula is equivalent to the special case of Waldspurger formula [2] for unramified $\pi$, relating the automorphic period integral to the central value of the $L$-function of $\pi_{F'}$

$$\left| \int_{T(F') \cap T(\mathbb{A})} \varphi(t)dt \right|^2 = \frac{1}{2} |\omega_X| \mathcal{L}(\pi_{F'}, 1/2),$$

where $\varphi \in \pi^K$ is normalized such that the Petersson inner product $(\varphi, \varphi) = 1$, and the measure on $G(\mathbb{A})$ is such that $\text{vol}(K) = 1$, and the measure on $T(\mathbb{A})$ is such that the maximal compact open subgroup has volume one.

**Remark 0.4.** In [3] we only consider the everywhere unramified situation where the $L$-function has nonzero Taylor coefficients in even degrees only. But the same construction with slight modifications should work in the ramified case as well, where the $L$-function may have nonzero Taylor coefficients in odd degrees. The case $r = 1$ would then give an analog of the Gross–Zagier formula [1] in the function field case.

**References**

