1. Problems

1.1. Compute the following integration

\[ \int \arcsin \sqrt{x} \, dx \]

via the following steps.
(1) Do the substitution to get rid of the square root.
(2) Integration by parts.
(3) Use trigonometrical substitution to compute the final result.

1.2. Compute the integral

\[ \int \arctan \sqrt{x} \, dx \]

1.3. Consider the series

\[ \sum_{n=1}^{\infty} nr^n \]

(1) When \(|r| \geq 1\), show that it is divergent.
(2) When \(|r| < 1\), show that it is convergent by both ratio and root test.
(3) Compute the value of it when it is convergent.

1.4. This problem asks you to give examples, you don’t have to prove your result. In this problem, we shall write \(\sum\) instead of \(\sum_{n=1}^{\infty}\).
(1) Give an absolutely convergent series.
(2) Give an conditionally convergent series.
(3) Give two divergent series \(\sum a_n\) and \(\sum b_n\), such that \(\sum (a_n + b_n)\) is convergent. Is it possible to make \(\sum (a_n - b_n)\) also convergent?
(4) Give two convergent series \(\sum a_n\) and \(\sum b_n\), such that \(\sum a_n b_n\) is divergent.

1.5. Determine whether the following series are convergent or divergent. Indicate briefly why it is so.

\[ \sum_{n=1}^{\infty} \frac{n}{n^3 + 1} \quad \sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{n^2} \]

1.6. Write the rational function

\[ \frac{2x}{(1 + x)^2(1 + x^2)} \]

as a sum of partial fractions, and compute the integral of it.
1.7. Use the following steps to compute the Maclaurin series of $\text{arctan } x$.
   (1) Compute Maclaurin series of $1/(1 + x^2)$.
   (2) Compute the indefinite integral on both sides.
   (3) Find out the final answer.
   (4) What is the radius of convergence of this power series?

1.8. Compute the first two terms of the Maclaurin series of (1) $e^x \sin x$; (2) $\tan x$.

1.9. (1) Find all values of $p$ such that the improper integral
   $$\int_1^\infty \frac{1}{x(\ln x)^p} \, dx$$
   is convergent.
   (2) Do the same thing for the series
   $$\sum_{n=1}^\infty \frac{1}{n(\ln n)^p}$$

1.10. (1) Show the Maclaurin series of the function
   $$f(x) = \frac{x}{1 - x - x^2}$$
   is
   $$\sum_{n=1}^\infty f_n x^n$$
   where $f_n$ is the Fibonacci sequence given by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
   (2) By writing $f(x)$ as a sum of partial fractions and thereby obtaining the Maclaurin series in a different way, find an explicit formula for the $n$th Fibonacci number $f_n$.

1.11. Compute the value of the series if it is convergent, indicate briefly why if you think it is divergent.
   $$\sum_{n=1}^\infty \frac{1}{n(n + 1)} \quad \sum_{n=1}^\infty \frac{1}{\sqrt{n} + \sqrt{n + 1}}$$

1.12. Suppose the sequence $\{a_n\}$ is given by the following reduction formula. $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$.
   (1) Show this sequence is increasing and bounded above.
   (2) Is this sequence convergent?
   (3) If it is convergent, what is the limit of the sequence?
1.13. Consider the series
\[ \sum_{n=1}^{\infty} \frac{1}{n} (x + 1)^n \]

(1) What is the radius \( R \) of convergence?
(2) Find all the values of \( x \) such that this series is convergent.
(3) When \( |x + 1| < R \), find the value of the series.

1.14. The arc of the parabola \( y = x^2 \) is rotated about the \( y \)-axis. Find the area of the resulting surface. (textbook example 8.2.2)

1.15. Evaluate \( \int_0^3 \frac{dx}{x-1} \) if possible.

1.16. Show that the series
\[ \sum_{n=0}^{\infty} (-1)^n \ln(1 + \frac{1}{n}) \]
is conditionally convergent.

1.17. Find all the values of \( c \) such that the series
\[ \sum_{n=1}^{\infty} \left( \frac{c}{n} - \frac{1}{n+1} \right) \]
is convergent.

1.18. Let \( f(x) = \sin(x^3) \). Find \( f^{(15)}(0) \).

1.19. Evaluate
\[ \lim_{x \to 0} \frac{e^{x^2} - \sin^2 x - 1}{\ln(1 + x^2)} \]

1.20. Compute the integral
\[ \int \frac{\sqrt{x}}{1 + \sqrt{x}} \, dx \]
2. Sketch of some of the solutions

1.5. For the first one use limit comparison, for the second one use absolute convergence and comparison.

1.9. We have
\[ \int_1^t \frac{1}{x(\ln x)^p} \, dx = (\ln t)^{1-p} \]
Now use the definition of convergence of improper integrals. For the series, use integral test.

1.10. We have
\[ x = (1 - x - x^2)(f_1 x + f_2 x^2 + f_3 x^3 + \cdots) \]
Expand the right hand side we get
\[ x = f_1 x + (f_2 - f_1)x^2 + (f_3 - f_2 - f_1)x^3 + (f_4 - f_3 - f_2)x^4 + \cdots \]
Compare the coefficients of both sides, we get (1).
For (2), we have
\[ \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) x^n \]
. Comparing the coefficients we get
\[ f_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \]
1.12. For (1) and (2), use induction. For the last one, take limits on both sides of the reduction formula.

1.15. Note that \( x = 1 \) is a discontinuous point.

1.16. Use limit comparison to show it is not absolutely convergent. It is an alternating series itself.

1.17. We have
\[ \frac{c}{n} - \frac{1}{n+1} = \left( \frac{1}{n} - \frac{1}{n+1} \right) + \frac{c-1}{n} \]
1.18. Expand \( \sin(x^3) \) as Maclaurin series.
1.19. Expand every transcendental function as a Maclaurin series.

1.20. Use the substitution \( u = \sqrt[6]{x} \).