THE $p$-ADIC COMPLEX NUMBERS

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1. Basic properties

The residue field of $\mathbb{Q}_p$ is $\mathbb{F}_p$, which is not algebraically closed. Therefore $\mathbb{Q}_p$ is not algebraically closed. We extend the $p$-adic valuation and absolute value on $\mathbb{Q}_p$ to $\bar{\mathbb{Q}}_p$, denoted by $|\cdot|$ and $v$. Note that $v$ on $\mathbb{Q}_p$ is no longer discrete. By definition, we have $v(x) = [L : \mathbb{Q}_p]^{-1}v(N_L/\mathbb{Q}_p x)$, if $x \in L$ with $L/\mathbb{Q}_p$ a finite extension. We normalize so that $v(p) = 1$, $|p| = p^{-1}$.

Lemma 1.1. Let $m$ be a positive integer coprime to $p$. The $m$-th roots of unity \{\(\zeta_i, 1 \leq i \leq m\)\} in $\bar{\mathbb{Q}}_p$ are pairwise non-congruent modulo $v$ i.e. $v(\zeta_i - \zeta_j) = 0, i \neq j$.

Proof. Suppose \{\(\zeta_i, 1 \leq i \leq m - 1\)\} are the $m$-th roots of unity apart from 1. Then
$$\prod_{1 \leq i \leq m-1} (1 - \zeta_i) = \frac{X^m - 1}{X - 1}|_{X=1} = m.$$ But $v(m) = 0$, so $v(1 - \zeta_i) = 0$. \hfill \Box

Proposition 1.2. $\bar{\mathbb{Q}}_p$ is not complete.

Proof. Suppose $\bar{\mathbb{Q}}_p$ is complete. Then the following series should converge to an element $\alpha \in \bar{\mathbb{Q}}_p$.

(1) $$\alpha = \sum_{n=1}^{\infty} \zeta_n p^n,$$

where $\zeta_n$ is a primitive $n$-th root of unity in $\bar{\mathbb{Q}}_p$ if $p \nmid n$, and $\zeta_n := 1$ if $p | n$. Let $K/\mathbb{Q}_p$ be a finite extension such that $\alpha \in K$. We prove by induction that $K$ contains all the $\zeta_n$’s. But then since the residue field of $K$ is finite, we have a contradiction, by the previous Lemma.

To show that $K$ contains all the $\zeta_n$, suppose $p \nmid m$ and $K$ contains $\zeta_n$ for $n < m$. Then $K$ contains the element

(2) $$\beta = p^{-m}(\alpha - \sum_{n<m} \zeta_n p^n).$$

But $\beta \equiv \zeta_m \mod p$, so by Hensel’s lemma, the element $\zeta_m \mod p$, which is contained in the residue field of $K$, lifts to an $m$-th root of unity in $K$. But the latter has to be $\zeta_m$ itself by the previous Lemma. \hfill \Box

We let $\mathbb{C}_p$ be the $p$-adic completion of $\bar{\mathbb{Q}}_p$, called the field of $p$-adic complex numbers.

Proposition 1.3. $\mathbb{C}_p$ is algebraically closed.

Proof. Let $f(x) \in \mathbb{C}_p[x]$, we need to show $\mathbb{C}_p$ contains a root of $f(x)$. Without loss of generality, we may assume $f(x)$ is monic with coefficients in $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$. We pick a sequence of monic polynomials $f_n(X) \in \mathcal{O}_{\bar{\mathbb{Q}}_p}[x]$ that coefficient-wise converge to $f(X)$. Say $f_{n+1} - f_n$ has coefficients in \{\(v \geq N_n\)\}, $N_n \to \infty$. Let $\alpha_n$
be a root of $f_n(X)$ in $\bar{\mathbb{Q}}_p$. Then necessarily $v(\alpha_n) \geq 0$. We have $v(f_{n+1}(\alpha_n)) = v(f_{n+1}(\alpha_n) - f_n(\alpha_n)) \geq N_n \to \infty$. But if $f_{n+1}(X) = \prod_i (X - \beta_i)$, then

$$v(f_{n+1}(\alpha_n)) = \sum_i v(\alpha_n - \beta_i) \geq N_n,$$

so $f_{n+1}$ has a root $\alpha_{n+1}$ with $v(\alpha_{n+1} - \alpha_n) \geq N_n/\deg f$. Thus we get a Cauchy sequence $\{\alpha_n\} \subset \mathbb{Q}_p$, whose limit in $\mathbb{C}_p$ is a root of $f(X)$.

Alternatively, let $\alpha$ be a root of $f(X)$ in $\mathbb{C}_p$. Find a monic polynomial $g(X) \in \mathcal{O}_\mathbb{Q}_p[X]$ that is coefficient-wise close to $f(X)$ and let $\beta$ be a root of $g(X)$. Let $\sigma \in \text{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$. Then $v(\alpha - \sigma \alpha) \geq \min \{v(\alpha - \beta), v(\sigma \alpha - \beta)\} = v(\alpha - \beta)$. If $\sigma \neq 1$, then we get an upper bound of $v(\alpha - \beta)$ that only depends on $\alpha$. Let $\beta_i$ be the roots of $g$, then $v(g(\alpha)) = \sum_i v(\alpha - \beta_i)$ has an upper bound. But $v(g(\alpha)) = v(g(\alpha) - f(\alpha))$ can be made arbitrarily large if we choose $g$ to be close to $f$. The argument in this paragraph implicitly proves what is called Krasner’s Lemma. \hfill $\square$

### 2. The Theorem of Tate and Ax

The main reference for the following material is the paper *Zeros of Polynomials over Local Fields - The Galois Action* by James Ax, 1969.

The Galois group $G = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $\mathbb{C}_p$ by isometries. Let $H$ be a closed subgroup of $G$. We want to determine the fixed field $\mathbb{C}_p^H$. Of course $K := (\bar{\mathbb{Q}}_p)^H \subset \mathbb{C}_p$, and also $\bar{K} \subset \mathbb{C}_p^H$ because $H$ acts continuously on $\mathbb{C}_p$. Here $\bar{K}$ is the completion (closure) of $K$ inside $\mathbb{C}_p$.

**Theorem 2.1 (Tate-Ax).** $\mathbb{C}_p^H = \bar{K}$.

**Proof.** Let $x \in \mathbb{C}_p^H$. Without loss of generality we may assume $v(x) \geq 0$. Let $\{x_n\} \subset \bar{\mathbb{Q}}_p$ be a sequence converging to $x$. We may assume $v(x_n - x) > n$. For $g \in H$ we have

$$v(gx_n - x_n) = v(gx_n - x + x - x_n) \geq \min \{v(gx_n - x), v(x_n - x)\} = v(x_n - x) > n.$$

By the following proposition, this implies that there exists $y_n \in K$ such that $v(y_n - x_n) \geq n - p/(p-1)^2$. Then we have $y_n \to \alpha$. \hfill $\square$

**Proposition 2.2.** Let $K$ be an algebraic extension of $\mathbb{Q}_p$. Let $v$ be the $p$-adic valuation on $K$ normalized by $v(p) = 1$. Let $x \in \bar{K}$ be such that for all $g \in G_K = \text{Gal}(\bar{K}/K)$, $v(gx - x) \geq n$. Then there exists $y \in K$ such that $v(x - y) \geq n - p/(p-1)^2$. Simply put, if a small ball of $\bar{K}$ contains a whole Galois orbit, then by enlarging the small ball by a constant scalar, we get a ball that contains an element of $K$.

We need the following lemma to prove the proposition.

**Lemma 2.3.** Let $f(X) \in \bar{K}[X]$ be a monic polynomial of degree $d > 1$. If $d$ is not a power of $p$, let $q$ be the $p$-part of $d$. If $d$ is a power of $p$, let $q = d/p$. Suppose $D = \{x | v(x - x_0) \geq \lambda \} \subset K$ is a ball containing all the roots of $f$.

1. If $d$ is not a $p$ power, $D$ contains a root of $f^{[q]} := f^{(q)}/q^!.$
2. If $d$ is a $p$ power, let $D' = \{x | v(x - x_0) \geq \lambda - 1/(d - q)\}$, an enlargement of $D$. Then $D'$ contains a root of $f^{[q]}$.

**Remark 2.4.** This is a $p$-adic analogue of Gauss’ Theorem: If a ball in $\mathbb{C}$ contains all the roots of a polynomial $f$, then it contains all the roots of $f'$. 


**Proof of Lemma.** Assume we are in case (1). Without loss of generality we may assume the ball \( D \) is centered around \( x_0 = 0 \). Write \( f(X) = \sum_{i=0}^{n} a_i X^i \). Then \( f^{[q]}(0) = a_q \), so \( v(f^{[q]}(0)) = v(a_q) \geq (d-q)\lambda \). Let \( \beta_i \) be the roots of \( f^{[q]} \), then

\[
f^{[q]}(0) = \left( \frac{d}{q} \right) \prod_{i=1}^{d-q} \beta_i
\]

since the leading coefficient of \( f^{[q]} \) is \( \left( \frac{d}{q} \right) \). But \( v\left( \left( \frac{d}{q} \right) \right) = 0 \), so there is some \( \beta_i \) for which \( v(\beta_i) \geq \lambda \).

In case (2), the argument is the same, the only difference being that now \( v\left( \left( \frac{d}{q} \right) \right) = 1 \). □

**Proof of Proposition.** We prove the following statement by induction on the degree \( d \) of \( x \).

**Statement:** If \( x \in \bar{K} \) is such that for all \( g \in G_K \), \( v(x-gx) \geq n \), then there exists \( y \in K \) such that

\[
v(x-y) \geq n - \sum_{i=1}^{[\log_p d]} \frac{1}{p^i - p^i - 1}.
\]

Note that this inequality is stronger than that asserted in the Proposition. Let \( f(X) \) be the monic minimal polynomial of \( x \) over \( K \), of degree \( d \).

For \( d = 1 \) we can take \( y = x \). For the induction step, let \( d > 1 \). First suppose \( d \) is not a \( p \) power. Let \( q \) be the \( p \)-part of \( d \). By Lemma, \( f^{[q]} \) has a root \( \beta \) satisfying \( v(x-\beta) \geq n \). For any \( g \in G_K \), we have

\[
v(\beta-g\beta) \geq \min \{ v(\beta-x), v(x-gx), v(gx-g\beta) \} \geq n.
\]

Let \( d(\beta) \) be the degree of \( \beta \), then \( d(\beta) \leq d - q \). By induction hypothesis, there is an element \( y \in K \) with

\[
v(\beta-y) \geq n - \sum_{i=1}^{[\log_p d(\beta)]} \frac{1}{p^i - p^{i-1}}.
\]

But \( v(x-y) \geq \min \{ v(x-\beta), v(\beta-y) \} \). So the statement is true for \( d \).

Suppose \( d \) is a \( p \) power. Let \( q = d/p \). Then by Lemma \( f^{[q]} \) has a root \( \beta \) satisfying \( v(x-\beta) \geq n - 1/(d-q) \). As before we see that for all \( g \in G_K \),

\[
v(g\beta-\beta) \geq n - 1/(d-q).
\]

By induction hypothesis we find a \( y \in K \) such that

\[
v(\beta-y) \geq n - 1/(d-q) - \sum_{i=1}^{[\log_p d(\beta)]} \frac{1}{p^i - p^{i-1}}.
\]

Then

\[
v(x-y) \geq n - 1/(d-q) - \sum_{i=1}^{[\log_p d(\beta)]} \frac{1}{p^i - p^{i-1}} \geq n - \sum_{i=1}^{[\log_p d]} \frac{1}{p^i - p^{i-1}}.
\]

\[\square\]

\[1\]In the two cases when computing \( v\left( \left( \frac{d}{q} \right) \right) \), simply use the formula \( v(n!) = \sum \{ n/p^i \} \).
Remark 2.5. In the above proof, the element $y$ that we found is in fact the root of the linear polynomial $f^{(d-1)}$, as can be seen from the induction process. This is none other than the arithmetic average of the conjugates of $x$. However if one makes the naive estimate $v(y) = v(\sum g x - x) - v(d) \geq \min_g v(g x - x) - v(d) \geq n - v(d)$, the lower bound depends on the degree $d$ of $x$, which is not good enough for our purpose.