1. The modular function $j$

In this section we study modular functions on $\mathfrak{h}$, these correspond to meromorphic functions on the modular curve $Y(1)$ but with a special condition at $\infty$.

1.1. Fourier coefficients. Consider the function

$$z \mapsto e^{2\pi iz}$$

It is a surjection from $\mathfrak{h}$ to the punctured disc

$$D_0 = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$$

moreover it is locally biholomorphic (bijective and holomorphic with holomorphic inverse). We define a new variable $q$ by setting

$$q = e^{2\pi iz}$$

Suppose now $f$ is a meromorphic function on $\mathfrak{h}$ such that

$$f(z) = f(z + n), \forall n \in \mathbb{Z}$$

then the value of $f$ only depends on $q = e^{2\pi iz}$ so that there exists a unique meromorphic function

$$F : D_0 \to \mathbb{C}$$

such that

$$f(z) = F(q)$$

We can consider $0$ as a singularity of $F$, and if this singularity is at worst a pole, then $F$ has a Laurent series expansion about $0$:

$$F(q) = \sum_{n >> 0} a_n q^n$$

**Definition 1.1.** Let $f$ be a meromorphic function on $\mathbb{C}$ such that

$$f(z) = f(z + n), \forall n \in \mathbb{Z}$$

Then $f$ is holomorphic (resp. meromorphic) at $\infty$ if the function $F(q)$ defined above is holomorphic (resp. meromorphic) at $0$.

If $f$ is meromorphic at $\infty$ the Fourier series expansion of $f$ or its $q$-expansion is the power series $F(q) = \sum_{n >> 0} a_n q^n$.

Recall the group $\Gamma = SL_2(\mathbb{Z})$ acts on $\mathfrak{h}$. Suppose $f$ is a meromorphic function on $\mathfrak{h}$ such that $f(\gamma z) = f(z) \forall \gamma \in \Gamma$, then in particular since $\zeta z = z + 1$ so that $f(z) = (z + n), \forall n \in \mathbb{Z}$ so we can apply the above constructions

**Definition 1.2.** A modular function is a meromorphic function $f : \mathfrak{h} \to \mathbb{C}$ satisfying the following two properties:

i) $f(\gamma z) = f(z) \forall \gamma \in \Gamma$

ii) $f$ is holomorphic at $\infty$.

The space of all modular functions will be denoted $\mathcal{M}_0'$. 
It is straightforward to check that $\mathcal{M}'_0$ is a field and contains all the constant functions.

**Remark 1.3.** If we add a point $\infty$ to the Riemann surface $Y(1) = \Gamma \setminus \mathbb{H}$, where we consider $\infty$ as the center of the disc $\mathbb{D}_0$ in the new coordinate $q$. The new coordinate $q$ defines a chart around this point, hence we may consider $Y(1) \coprod \{\infty\}$ as Riemann surface which we denote by $X(1)$. This construction is analogous to the adding $\infty$ to $\mathbb{C}$ to obtain $\mathbb{P}^1(\mathbb{C})$ and one sees that $X(1)$ is also compact.

1.2. **Modular functions and modular forms.** The main theorem of this section gives a very simple description of $\mathcal{M}_0$.

**Theorem 1.4.** There exists a unique modular function $j$ satisfying the following conditions:

1. $j$ is holomorphic on $\mathbb{H}$, has a simple pole at $\infty$ and $j(\omega) = 0$
2. $j$ induces by passage to the quotient a bijection $\Gamma \setminus \mathbb{H} \rightarrow \mathbb{C}$
3. The first term in the $q$-expansion of $j$ is $\frac{1}{q}$

Moreover we have an isomorphism $\mathcal{M}'_0 = \mathbb{C}(j)$, i.e. any modular function is a rational function of $j$.

In order to construct $j$ we need to consider a larger class of functions than modular functions.

**Definition 1.5.** Let $k \in \mathbb{N}$. A modular function of weight $k$ is a meromorphic function on $\mathbb{H}$ satisfying the following two properties

1. $f(\gamma z) = (cz + d)^k f(z)$
2. $f$ is meromorphic at $\infty$

If in addition $f$ is holomorphic and holomorphic at $\infty$ then we say $f$ is a modular form. If $f$ is 0 at $\infty$, i.e. the $q$-expansion has all non-zero coefficients in positive degree, then $f$ is a a cusp form. We will use $\mathcal{M}'_k, \mathcal{M}_k$ and $\mathcal{S}_k$ to denote the spaces of modular functions, modular forms and cusp forms of weight $k$ respectively.

Note that condition i) in the above definition ensures $f(z) = f(z + n), \; \forall n \in \mathbb{Z}$ so that condition ii) makes sense. Then modular functions of weight 0 are just modular functions in the sense of Definition 1.2.

**Exercise:** Let $f$ and $g$ be modular functions of weights $k_1$ and $k_2$ respectively. Show that $fg$ is a modular function of weight $k_1 + k_2$ and $f/g$ is a modular function of weight $k_1 - k_2$ ($g \neq 0$) and that if $f$ and $g$ are both modular forms, then $fg$ is also a modular form.

The exercise shows that if we let

$$\mathcal{M} := \bigoplus_{i=0}^{\infty} \mathcal{M}_k$$

then $\mathcal{M}$ has a natural structure of a graded ring.

**Remark 1.6.** Strictly speaking these should be modular functions of weight $k$ and level $\Gamma$. However since we do not consider other levels in this course, we will ignore this point.
Recall the lattice $\Lambda_\tau = \langle 1, \tau \rangle$ of the last lecture. For $k \geq 2$ an integer, define the *Eisenstein series of weight $2k$* to be the function

$$E_k(z) := \sum_{\omega \in \Lambda_\tau, \omega \neq 0} \frac{1}{\omega^{2k}} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(nz + m)^{2k}}$$

We also define the following functions

$$g_2(z) = 60E_2(z), \quad g_3(z) = 140E_3(z), \quad \Delta = g_3^2(z) - 27g_2^3(z)$$

**Proposition 1.7.** For $k \geq 2$ an integer

i) The series defining $E_k$ above converges to a holomorphic function on $\mathfrak{h}$. Moreover $E_k(z)$ is a modular form of weight $2k$.

ii) $E_k(\infty) = 2\zeta(2k)$ where $\zeta$ is the Riemann zeta function.

iii) The function $j := 1728\frac{g_3^3}{\Delta}$ is a modular function (of weight 0)

**Proof.** i) As each of the terms in the series is a holomorphic function on $\mathfrak{h}$ it suffices to prove that the series converges absolutely and uniformly on compact subset of $\mathfrak{h}$. Let $P_n$ denote the parallelogram with vertices $\{\pm nz, \pm n\}$ so that $\Lambda \cap P_n$ has $8n$ points. Let $r$ be the minimum distance from 0 to a point on the parallelogram $P_1$, we then have the estimate

$$\sum_{\omega \in \Lambda_\tau, \omega \neq 0} \frac{1}{|\omega|^{2k}} = \sum_{n=1}^{\infty} \sum_{\omega \in \Lambda_\tau \cap P_n, \omega \neq 0} \frac{1}{|\omega|^{2k}} < \sum_{n=1}^{\infty} \frac{8n}{(nr)^{2k}}$$

$$= \frac{8}{r^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k-1}} = \frac{8}{r^{2k}} \zeta(2k-1)$$

For $z$ in a compact subset of $\mathfrak{h}$, the value $r$ has a positive infimum, and since $\zeta(2k-1)$ covers for $k > 1$, it follows easily that the series converges absolutely and uniformly in any compact subset of $\mathfrak{h}$.

We leave it as an exercise to check that $E_k(z)$ has the required translation property with respect to elements of $\Gamma$.

For holomorphicity at $\infty$, this is equivalent to checking that $E_k(z)$ is bounded as $z \to \infty$. But this follows from the above estimate by noting that for $C > 0$, the value $r$ has an infimum on the set $\{z \in \mathfrak{h} : |z| > C\}$.

ii) By absolute convergence we may rearrange the order of summation so that

$$E_k(z) = \sum_{n \in \mathbb{Z} \setminus 0} \frac{1}{n^{2k}} + \sum_{m \in \mathbb{Z} \setminus 0} \sum_{n \in \mathbb{Z}} \frac{1}{mz + n}$$

All the terms in the right term tend to 0 as $\Im z \to \infty$, and the first term is just $2\zeta(2k)$.

iii) Since $g_2$ has weight 2 and $g_3$ has weight 3, then $\Delta$ is a modular form of weight 6 which is non-zero by Corollary 1.9. It follows from the exercise that $j$ is a modular function of weight 0.

Remarkably, we will actually show that the above construction in some sense exhausts all modular forms (of any weight). More precisely, we will show that the space of modular forms $\mathcal{M}$ is isomorphic to $\mathbb{C}[E_2, E_3]$. This is something of a miracle.
as what the statement is really saying is that there are many algebraic relations between these modular forms, whose definitions a priori are based completely in analysis.

The following is the key result we need to prove the above statements. Given a meromorphic function \( f \) and \( z \in \mathbb{C} \) we denote by \( v_z(f) \) the order of the function \( f \) at \( z \). If \( f \) is a modular function, then \( v_\infty(f) \) is defined in the obvious way, and it follows from the isolation of zeros/poles theorem that there are only finitely many \( z \in D \cup \{\infty\} \) for which \( v_z(f) \) is non-zero.

**Proposition 1.8.** Let \( f \) be a modular function of weight \( k \) and \( e_z = |\Gamma z|/2 \), then we have the relation

\[
v_\infty(f) + \sum_{z \in \Gamma \setminus \mathbb{H}} \frac{v_z(f)}{e_z} = \frac{k}{6}
\]

**Proof.** Let \( C \) be the path in the diagram on page 87 of Serre’s "A Course in Arithmetic." It follows from the argument principle that

\[
\frac{1}{2\pi i} \int f' f dz = \sum_{z \in \text{interior of } C} v_z(f)
\]

Now we compute the left hand side of the expressions. The integrals along the segments \( AB \) and \( D'E \) cancel since \( f(z) = f(z + 1) \). To compute the contribution of \( EA \), we note that changing the variable to \( q \), this segment is mapped to a path which loops once around 0, hence taking \( z \to \infty \) this contribution will be \(-v_\infty(f)\).

For the contribution \( DD' \) at \( \omega = \frac{1 + \sqrt{-3}}{2} \), note that as we take the arc smaller and smaller, we may assume that the only possible zero/pole occurs at \( \omega \). Also for a small enough arc, the value of the \( f'/f \) on the arc is roughly equal on the arc, so that in the limit

\[
\frac{1}{2\pi i} \int_{DD'} f' f dz \to -\frac{1}{6} v_\omega(f)
\]

The \(-\) sign comes about from the orientation of the arcs, the same remarks reply for the point \( i \) and \( \omega^2 \). The contribution along \( BB' \) is equal to the one along \( DD' \) and the contribution along \( CC' \) is \(-\frac{v_\omega(f)}{2}\).

Finally we must compute the contribution along the arcs \( B'C \) and \( C'D \). Let

\[
\mu = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

then \( \mu \) maps \( B'C \) to \( C'D \) with the opposite orientation. The relation

\[
f(\mu z) = f(-1/z) = z^{2k} f(z)
\]

gives us

\[
\frac{f'(-1/z)}{z^2} = z^{2k} f'(z) + 2kz^{2k-1} f(z)
\]

We thus obtain

\[
\int_{DC'} f' f dz = -\int_{BC'} \frac{f'(\mu z)}{f(\mu z)} d\mu z
\]

\[
= -\int_{BC'} \frac{f'(z)}{f(z)} + \frac{2k}{z} dz
\]
Now \( \int_{BC} \frac{2k}{z} \, dz \) can be computed explicitly, by eg. making the substitution \( z = e^{2\pi ix} \), and we obtain \( \frac{k}{z} \) for this integral. Taking the limit for smaller and smaller arcs the formula above becomes:

\[
\frac{k}{6} - v_\infty(f) - \frac{v_i(f)}{2} - \frac{v_\omega(f)}{3} = \sum_{z \in \text{interior } D} v_z(f)
\]

The result follows from the the fact that \( e^\omega = 3 \) and \( e^i = 2 \) and \( e^z = 1 \) for all \( z \) in the interior of \( D \).

**Corollary 1.9.** i) \( E_2 \) and \( E_3 \) can only have zeros at the points \( \omega \) or \( i \). \( \Delta \) is non-zero on \( \mathfrak{h} \) and has zero of order 1 at \( \infty \).

ii) \( \mathcal{M} = \oplus_{k=0}^\infty \mathcal{M}_k \cong \mathbb{C}[E_2, E_3] \) as rings.

**Proof.** i) Since \( E_2 \) and \( E_3 \) are holomorphic, it follows that they have non-negative orders everywhere, hence since \( k/6 < 1 \), it follows they can only have zeros at the specified points.

By part ii) of Proposition 1.7 we have

\[
g_2(\infty) = 60E_2(\infty) = 120\zeta(4) = 4/3\pi^4
\]

\[
g_3(\infty) = 140E_3(\infty) = 280\zeta(6) = 8/27\pi^6
\]

Thus \( v_\infty(\Delta) \geq 1 \) and since \( \Delta \in \mathcal{M}_6 \), the above formula shows \( v_\infty(\Delta) = 1 \) and that it cannot have any other zeros.

ii) The map

\[
\mathcal{M}_k \to \mathbb{C}
\]

given by \( f \mapsto f(\infty) \) has kernel \( \mathcal{S}_k \) and is surjective for \( k \geq 6 \). The surjectivity follows from the fact that we can write \( k = 2\alpha + 3\beta \) for some positive integers \( \alpha, \beta \). Then \( E_2^\alpha E_3^\beta \in \mathcal{M}_k \), is non-zero at \( \infty \). Therefore \( \mathcal{M}_k = \mathcal{S}_k \oplus (E_2^\alpha E_3^\beta) \), so that it suffices to show \( \mathcal{S}_k \) is spanned by \( E_2 \) and \( E_3 \).

Let \( f \in \mathcal{S}_k \), then \( f/\Delta \in \mathcal{M}_k \) since \( \Delta \) is non-zero on \( \mathfrak{h} \) and has a simple pole at \( 0 \). Thus by induction we need to show for \( k = 0, 1, 2, 3, 4, 5 \) that \( \mathcal{M}_k \subset \mathbb{C}[E_2, E_3] \).

We leave this easy exercise, which follows from Proposition 1.8 for the reader to check.

**Exercise: Use Proposition 1.8 to show that \( \mathcal{M}_k \subset \mathbb{C}[E_2, E_3] \) for \( k = 0, \ldots, 5 \)**

**Proof.** (of Theorem 1.4) It is clear any function is unique as the difference would be a holomorphic map on a compact Riemann surface, hence constant and since \( j(\omega) = 0 \) this constant must be 0. Thus it suffices to show \( j \) satisfies the conditions of the theorem.

i) This follows immediately from Corollary 1.9.

ii) We need to show for \( a \in \mathbb{C} \), \( j(z) - a \) has a zero of order 1. Since \( j(z) - a \) is a modular function of weight 0 with a simple pole at \( \infty \), Proposition 1.8 shows that

\[
\sum_{z \in \Gamma \backslash \mathfrak{h}} \frac{v_z(j(a) - a)}{e_z} = 1
\]

It follows that \( j - a \) has a unique zero on \( \Gamma \backslash \mathfrak{h} \).

iii) This follows from an explicit computation, see Serre’s ”A course in Arithmetic.”
We now show that any \( f \in M'_0 \) is a rational function of \( j \). Suppose \( f \in M'_0 \) has poles at \( a \). We may multiply \( f \) by a suitable power of \( j - j(a) \) so that \( f \) is holomorphic at \( a \), hence we may assume \( f \) is holomorphic on \( \Gamma \setminus \mathfrak{h} \). Suppose \( f \) has a pole of order \( n \) at \( \infty \), then \( f^{n} \in M_{6n} \), but we know \( M_{6n} \) is spanned by \( E_2^\alpha E_3^\beta \) where \( 2\alpha + 3\beta = 6n \). Thus it suffices to show the result for \( E_2^\alpha E_3^\beta \).

But \( 2\alpha + 3\beta = 6n \) implies \( 3|\alpha \) and \( 2|\beta \), so that it suffices to show the result for \( E_2^3 \) and \( E_3^3 \). This follows from the identities

\[
\frac{E_2^3}{\Delta} = \frac{g_3^2}{60^3 \Delta} = \frac{1}{1728.60^3 j} \quad \frac{E_3^3}{\Delta} = \frac{g_3^2}{140^2 \, \Delta} = \frac{1}{140^2 \, 27} \left( \frac{g_3^3}{\Delta} - 1 \right) = \frac{1}{1728.140^2 \, 27} j - \frac{1}{140^2 \, 27} \]

\( \square \)