Exercise 1. For \( p \) an odd prime, let \( \zeta \in \overline{\mathbb{F}}_p \) be a primitive 8-th root of unity. Show that \( \zeta + \zeta^{-1} \) represents \( \sqrt{2} \). Use this to prove \( \left( \frac{2}{p} \right) = (-1)^{\frac{p^2-1}{8}} \).

Exercise 2. Let \( p \) be an odd prime. We view the Legendre symbol \( \left( \frac{\cdot}{p} \right) \) as an element \( \sigma \in \text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \{ \pm 1 \} \), where \(-1 \) acts on \( \mathbb{Q}(i) \) via \( i \mapsto -i \). Show that \( \sigma a \equiv a^p \mod p \) for \( a \in \mathbb{Z}[i] \) and any prime factor \( p \in \mathbb{Z}[i] \) of \( p \). (Recall the different ways \( p \) factorizes into prime factors inside \( \mathbb{Z}[i] \))

Exercise 3. Prove that a UFD is integrally closed.

Exercise 4. Show that the identity \( 3 \cdot 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}) \) gives two essentially different factorizations of 21 into irreducible elements in the ring \( \mathcal{O}_{\mathbb{Q}(\sqrt{-5})} = \mathbb{Z}[\sqrt{-5}] \). Therefore \( \mathbb{Z}[\sqrt{-5}] \) is not a UFD.

Exercise 5. In the previous exercise, we saw that unique factorization fails in \( \mathbb{Z}[\sqrt{-5}] \) because for instance
\[
3 \cdot 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}).
\]
But we have the factorizations into prime ideals
\[
(3) = (3, \sqrt{-5} + 1)(3, \sqrt{-5} - 1),
\]
\[
(7) = (7, \sqrt{-5} + 3)(7, \sqrt{-5} - 3),
\]
\[
(1 + 2\sqrt{-5}) = (3, \sqrt{-5} - 1)(7, \sqrt{-5} - 3)
\]
\[
(1 - 2\sqrt{-5}) = (3, \sqrt{-5} + 1)(7, \sqrt{-5} + 3).
\]
Prove these identities.