1. Tamagawa number of $\text{SL}_2$

Consider the action of $\text{SL}_2(\mathbb{R})$ on the upper half plane $\mathcal{H}$ via Möbius transformations. The subgroup $\text{SL}_2(\mathbb{Z})$ has a famous fundamental domain whose closure is

$$F = \left\{ z = x + iy \mid -1/2 \leq x \leq 1/2, y \geq \sqrt{1-x^2} \right\}.$$

Consider the Poincaré metric on $\mathcal{H}$, given by $d\sigma^2 = dx^2 + y^2 dy^2$. Under this metric $\mathcal{H}$ has constant curvature $-1$, and $F$ is a (generalized) geodesic triangle with angles $\pi/3, \pi/3, 0$. By Gauss-Bonnet or direct calculation the area of $F$ is equal to $\pi/3$ under the measure $y^{-2} dx dy$ induced by $d\sigma^2$.

Let $K = \text{SO}_2(\mathbb{R}) \subset \text{SL}_2(\mathbb{R})$. We have $\text{SL}_2(\mathbb{R})/K \sim \to \mathcal{H}, g \mapsto gi$. From this we see that $F$, which is roughly $\text{SL}_2(\mathbb{Z}) \setminus \mathcal{H} = \text{SL}_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{R})/K$, is closely related to the quotient $\text{SL}_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{R})$.

We now make this more precise.

We have the Iwasawa decomposition for $\text{SL}_2(\mathbb{R})$:

$$N \times A^+ \times K \sim \to \text{SL}_2(\mathbb{R})$$

$$(u \in \mathbb{R}, a \in \mathbb{R}_{>0}, \phi \in [0, 2\pi[) \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & \phi \\ a^{-1} \sin \phi & \cos \phi \end{pmatrix}.$$  

Thus $N \times A^+ \sim \to \mathcal{H}$. We introduce a Haar measure on $\text{SL}_2(\mathbb{R})$ in the following way. Consider the differential form

$$\omega = x^{-1} \, dx \wedge dy \wedge dz,$$

where $x, y, z$ are the coordinates in $\begin{pmatrix} x & y/2 \\ z & 1+yz \end{pmatrix}$. It is easy to see that $\omega$ is bi-invariant. Thus we get a Haar measure on $\text{SL}_2(\mathbb{R})$:

$$\mu_\infty = |\omega| := |x^{-1}| \, dx \, dy \, dz,$$

where $dx, dy, dz$ mean the Lebesgue measures on $\mathbb{R}$.

Exercise 1.0.1. $\mu = a^{-3} \, du \, da \, d\phi$ in the $(u, a, \phi)$ coordinates.

The quotient measure on $\mathcal{H}$ induced by $d\mu$ is $2^{-1} y^{-2} \, dx \, dy$. Moreover, it is not hard to see that a fundamental domain for the left action of $\text{SL}_2(\mathbb{Z})$ on $\text{SL}_2(\mathbb{R})$ is given by $F \times \{ \phi \in [0, \pi[ \}$ . Thus the knowledge that $F$ has area $\pi/3$ is equivalent to

$$\mu_\infty(\text{SL}_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{R})) = \pi^2/6.$$  

This number is of course the value of Riemann’s zeta function $\zeta(s)$ at $s = 2$.
Recall we got the measure $\mu_\infty$ on $\text{SL}_2(\mathbb{R})$ from the invariant differential form $\omega$ by taking absolute values. This still makes perfect sense if we replace the archimedean absolute value by the $p$-adic ones. We write

$$\mu_p = |\omega|_p = |x^{-1}|_p \, dx \, dy \, dz,$$

where $|\cdot|_p$ is the $p$-adic absolute value and $dx, dy, dz$ are the Haar measures on the additive group $\mathbb{Q}_p$ normalized by requiring $\mathbb{Z}_p$ has volume 1. This expression gives a Haar measure $\mu_p$ on $\text{SL}_2(\mathbb{Q}_p)$. Let’s compute the volume of $\text{SL}_2(\mathbb{Z}_p)$ under this measure. Firstly, consider the reduction map $\text{SL}_2(\mathbb{Z}_p) \to \text{SL}_2(\mathbb{F}_p)$.

It is surjective. Call the kernel $\Gamma$. Then we have $\mu_p(\text{SL}_2(\mathbb{Z}_p)) = \# \text{SL}_2(\mathbb{F}_p) \cdot \mu_p(\Gamma)$, and

$$\mu_p(\Gamma) = \int_{x,y,z \in p\mathbb{Z}_p} dx \, dy \, dz = p^{-3}.$$ 

But $\# \text{SL}_2(\mathbb{F}_p) = p(p^2 - 1)$, so we get

$$\mu_p(\text{SL}_2(\mathbb{Z}_p)) = 1 - p^{-2}.$$ 

These numbers relate to the number $\pi^2/6$ in (2) by

$$\pi^2/6 = \sum_n 1/n^2 = \prod_p (1 - p^{-2})^{-1},$$

and consequently

(3) \hspace{1cm} \mu_\infty(\text{SL}_2(\mathbb{Z})\backslash \text{SL}_2(\mathbb{R})) \cdot \prod_p \mu_p(\text{SL}_2(\mathbb{Z}_p)) = 1.

Does (3) result from smart normalizations of Haar measures? No. In fact, the measures $\mu_v$ all come from the same invariant differential form $\omega$ defined over $\mathbb{Q}$. Any other such choice will be $c\omega, c \in \mathbb{Q}^\times$, and each measure $\mu_v$ will be scaled by $|c|_v$. But $\prod_v |c|_v = 1$, so (3) will remain true.

We now rewrite (3) in an even neater way. Recall the adelic group $\text{SL}_2(\mathbb{A})$ is the restricted product of $\text{SL}_2(\mathbb{Q}_v)$ with respect to the compact open subgroups $\text{SL}_2(\mathbb{Z}_v)$. $\text{SL}_2(\mathbb{Q})$ sits diagonally inside $\text{SL}_2(\mathbb{A})$ and is discrete. We define the Tamagawa measure on $\text{SL}_2(\mathbb{A})$ to be the product measure of the $\mu_v$’s, which is Haar measure canonically defined independent of choices.

**Definition 1.0.2.** The Tamagawa number $\tau(\text{SL}_2)$ of $\text{SL}_2$ is the volume of $\text{SL}_2(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{A})$, under the Tamagawa measure.

Recall strong approximation: $\text{SL}_2(\mathbb{A}) = \text{SL}_2(\mathbb{Q})(\text{SL}_2(\mathbb{R}) \times \prod_p \text{SL}_2(\mathbb{Z}_p))$. Consequently $\text{SL}_2(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{A})$ has a fundamental domain of the form $\Omega \times \prod_p \text{SL}_2(\mathbb{Z}_p)$, where $\Omega \subset \text{SL}_2(\mathbb{R})$ is a fundamental domain for $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})$. Thus (3) can be reinterpreted as

**Proposition 1.0.3.** $\tau(\text{SL}_2) = 1.$
2. Weil’s conjecture

Let $G$ be any semisimple group over $\mathbb{Q}$. The construction above carries over, and we define the Tamagawa measure, and Tamagawa number for $G$. More precisely, we take a top degree invariant differential form $\omega$ on $G$, defined over $\mathbb{Q}$. For each $\mathbb{Q}_v$ choose a Haar measure on $\left( \mathbb{Q}_v, + \right)$ s.t. the volume of $\mathbb{Z}_v$ is 1 when $v$ is finite and the volume of $[0,1]$ is 1 when $v = \infty$. With these choices $|\omega_v|$ defines a Haar measure $\mu_v$ on $G(\mathbb{Q}_v)$. Define $\mu = \prod_v \mu_v$, a Haar measure on $G(\mathbb{A})$, called the Tamagawa measure. Since two choices of $\omega$ differ by $\mathbb{Q} \times \mathbb{Q}$, by the product formula $\mu$ is well defined. Define the Tamagawa number to be

$$\tau(G) = \mu(G(\mathbb{Q}) \backslash G(\mathbb{A}))$$

However note that it is a non-trivial fact that the local measures $\mu_v$ multiply together to give a measure on $G(\mathbb{A})$, i.e. that the infinite product $\prod_p \mu_p(G(\mathbb{Z}_p))$ absolutely converges.

**Theorem 2.0.1** (Conjecture of Weil, Theorem of Langlands–Lai–Kottwitz). If $G$ is simply connected, then $\tau(G) = 1$.

This was originally conjectured by Weil, who examined many cases.

For a general semisimple group $G$, $\tau(G)$ can be related to $\tau(\tilde{G})$ in a certain fashion (due to Ono), where $\tilde{G}$ is the simply connected cover of $G$. The formulation can also be extended to reductive groups, although in that case the Tamagawa number is defined in a more subtle way (always a finite number but in general no longer the volume of $G(\mathbb{Q}) \backslash G(\mathbb{A})$ under any Haar measure, which can be infinite.)

We have the following

**Theorem 2.0.2** (Kottwitz-Sansuc). Let $G$ be a reductive group over a number field, then

$$\tau(G) = \frac{|\text{Pic} G|}{|\text{IHG}|}.$$  

This Theorem follows from Theorem 2.0.1 and the understanding of Tamagawa numbers of tori (due to Ono).

**Remark 2.0.3.** One could also consider Tamagawa numbers for reductive groups over a number field $F$. The problem reduces to $F = \mathbb{Q}$ by considering Weil restriction of scalars.

3. Mass formulas

From the knowledge of Tamagawa numbers, one can deduce a lot of interesting formulas. The classical example is the Smith–Minkowski–Siegel mass formula for quadratic forms.

Let $q$ be a $\mathbb{Z}$-valued quadratic form on $\mathbb{Z}^n$. We get an algebraic group $G = O_q$ defined over $\mathbb{Z}$. Let $q'$ be another $\mathbb{Z}$-valued quadratic form on $\mathbb{Z}^n$. We say $q'$ is isomorphic to $q$, if they are related by changing coordinates using a matrix in $\text{GL}_n(\mathbb{Z})$. We say $q'$ and $q$ are in the same genus, if they are equivalent over $\mathbb{R}$ and equivalent over $\mathbb{Z}/N\mathbb{Z}$ for all $N \in \mathbb{Z}$. By a compactness argument, $q'$ and $q$ are in the same genus if and only if they are related by a matrix in $\text{GL}_n(\mathbb{R}) \times \text{GL}_n(\hat{\mathbb{Z}}) = \text{GL}_n(\mathbb{R}) \times \prod \text{GL}_n(\mathbb{Z}_p)$. By the Hasse principle, if $q$ and $q'$ are in the same genus, then they are also equivalent over $\mathbb{Q}$.
Let $X_q$ be the set of isomorphism classes in the genus of $q$. Claim: we have a bijection of finite sets

$$X_q \xrightarrow{\sim} Y = G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}}).$$

What is the map here? Given $q' \in X_q$, by assumption $\exists a \in \text{GL}_n(\mathbb{Q}), b \in \text{GL}_n(\hat{\mathbb{Z}})$ such that $q' = q(a \cdot), q = q'(b)$. Then $ab \in G(\mathbb{A}_f)$ gives a well-defined element of $Y$. Injectivity is easy to show. For example, if $ab = [1]$ in $Y$, then we may re-choose $a, b$ such that $ab = 1 \in G(\mathbb{A}_f)$. Then $a = b^{-1} \in \text{GL}_n(\mathbb{Q}) \cap \text{GL}_n(\hat{\mathbb{Z}}) = \text{GL}_n(\mathbb{Z})$, which shows that $q, q'$ are isomorphic. Surjectivity follows from the fact that the analogue of the set $Y$ for $\text{GL}_n$ is trivial (which follows from strong approximation for $\text{SL}_n$, and the fact that $\mathbb{Q}$ has class number 1). The finiteness of $Y$ is called “finiteness of the class number of $G$”, and is a consequence of reduction theory.

We are interested in computing

$$M(q) = \sum_{q' \in X_q} |\text{Aut } q'|^{-1} = \sum_{q' \in X_q} |\text{O}_q(\mathbb{Z})|^{-1},$$

called the mass of the genus of $q$. From the claim, it easily follows

$$M(q) = \sum_{\gamma \in Y} |\text{G(\hat{\mathbb{Z}}) \cap \gamma^{-1}\text{G(\mathbb{Q})}\gamma}|^{-1}. $$

The above formula is also equivalent to:

$$M(q) = \frac{\mu(G(\mathbb{A})/G(\mathbb{Q}))}{\mu(G(\hat{\mathbb{Z}} \times \mathbb{R}))},$$

where $\mu$ is any Haar measure on $G(\mathbb{A})$.

Note that the algebraic group $G$ is not connected: it has two connected components, and the identity component $G^0 = \text{SO}_q$ (which is the kernel of $\text{det} : G \to \mu_2$) is semi-simple. Define

$$M^0(q) = \frac{\mu(G^0(\mathbb{A})/G^0(\mathbb{Q}))}{\mu(G^0(\hat{\mathbb{Z}} \times \mathbb{R}))},$$

where $\mu$ is any Haar measure on $G^0(\mathbb{A})$. If we take $\mu$ to be the Tamagawa measure, then we get

$$M^0(q) = \frac{\tau(G^0)}{\mu(G^0(\hat{\mathbb{Z}} \times \mathbb{R}))}.$$

But $\tau(G^0) = 2$, and $\mu(G^0(\hat{\mathbb{Z}} \times \mathbb{R}))$ can be calculated explicitly, so we obtain an explicit formula for $M^0(q)$.

In order to go back to $M(q)$, we let $d(R) \in \{1, 2\}$ denote the index $[G(R) : G^0(R)]$ for any $\mathbb{Z}$-algebra $R$. Observe that if $x \in \mathbb{Z}^n$ is any element satisfying $q(x) \neq 0$, then the reflection along $x$ defines a $\mathbb{Z}[\frac{1}{q(x)}]$-valued point of the component $G - G^0$.

It easily follows that $d(\mathbb{Q}) = d(\mathbb{R}) = 2$, and $d(\mathbb{Z}_p) = 2$ for almost all primes $p$. In particular, $\prod_p \text{det} : G(\mathbb{A}) \rightarrow \prod_p \{\pm 1\}$ is surjective. Comparing (4) and (5), we get

$$M(q) = 2^{k-1} M^0(q),$$

where $k$ is the number of primes $p$ such that $d(\mathbb{Z}_p) = 1$. Finally, note that if $q$ is self-dual over $\mathbb{Z}_p$, then there exists $x \in \mathbb{Z}_p^n$ such that $q(x) = \mathbb{Z}_p^\times$, and reflection along $x$ is a $\mathbb{Z}_p$-point of $G - G^0$, i.e., $d(\mathbb{Z}_p) = 2$. Hence if $q$ is unimodular, then $k = 0$. 

**Example 3.0.1.** Assume \( q \) is unimodular and positive definite of rank \( n \) (where \( n \) is necessarily divisible by \( 8 \)). Then

\[
M(q) = \left| \frac{B_{n/4}}{n} \right| \prod_{1 \leq j < n/2} \frac{B_{2j}}{4j}.
\]

In fact, in this case we also know that the genus of \( q \) consists of all the unimodular and positive definite quadratic lattices of rank \( n \), so we may write \( X_n \) for \( X_q \) and write \( M(n) \) for \( M(q) \). For \( n = 8 \), the above formula gives \( M(8) = 696729600^{-1} \).

On the other hand, we have \( E_8 \in X_8 \), and \( |\text{Aut} E_8| = 696729600 \). It follows that \( X_8 = \{ E_8 \} \). For \( n = 32 \), the formula implies \( |X_n| > \) eighty million.

Similar arguments as before can be used to obtain mass formulas for other arithmetic objects. We consider one more example.

Recall an elliptic curve \( E \) over \( \overline{\mathbb{F}_p} \) is said to be supersingular if one of the following two equivalent conditions holds:

1. \( E(\overline{\mathbb{F}_p})[p] = 0 \).
2. \( \text{End}(E) \) has rank 4 over \( \mathbb{Z} \).

For any such \( E \), \( \text{End}(E) \otimes \mathbb{Q} \) is the Quaternion algebra \( D \) over \( \mathbb{Q} \) ramified at \( \infty \), \( p \), and \( \text{End}(E) \) is a maximal order of \( D \). Fix an \( E_0 \). Let \( G = D^\times \) as an algebraic group over \( \mathbb{Q} \). The order \( \text{End} E_0 \subset D \) gives a \( \mathbb{Z} \)-structure of \( G \). Similar to the situation before we have a bijection

\[
\{ E/\overline{\mathbb{F}_p} \text{ s.s.} \} / \cong \rightarrow G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\mathbb{R} \times \hat{\mathbb{Z}}).
\]

The map here is defined using the Tate module away from \( p \) and the Dieudonné module at \( p \). Analogous as before, the input about \( \tau(G) \) (although here \( G \) is only reductive, not semi-simple) gives us a mass formula, due to Eichler-Deuring:

\[
\sum_{E/\overline{\mathbb{F}_p} \text{ s.s.}} |\text{Aut} E|^{-1} = \frac{p^{-1}}{24}.
\]

Note that since \( \text{Aut} E \) are easy to understand, we can get the precise number of supersingular elliptic curves from the above formula, which is

\[
\left[ \frac{p}{12} \right] + \begin{cases} 0, & p \equiv 1 \mod 12 \\ 1, & p \equiv 5, 7 \mod 12 \\ 2, & p \equiv 11 \mod 12. \end{cases}
\]

**4. Relation to the BSD conjecture**

The equality \( \tau(G) = |\text{Pic}(G)_{\text{tors}}| / |\text{III}G| \), which holds for reductive groups \( G \), may still make sense for more general \( G \), e.g. abelian varieties. Of course the finiteness of \( \text{III}G \) is very non-trivial, and only known for some elliptic curves. Nevertheless we have the following interesting relation to the BSD conjecture (for abelian varieties) discovered by S. Bloch [A Note on Height Pairings, Tamagawa Numbers, and the Birch and Swinnerton-Dyer Conjecture].

Let \( A \) be an abelian variety over a number field \( K \). We assume \( A(K) \) is a finite group. Let \( L(s, A) \) be the Hasse-Weil zeta function of \( A \) away from the bad primes. Then the BSD conjecture predicts that \( L(s) \) is non-zero, holomorphic at \( s = 1 \), and

\[
L(1, A) = \frac{|\text{III}A| \cdot \det(\langle \cdot, \cdot \rangle) \cdot V_{\text{bad}} V_{\infty}}{|A(K)| \cdot |\text{Pic}(A)_{\text{tors}}|}.
\]
Block constructs a torus extension of $A$:

$$0 \to T \to X \to A \to 0,$$

and deduces \[\text{[6]}\] from the following hypotheses

- $\text{III}_X$ is finite.
- $\tau(X) = |\text{Pic}(X)_{\text{tor}}| / |\text{III}_X|.$

5. Sketch of the proof of Theorem 2.0.1

Throughout we admit the following result, whose proof uses the Langlands theory of Eisenstein series:

**Theorem 5.0.1** (Langlands, Lai). *The conjecture holds if $G$ is quasi-split.*

Recall that for $G$ arbitrary, it has a (essentially unique) inner form $G_0$ which is quasi-split. Here being an inner form means that there is an isomorphism

$$\psi : (G_0)_\bar{\mathbb{Q}} \xrightarrow{\sim} G_{\bar{\mathbb{Q}}},$$

s.t. for any $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the automorphism $\sigma(\psi)^{-1} \psi : G_{\bar{\mathbb{Q}}} \to G_{\bar{\mathbb{Q}}}$ is inner, i.e. given by $\text{Int}(g)$ for some $g \in G(\bar{\mathbb{Q}})$.

We will sketch the proof of the following

**Theorem 5.0.2** (Kottwitz). $\tau(G) = \tau(G_0)$.

The proof inducts on $\text{dim} \ G$. Note that for a given dimension $n$, Theorem \[5.0.2\] for $G$ with $\text{dim} \ G \leq n$ implies Theorem \[2.0.1\] (since we admit Theorem \[5.0.1\]), which in turn implies Theorem \[2.0.2\] for reductive groups whose derived subgroups have dimensions $\leq n$ (with anisotropic center). In particular, when we prove Theorem \[5.0.2\] for $G$, we may assume the following is known: For any $\gamma \in G(\mathbb{Q})$ semi-simple and non-central, (s.t. $G_{\gamma}$ has anisotropic center), we know that $G_{\gamma}$ and any inner form of it have equal Tamagawa numbers, given by Theorem \[2.0.2\].

**Example 5.0.3.** Let $V$ be an $n$-dimensional quadratic space over $\mathbb{Q}$, with $n$ odd. Let $V_0$ be the $n$-dimensional quadratic space with quadratic form $X_1X_2 + X_3X_4 + \cdots + X_{n-2}X_{n-1} + X_n^2$. Then $G = \text{SO}(V)$ or $\text{Spin}(V)$, $G_0 = \text{SO}(V_0)$ or $\text{Spin}(V_0)$. The group $G$ is anisotropic iff $V$ is anisotropic.

**Example 5.0.4.** Let $D$ be a central simple algebra of dimension $n^2$ over $\mathbb{Q}$. Then $G = D^\times$ or $D^\times/\mathbb{Q}^\times$ or $D^\times, \text{Nrd}=1$, and $G_0 = \text{GL}_n$ or $\text{PGL}_n$ or $\text{SL}_n$. The group $G$ is anisotropic mod center iff $D$ is a division algebra.

The idea of the proof of Theorem \[5.0.2\] is to compare the Arthur-Selberg trace formulas for $G$ and $G_0$. Since $G_0$ is not anisotropic, we need a version of simple trace formulas which is valid for special test functions.

5.1. **Anisotropic trace formula.** As a motivation, recall the Arthur-Selberg trace formula for $G$ anisotropic. In this case $G(\mathbb{Q}) \backslash G(\mathbb{A})$ is compact. Let $\rho$ denote the action of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. Let $d \gamma$ be the Tamagawa measure on $G(\mathbb{A})$. For $f \in C_c^\infty(G(\mathbb{A}))$, define

$$\rho(f) = \rho(f, d \gamma) : L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) \to L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$$

$$v \mapsto \int_{g \in G(\mathbb{A})} \rho(g) v f(g) \, d \gamma.$$
Theorem 5.1.1. $\rho(f)$ is of trace class, and

$$\text{Tr} \rho(f) = \sum_{\gamma} a_{\gamma}(d\,i)O_{\gamma}(f, d\,g, d\,i).$$

Here $\gamma$ runs through the elements of $G(\mathbb{Q})$ (necessarily semi-simple) up to $G(\mathbb{Q})$-conjugacy, and

$$a_{\gamma}(d\,i) := \text{vol}_{d\,i}(G_{\gamma}(\mathbb{Q}) \setminus G_{\gamma}(\mathbb{A}))$$

$$O_{\gamma}(f, d\,g, d\,i) := \int_{g \in G_{\gamma}(\mathbb{A}) \setminus G(\mathbb{A})} f(g^{-1}g_{\gamma}) \frac{dg}{d\,i},$$

where $d\,i$ is any chosen Haar measure on $G_{\gamma}$.

5.2. Simple trace formula. We want to write down trace formulas for both $G$ and $G_0$. Even if we are only interested in anisotropic $G$, the group $G_0$ is in general not anisotropic. Therefore we need the Arthur-Selberg trace formula for general semi-simple groups. In general it is very complicated, but for special test functions, it assumes a shape almost identical to the anisotropic case.

Theorem 5.2.1 (Arthur). Let $G$ be semi-simple simply connected. Let $f \in C_c^\infty(G(\mathbb{A}))$ be of the form $f = f_{v_1}f_{v_2}f_{v_1}^{\nu_1\nu_2}$, where $v_1 \neq v_2$ are two finite places, such that

A1: $f_{v_1}$ is the coefficient of a supercuspidal representation of $G(\mathbb{Q}_{v_1})$.

A2: $f_{v_2}$ has the property that for all $\gamma \in G(\mathbb{Q}_{v_2})$, $O_{\gamma}(f_{v_2}) = 0$ unless $\gamma$ is semi-simple elliptic. Here elliptic means that $G_{\gamma}$ contains a maximal torus that is elliptic (i.e., compact) in $G_{\mathbb{Q}_{v_2}}$. Then the operator $\rho(f) : L^2(G(\mathbb{Q}) \setminus G(\mathbb{A})) \to L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ has image in $L^2_{\text{cusp}}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ and is of trace class, and its trace is given by

$$\text{Tr} \rho(f) = \sum_{\gamma} a_{\gamma}(d\,i)O_{\gamma}(f, d\,g, d\,i).$$

Here $\gamma$ runs through conjugacy classes in $G(\mathbb{Q})$ that are semi-simple and elliptic.

5.3. Comparison. To proceed, consider $G$ and $G_0$ as before, with an inner twisting $\psi : G_0 \to G$. Take $S$ to be a large enough finite set of places, including $\infty$ at least one finite place, s.t. for all $v \notin S$, $G_{\mathbb{Q}_v} \cong G_{0,\mathbb{Q}_v}$ as $G_{\mathbb{Q}_v}$ groups. More precisely, we assume that for all $v \notin S$, $\psi$ is the composition of a $\mathbb{Q}_v$-isomorphism $\phi_v : G_{0,\mathbb{Q}_v} \cong G_{\mathbb{Q}_v}$ with an inner automorphism of $G_{\mathbb{Q}_v}$. We think of $\phi_v$ as the identity and omit it from the notation. We will define $f = \prod_v f_v$ on $G(\mathbb{A})$ and $f_0 = \prod_v (f_0)_v$ on $G_0(\mathbb{A})$ satisfying the hypotheses of the above Theorem 5.2.1 and write down the two trace formulas for $G$ and $G_0$. More precisely, we fix a place $v_{\text{sc}} \notin S$ s.t. the group $G(\mathbb{Q}_{v_{\text{sc}}}) = G_0(\mathbb{Q}_{v_{\text{sc}}})$ admits a supercuspidal representation. (e.g. choose $v_{\text{sc}}$ to split $G$.) We will choose $f_{v_{\text{sc}}} = (f_0)_{v_{\text{sc}}}$ to be the coefficient of such a supercuspidal representation s.t. $f_{v_{\text{sc}}}(1) \neq 0$.

Let

$$S_1 = S \cup \{v_{\text{sc}}\}.$$ 

We choose $f^{S_1} = (f_0)^{S_1}$ arbitrarily, on the same group $G(\mathbb{A}^{S_1}) = G_0(\mathbb{A}^{S_1})$ (with the usual condition that for a.a. $v$, $f_v = (f_0)_v = 1_{G(\mathbb{Q})}$). We postpone the definition.

\footnote{Since reductive groups over $\mathbb{Q}_{v_2}$ have elliptic maximal tori, it is equivalent to the condition that $Z_{G_{\gamma}}$ is anisotropic over $\mathbb{Q}_{v_2}$.}
of $f_S$ and $(f_0)_S$ for a moment, but suppose for now that for a place $v \in S$ the condition $\textbf{A2}$ holds for $f_v$ and $(f_0)_v$.

Then by Theorem $5.2.1$ we have

\[
\text{Tr} \rho_{\text{cusp}}(f) = \tau(G) \sum_{\zeta \in Z(\mathbb{Q})} f(\zeta) + \sum_{\gamma \in G(F) \setminus Z(F), \text{ell ss./conj}} a_{\gamma}(d \, i) \rho_{\gamma}(f, d \, g, d \, i)
\]

and similarly

\[
\text{Tr} \rho_{\text{cusp}}(f_0) = A_0 + B_0.
\]

Here $\rho_{\text{cusp}}(f)$ denotes the restriction of $\rho(f)$ to $L^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$, which is an invariant subspace of $\rho(f)$ according to Theorem $5.2.1$.

**Proposition 5.3.1.** Assume, as an induction hypothesis, that Theorem 5.0.2 is true for all $G_\gamma$ and $(G_0)_\gamma_0$ that appear in $B$ and $B_0$. It is possible to choose $f_S$ and $(f_0)_S$, s.t. for any choice of $f_S$ and $(f_0)_S$, the following statements hold for $f := f_S f_v \in S$ and $f_0 := (f_0)_S (f_0)_v f_S$.

(I): $B(f) = B_0(f_0)$

(II): $f_S(1) = (f_0)_S(1) \neq 0$. In particular $f(1) = f_0(1)$.

(III): For one (actually all) $v \in S - \{1\}$, the functions $f_v$ and $(f_0)_v$ both satisfy the condition $\textbf{A2}$.

**Proof of Theorem 5.0.2 assuming Proposition 5.3.1** Choose $f_S$ and $(f_0)_S$ as in Proposition 5.3.1. Fix a place $v_0 \not\in S_1$ to be used later, s.t. $G(\mathbb{Q}_{v_0}) = G_0(\mathbb{Q}_{v_0})$ is non-compact and unramified. For all $v' \not\in S_1 \cup \{v_0\}$, fix the choice of $(f_0)_v$ s.t. $f_v(1) \neq 0$. Moreover for each such $v' \not\in S_1 \cup \{v_0\}$, shrink the support of $f_v$ if necessary to assume that for $\zeta \in Z(F) - \{1\}$, $f_v(\zeta) = 0$. Then we have

\[
\text{Tr} \rho_{\text{cusp}}(f) - \text{Tr} \rho_{\text{cusp}}(f_0) = (\tau(G) - \tau(G_0)) f(1).
\]

Let $f_{v_0} = (f_0)_{v_0}$ vary. Suppose $\tau(G) \neq \tau(G_0)$. Recall that we assumed that $f_{v_0}(1) = (f_0)_{v_0}(1) \neq 0$. Thus by (II), \eqref{eq:tracial} expresses a non-zero multiple of the functional $f_{v_0} \mapsto f_{v_0}(1)$ in terms of a discrete sum $f_{v_0} \mapsto \sum c_i \text{Tr} \pi_i(f_{v_0})$, where $\pi_i$ are the unitary representations of $G(\mathbb{Q}_{v_0})$. This contradicts with the fact that the Plancherel formula for the non-compact group $G(\mathbb{Q}_{v_0})$ has a continuous part. \qed

The rest of this exposition is devoted to sketching a proof of Proposition 5.3.1.

### 5.4. Recall of stable conjugacy

Let $G$ be a reductive group over a field $F = \mathbb{Q}$ of $\mathbb{Q}_v$. Assume $G^{\text{der}}$ is simply connected.

**Definition 5.4.1.** Two semi-simple elements of $G(F)$ are stably conjugate if they are conjugate in $G(\bar{F})$.

Let $\psi : G_0 \to G$ be the inner twist from the quasi-split inner form $G_0$. If $\gamma \in G(F)$ is semi-simple, the $G(\bar{F})$-conjugacy class of $\gamma$ is a subvariety of $G_F$ (which can be identified with $G(\bar{F})$) defined over $F$. The image under $\psi^{-1}$ of this subvariety is a subvariety of $(G_0)_F$ defined over $F$, and a theorem of Steinberg implies that this subvariety has an $F$-point. In this way we get a multi-valued map $G(F)_{ss} \to (G_0(F))_{ss}$. In fact this induces an injection

\[
\iota : G(F)_{ss/\text{stab. conj.}} \to G_0(F)_{ss/\text{stab. conj.}}
\]
Unfortunately, stable conjugacy among elements in $G(F)$ is not the same as conjugacy in $G(F)$, in general. The summation index sets for $B$ and $B_0$ however involves conjugacy in $G(\mathbb{Q})$. This difficulty, called endoscopy, adds complication to comparing $B$ with $B_0$. The general strategy to overcome this difficulty is to stabilize the expression $B$. There are two steps. The first step, sometimes called pre-stabilization, uses Galois cohomology to rewrite $B$ in terms of stable orbital integrals plus error terms. The second step equates the error terms with stable stabilized the expression to comparing $B$ in involves conjugacy in $G$. The second step assumes the Langlands-Shelstad transfer conjecture and the Fundamental Lemma. For the proof of Proposition 5.3.1, the second step is not needed, as the functions $f_{S_1}$ and $(f_0)_{S_1}$ to be chosen automatically kill the error terms in the first step.

Remark 5.4.2. In some cases, for instance when $G = D^\times/\mathbb{Q}^\times$ and $G_0 = \text{PGL}_n$ as in Example 5.0.4, there is no endoscopy, i.e. stable conjugacy is the same as conjugacy. In this particular example, $\iota$ induces a bijection between the summation index sets for $B$ and $B_0$. However in this case $G$ is not simply connected, and the subtlety that $G_\gamma$ is no longer connected intervenes.

5.5. The Euler-Poincaré measure and function. To prove Proposition 5.3.1, we choose $f_{S_1\setminus\{\infty\}}$ and $(f_0)_{S_1\setminus\{\infty\}}$ to be the Euler-Poincaé functions. We recall this concept below, and we use a local notation. Let $F = \mathbb{Q}_v$ be a local field, and $\psi : G_0 \to G$ an inner twist over $F$, with $G_0$ quasi-split.

Theorem 5.5.1 (Serre). There is a Haar measure $\text{d}\, g_{EP}$ on $G(F)$, called the Euler-Poincaré measure (positive or negative), characterized by the following condition: For all subgroup $\Lambda \subset G(F)$ which is discrete, cocompact, and torsion free, we have

$$\text{d}\, g_{EP}(\Lambda \backslash G(F)) = \chi(H^\ast(\Lambda, \mathbb{Q})).$$

Here the RHS is the Euler-Poincaré characteristic of the group cohomology of $\Lambda$.

When $F = \mathbb{R}$, the Theorem is a consequence of the Gauss-Bonnet-Chern Theorem, and when $F = \mathbb{Q}_p$, it uses the Bruhat-Tits theory of buildings.

Theorem 5.5.2 (Kottwitz). Assume $F$ is $p$-adic. There is a function $f_{EP} \in C_c^\infty(G(F))$, called an Euler-Poincaré function, s.t.

(1) $f_{EP}$ satisfies A2.

(2) Let $\gamma \in G(F)$ be elliptic semi-simple. Let $\text{d}\, g_{EP}$ and $\text{d}\, i_{EP}$ be the Euler-Poincaré measures on $G(F)$ and $G_\gamma(F)$ respectively. Then

$$O_\gamma(f_{EP}, \text{d}\, g_{EP}, \text{d}\, i_{EP}) = 1.$$

Definition 5.5.3. Let $\text{d}\, g$ on $G(F)$ and $\text{d}\, g_0$ on $G_0(F)$ be Haar measures. We say that they are associated, if there is a non-zero constant $\lambda \in \mathbb{R}$ s.t. $\text{d}\, g = \lambda |\omega|_F$ and $\text{d}\, g_0 = \lambda |\psi^\ast \omega|_F$, for some top degree invariant differential form $\omega$ on $G$ defined over $F$. ($\psi^\ast$ is also an invariant differential form on $G_0$ defined over $F$).

Theorem 5.5.4. Let $F$ be $p$-adic. The measures $e(G) \text{d}\, g_{EP}$ and $e(G_0) \text{d}\, g_{EP}$ are associated. Here $e(G) \in \{\pm 1\}$ is a sign canonically associated to the group $G$.

---

We have $e(G) = (-1)^{\text{dim}(G) - \text{dim}(G_0)}$, where by definition $\text{dim}(G)$ is one half of the $\mathbb{R}$-dimension of the symmetric space of $G(F)$ when $F = \mathbb{R}$, and $\text{dim}(G)$ is the $F$-rank of $G$ when $F$ is non-archimedean.
5.6. **Pre-stabilization.** The goal is to write $B$ in terms of the so-called $\kappa$-orbital integrals, which are local in nature. Here $\kappa$ is an element in a certain finite abelian group. For $\kappa = 1$, we get stable orbital integrals, which are nice objects; for other $\kappa$ we get what we think of as error terms. In the proof of Proposition 5.3.1 these error terms will automatically vanish.

**Definition 5.6.1.** Let $F = \mathbb{Q}$ or $\mathbb{Q}_v$. Let $G$ be a semi-simple, simply connected group over $F$. Let $\gamma \in G(F)_{ss}$. Let $I := G_{\gamma}$. Define

$$\mathcal{R}(\gamma/F) = \mathcal{R}(I/F) := \pi_0(Z(\hat{I})^{\text{Gal}F}).$$

It is a finite abelian group.

We come back to the global setting with $\psi : G_0 \to G$ an inner twisting between semi-simple simply connected groups over $\mathbb{Q}$ with $G_0$ quasi-split. Fix $\gamma_0 \in G_0(\mathbb{Q})_{ss}$. Let $I_0 := G_{0,\gamma_0}$. Let $\gamma \in G(\hat{\mathbb{A}})$ that is conjugate to $\psi(\gamma_0)$ in $G(\hat{\mathbb{A}})$. Kottwitz constructs for such $\gamma$ an element $\text{obs}(\gamma) \in \mathcal{R}(I_0/\mathbb{Q})^D := \text{Hom}(\mathcal{R}(I_0/\mathbb{Q}), \mathbb{C}^\times)$.

**Lemma 5.6.2** (Kottwitz). $\text{obs}(\gamma) = 1$ iff there is an element $\delta \in G(\mathbb{Q})$ s.t. $\delta$ is conjugate to $\gamma$ in $G(\hat{\mathbb{A}})$.

**Definition 5.6.3.** For $\kappa \in \mathcal{R}(I_0/\mathbb{Q})$ and $f \in C_c^\infty(G(\hat{\mathbb{A}}))$, define the $\kappa$-orbital integral:

$$\mathcal{O}^\kappa_{\gamma_0}(f, dg_0, d i_0) := \sum_{\gamma \in G(\hat{\mathbb{A}})/\text{conj. } \gamma \text{ conj. to } \psi(\gamma_0) \text{ in } G(\hat{\mathbb{A}})} e(\gamma) \langle \text{obs}(\gamma), \kappa \rangle O_{\gamma}(f).$$

Here $dg_0$ is a Haar measure on $G_0(\mathbb{A})$, $di_0$ is a Haar measure on $I_0(\mathbb{A})$, and each $O_{\gamma}(f) = O_{\gamma}(f, dg, di)$ is defined using $dg$ on $G(\mathbb{A})$ associated to $dg_0$ and $di$ on $G(\hat{\mathbb{A}})$ associated to $di_0$ (i.e. w.r.t. the inner twisting $\psi : G_0 \to G$ and a natural inner twisting $(I_0)_{\mathbb{Q}_v} \to G_{v,\gamma_0}$ for each $v$). The sign $e(\gamma) := \prod_v e(G_{v,\gamma_0})$. When $dg_0$ and $di_0$ are Tamagawa measures we omit them from the notation. Define

**Remark 5.6.4.** $\mathcal{O}^1_{\gamma_0}(f)$ is also known as $SO_{\gamma_0}(f)$, the stable orbital integral. When $\kappa \neq 1$, it follows from landmark theorems of Waldspurger and Ngô that $\mathcal{O}^\kappa_{\gamma_0}(f)$ is equal to some stable orbital integral on an endoscopic group.

**Definition 5.6.5.** Define $E^\kappa_0$ to be (a set of representatives in $G_0(\mathbb{Q})$ of) the set of stable conjugacy classes in $G_0(\mathbb{Q})$ which are semi-simple elliptic and non-central.

**Proposition 5.6.6** (Kottwitz). Assume the induction hypothesis as in Proposition 5.3.1. The expression $B$ is equal to

$$B = \sum_{\gamma_0 \in E^\kappa_0} \sum_{\kappa \in \mathcal{R}(G_0,\gamma_0/\mathbb{Q})} \mathcal{O}^\kappa_{\gamma_0}(f).$$

5.7. **The local nature of $\kappa$-orbital integrals.**

**Definition 5.7.1.** Let $v$ be a place of $\mathbb{Q}$. Let $\gamma_v \in G(\mathbb{Q}_v)_{ss}$. Let $\kappa_v \in \mathcal{R}(G_{\gamma_v}/\mathbb{Q}_v)$. For $f_v \in C_c^\infty(G(\mathbb{Q}_v))$, define

$$\mathcal{O}^\kappa_{\gamma_v}(f_v) = \sum_{\gamma'} e(G_{v,\gamma'}) \langle \text{inv}(\gamma_v, \gamma'), \kappa_v \rangle O_{\gamma'}(f_v).$$

\footnote{For $G$ reductive it is defined to be the subgroup of $\pi_0(\mathcal{H}(\hat{I}/\mathcal{H}(\hat{G}))^{\text{Gal}F})$ consisting of elements whose image in $\mathcal{H}(F, Z(\hat{G}))$ is (locally) trivial. Note that for $G$ semi-simple simply connected, $Z(\hat{G}) = 1$.}
Here $\gamma'$ runs through conjugacy classes in $G(Q_v)$ that are stably conjugate to $\gamma_v$. Note that the set of such conjugacy classes is in bijection with the set $\mathcal{D}(G_{\gamma_v}/Q_v) := \ker(H^1(Q_v,G_{\gamma}) \to H^1(Q_v,G))$, which has a map to $R(G_{\gamma_v}/Q_v)^D$. (This map is a bijection when $v$ is finite). Denote by $\imath(\gamma',\gamma)$ the image of $\gamma'$ in $\mathcal{D}(G_{\gamma}/\bar{K})$.

**Lemma 5.7.2.** Suppose the summation index set in the definition of the $\kappa$-orbital integral in Definition 5.6.3 is non-empty, (in which case we say $\gamma_v$ comes from $G(\bar{K})$). Let $\gamma \in G(\bar{K})$ be an element in that summation index set. Write $\gamma = (\gamma_v)_v$ and for each $v$ let $\kappa_v$ be the image of $\kappa$ under the natural map $\mathcal{R}(I_0/Q) \to \mathcal{R}(G_{\gamma_v}/Q_v)$. For $f = \prod f_v$, we have

$$O_{\gamma_v}(f) = \langle \imath(\gamma_v),\kappa_v \rangle \prod_v O_{\gamma_v}^\kappa_v(f_v)$$

**Proof.** We use the relation

$$\langle \imath(\gamma'),\kappa \rangle = \langle \imath(\gamma),\kappa \rangle \prod_v \langle \imath(\gamma_v,\gamma_v'),\kappa_v \rangle$$

for $\gamma' = (\gamma'_v)$ another element in $G(\bar{K})$ that is conjugate to $\psi(\gamma_0)$ in $G(\bar{K})$. \hfill $\square$

**Lemma 5.7.3.** Let $v$ be a finite place of $Q$. Let $\gamma_0 \in G(Q_v)_{ss}$ and $\kappa_v \in \mathcal{R}(G_{\gamma_v}/Q_v)$. Assume either $\kappa_v \neq 0$ or $\gamma_v$ is non-elliptic. Let $f_v$ be an Euler-Poincaré function on $G(Q_v)$. Then $O_{\gamma_v}^\kappa_v(f_v) = 0$.

**Proof.** In the summation (10) the elements $\imath(\gamma_v,\gamma_v')$ runs precisely through $\mathcal{R}(G_{\gamma_v}/Q_v)^D$. All $\gamma_v'$ are simultaneously elliptic or simultaneously non-elliptic, because the center of $G_{\gamma_v}$ is the same as that of $G_{\gamma_v}$. When $\gamma_v$ is elliptic, the terms $e(G_v,\gamma_v')O_{\gamma_v'}(f_v)$ is independent of $\gamma_v$, since for various $\gamma_v$ the orbital integrals $O_{\gamma_v}$ are computed using associated measures (i.e. for $\gamma_v$, there is a natural inner twisting $G_{\gamma_v} \to G_{\gamma_v}$).

**Corollary 5.7.4.** Assume the induction hypothesis as in Proposition 5.3.1. Assume $f = \prod f_v$ with one finite place $v_{EP}$ s.t. $f_{v_{EP}}$ is an Euler-Poincaré function. Then

$$B = \sum_{\gamma_0 \in E_{0,v}^*} O_{\gamma_0}^1(f).$$

**Proof.** By Proposition 5.6.6, it suffices to prove that for $\gamma_0 \in E_0^*$ and $\kappa_0 \in \mathcal{R}(I_0/Q) - \{0\}$ (where $I_0 = G_{\gamma_0}$), we have $O_{\gamma_0}^\kappa(f) = 0$. If $\gamma_0$ does not come from $G(\bar{K})$, there is nothing to prove. Assume the opposite, i.e. there is $\gamma = (\gamma_v)_v \in G(\bar{K})$ that is conjugate to $\psi(\gamma_0)$ in $G(\bar{K})$. If $\gamma_{v_{EP}}$ is elliptic, the map $\mathcal{R}(I_0/Q) \to \mathcal{R}(G_{\gamma_v}/Q_v)$ is injective. We are done by Lemma 5.7.2 5.7.3. \hfill $\square$

**5.8. Proof of Proposition 5.3.1**

**Proof.** For all $v \in S - \{\infty\}$ (assumed to be non-empty), choose $f_v$, resp. $(f_0)_v$, to be an Euler-Poincaré function on $G(F_v)$, resp. $G_0(F_v)$. Then (III) holds. By the work of Clozel-Delorme and Shelstad, it is possible to choose $f_\infty$ and $(f_0)_\infty$ s.t. they have $c(G_\mathbb{R})e(G_{0,\mathbb{R}})$-matching (i.e. matching or $-1$-matching) stable orbital integrals. This means the following:

- if $\gamma_{0,\infty} \in G_0(\mathbb{R})_{ss}$ and $\gamma_\infty \in G(\mathbb{R})_{ss}$ are s.t. $\gamma_\infty$ is conjugate to $\psi_0(\gamma_{0,\infty})$ in $G(\mathbb{C})$, then

$$e(G_\mathbb{R})O_{\gamma_\infty}(f_\infty) = e(G_{0,\mathbb{R}})O_{\gamma_{0,\infty}}^1(f_0)$$
• if \( \gamma_{0,\infty} \in G_0(\mathbb{R})_{ss} \) does not come from \( G(\mathbb{R}) \), in the sense that \( \gamma_{\infty} \) as above does not exist, then

\[
O^1_{\gamma_0}(f_{0,\infty}) = 0.
\]

Here \( O^1_{\gamma_0} \) and \( O^1_{\gamma_0} \) are defined as in (10).

By Corollary \[5.7.4\] we have

\[
B = \sum_{\gamma_0 \in E_0^*} O^1_{\gamma_0}(f)
\]

and

\[
B_0 = \sum_{\gamma_0 \in E_0^*} O^1_{\gamma_0}(f_0).
\]

To show \( B = B_0 \), it suffices to show that

\[
O^1_{\gamma_0}(f) = O^1_{\gamma_0}(f_0).
\]

If \( \gamma_0 \) does not come from \( G(\mathbb{A}) \), both sides are zero: \( O^1_{\gamma_0}(f) = 0 \) by definition and \( O^1_{\gamma_0}(f_0) = 0 \) because either (12) holds or there must be a finite place \( v \) in \( S_1 \) at which \( \gamma_0 \) is not elliptic. (Otherwise \( \gamma_0 \) comes from \( G_v \) for all \( v \in S_1 \), and we know \( \gamma_0 \) comes from \( G(\mathbb{A}^{an}) \).) Assume \( \gamma_0 \) comes from some \( \gamma = (\gamma_v) \in G(\mathbb{A}) \). Also write \( \gamma_{0,v} \) for the image of \( \gamma_0 \) in \( G_0(\mathbb{Q}_v) \). Then by Lemma \[5.7.2\]

\[
O^1_{\gamma_0}(f) = \prod_v O^1_{\gamma_v}(f_v)
\]

\[
O^1_{\gamma_0}(f_0) = \prod_v O^1_{\gamma_{0,v}}(f_{0,v}).
\]

If \( v \notin S \), we have \( O^1_{\gamma_v}(f_v) = O^1_{\gamma_{0,v}}(f_{0,v}) \), because \( f_v = f_{0,v} \) on \( G(\mathbb{Q}_v) \) and \( \gamma_v \) and \( \gamma_{0,v} \) are two elements of \( G(\mathbb{Q}_v) \) which are stably conjugate. For \( v = \infty \), the comparison is taken care of by (11). Let \( v \in S - \{\infty\} \). We may assume \( \gamma_v \) and \( \gamma_{0,v} \) are both elliptic, as otherwise they are both non-elliptic, and \( O^1_{\gamma_v}(f_v) = O^1_{\gamma_{0,v}}(f_{0,v}) = 0 \). If \( \mu \) is a Haar measure on a \( p \)-adic semi-simple group over \( \mathbb{Q}_p \), write \( \chi(\mu) \in \mathbb{R}^\times \) for the quotient of \( \mu \) by the Euler-Poincaré measure. By Theorem \[5.5.2\] and Theorem \[5.5.4\] we compute

\[
O^1_{\gamma_v}(f_v) = |D(G_{\gamma_v}/\mathbb{Q}_v)| e(\gamma_v) O^1_{\gamma_v}(f_v) = |D(G_{\gamma_v}/\mathbb{Q}_v)| e(\gamma_v) \chi(d g_v) \chi(d i_{0,v})^{-1}.
\]

Here \( d g_v \) is the local factor at \( v \) of the Tamagawa measure on \( G(\mathbb{A}) \), and \( d i_v \) is the Haar measure on \( G_{\gamma_v}(\mathbb{Q}_v) \) associated to the Haar measure on \( G_{0,\gamma_0}(\mathbb{Q}_v) \) which is the local factor of the Tamagawa measure on \( G_{0,\gamma_0} \). Also recall

\[
D(G_{\gamma_v}/\mathbb{Q}_v) := \ker(\text{H}^1(\mathbb{Q}_v, G_{\gamma_v}) \to \text{H}^1(\mathbb{Q}_v, G)).
\]

Similarly, we have

\[
O^1_{\gamma_{0,v}}(f_{0,v}) = |D(G_{0,\gamma_0}/\mathbb{Q}_v)| e(\gamma_{0,v}) \chi(d g_{0,v}) \chi(d i_{0,v})^{-1}.
\]

As a matter of fact we have \( |D(G_{\gamma_v}/\mathbb{Q}_v)| = |D(G_{0,\gamma_0,v}/\mathbb{Q}_v)| \). Here \( d g_{0,v} \) and \( d i_{0,v} \) are the local factors at \( v \) of the Tamagawa measures for \( G_0 \) and \( G_{\gamma_0} \) respectively. We may assume that \( d g_v \) is associated to \( d g_{0,v} \). By definition \( d i_v \) is associated to \( d i_{0,v} \). Therefore from Theorem \[5.5.4\] we conclude

\[
e(G_v) O^1_{\gamma_v}(f_v) = e(G_{0,v}) O^1_{\gamma_{0,v}}(f_{0,v}), \ \forall v \in S - \{\infty\}.
\]

\[4\] for chosen decompositions of the Tamagawa measures in question into local products.
In view of (11) and (15), to show (13) it suffices to show
\[ \prod_{v \in S} e(G_v)e(G_{0,v}) = 1, \]
which follows from the product formula. Therefore \( B = B_0 \), and we have proved (I).

We now prove (II). Note that for any \( v \in S - \{\infty\} \) we know that 1 is elliptic in \( G_v \) and \( G_{0,v} \). The assertion that \( f_{S}(1) = f_{0,S}(1) \) follows from (11) and (15) too, because \( f_v(1) = e(G_v)\Omega_1(f_v) \) for all \( v \) and similarly for \( f_{0,v}(1) \). Finally we need to show \( f_{S}(1) \neq 0 \). From (14) we see \( f_{S-(\infty)}(1) \neq 0 \). It could also be arranged that \( f_{\infty}(1) \neq 0 \). In fact, the work of Clozel-Delorme and Shelstad allows one to start from quite arbitrary \( f_{\infty} \) and asserts the existence of a corresponding \( f_{0,\infty} \). \( \square \)