The road so far:

We defined $\pi_n(X,A,x_0)$ and showed that there is the exact sequence, involving $\pi_n(X,A,x_0)$:

\[ \cdots \rightarrow \pi_n(A,x_0) \rightarrow \pi_n(Y,x_0) \rightarrow \pi_n(X,A,x_0) \rightarrow \pi_{n-1}(A,x_0) \rightarrow \cdots \]

One can write a similar for the triple:

\[ \cdots \rightarrow \pi_n(A,B,x_0) \rightarrow \pi_n(X,B,x_0) \rightarrow \pi_n(X,A,x_0) \rightarrow \pi_{n-1}(A,B,x_0) \rightarrow \cdots \]

Whitehead theorem:

If a map $X \rightarrow Y$ is between connected CW complexes, if a map $f: X \rightarrow Y$ induces an isomorphism $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ for all $n$, then $f$ is a homotopy equivalence. In case $f$ is the inclusion of a subcomplex $X \subseteq Y$, the conclusion is stronger: $X$ deforms retract onto $Y$.

Follows from Compression lemma:

Let $(X,A)$ be a $(W)$ pair and let $(Y,B)$ be the pair $B \neq \emptyset$. For each $n$ such that $X \setminus A$ has cells of dimension $n$, assume that $\pi_n(Y,B,y_0) = 0$ for $y_0 \in B$. Then every map $f: (X,A) \rightarrow (Y,B)$ is homotopic rel $A$ to a map $X \rightarrow B$. 


Lemma (extension). Given a CW pair \((X, A)\) and a map \(f: A \to Y\) with \(Y\) path-connected, then \(f\) can be extended to a map \(X \to Y\) if \(\partial_n \chi (Y) = 0\) for all \(n \geq 1\).

Proof. Assume inductively that \(f\) has been extended over the \((m-1)\)-skeleton. Then an extension over \(X\) exists if the composition of \(f\) with an \(n\)-cell \(x^n\) is well-defined on \(X^n\).

Theorem. \(\forall f: X \to Y\) of \(CW\) complexes is homotopic to a cellular map. If \(f\) is already cellular on a subcomplex \(A \subset X\), the homotopy may be taken to be stationary on \(A\).

Corollary. \(\pi_n(S^n) = 0\) for \(n < k\).

Elementary methods of computation.

Theorem. Let \(X\) a \(CW\) complex, \(A, B, C \subset X\), \(A \cup B = X\).

\(A \cap B = C\) connected, nonempty. If \((A, C)\) is \(m\)-conn.

\(A \cup B = C\) connected, nonempty. If \((A, C)\) is \(m\)-conn. and \((B, C)\) \(n\)-conn., then \(\pi_i(A, C) \to \pi_i(X, B)\) induced by inclusion is an isom. for \(i < m+n\) and surjection for \(i = m+n\).
Corollary \( \pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1}) \) is an isom.

for \( i < 2n-1 \) and a surjection for \( i = 2n-1 \).

More generally: \( \pi_i(X) \rightarrow \pi_{i+1}(SX) \) where \( X \) is an \((n-1)\)-connected CW complex.

Proof: Decompose \( SX = C_+ \times U \times C_- \times X \).

\( \pi_i(X) \cong \pi_i(C_+ \times X) \cong \pi_{i+1}(C_+ \times C_- \times X) \) by inclusion.

From long exact seq. of \( (C_+ \times X) \) we see that it is \( n\)-conn. if \( X \) is \((n-1)\)-conn.

Thus the preceding theorem says middle map is an isom for \( i+1 < 2n \) and surj for \( i+1 = 2n \).

Corollary \( \pi_n(S^n) \cong \mathbb{Z} \), generated by the id. map for all \( n \geq 1 \). In particular the deg. map \( \pi_n(S^n) \rightarrow \mathbb{Z} \) is an isom. and we proved from Hurewicz bundle.

\( \pi_1(S^1) \rightarrow \pi_2(S^2) \rightarrow \pi_3(S^3) \rightarrow \mathbb{Z} \)

isom surj.
More on computation

Proposition: If $(X,A)$ is an $(r,s)$-pair which is $r$-connected, then $\tilde{\omega}:(X/A)\to\tilde{\omega}:(X/A)$ induced by $X\to X/A$ is an isomorphism for $i\leq s$ and a surjection for $i=s+1$.

Proof: Consider $XUC_A$.

$CA$ is a contractible subcomplex of $XUC_A$.

Then $XUC_A \to (XUC_A)/CA = X/A$ - homotopy equivalences.

Apply the homology exact sequence using the fact that $(CA,A)$ is $(s+1)$-cannonical if $A$ is $s$-cannonical.

Example: $X$ obtained from $V_aS^2$ by attaching $S^2$ via $\phi: S^2 \to V_aS^2$ with $n \geq 2$.

Cellular approximation: $\tilde{\omega}:(X) = 0$ for $i<n$.

Now show that $\tilde{\omega}:(V_aS^2)$ is the quotient of $\tilde{\omega}:(V_aS^2) \times \mathbb{Z}^d$ by $[p]\mathbb{Z}$. 

let us return back to that example:

\[ \Lambda_n(S^n) \rightarrow \Lambda_n(V_\odot S^n) \] for \( n \geq 2 \) is free abelian with basis the hom. classes of \( S^1 \rightarrow V_\odot S^n \).

Suppose that it is a finite wedge:

Pair \( (P_\odot S^n, V_\odot S^n) \) - \( 2n-1 \) connected \( \Rightarrow \)

\( V_\odot S^n \rightarrow P_\odot S^n \) induces isom at \( n \) if \( n \geq 2 \)

\( \Rightarrow \) \( \Lambda_n(P_\odot S^n) \cong \bigoplus \Lambda_k(S^n) \) same is true for \( V_\odot S^n \)

Ex. Finish - prove fit. dim case.

to see that it as claimed

\[ \Lambda_\odot \Sigma(X, V_\odot S^n) \rightarrow \Lambda_n(V_\odot S^n) \rightarrow \Lambda_n(X) \rightarrow 0 \]

One object \( X / V_\odot S^n \) is a wedge of spheres \( S^1 \)

One object \( X / V_\odot S^n \) is a wedge of spheres \( S^1 \)

Prev. prop. then implies that \( \Lambda_\odot \Sigma(X, V_\odot S^n) \) is free with basis the char. class of \( E^1 \)

(\()\) takes them to \([\text{cf}].\)

Eilenberg-MacLane spaces

\[ X \text{ with nontrivial hom. group } \Lambda_n(x) \Rightarrow \]

is called E-M space \( k(G,x) \)
Construction \( CP^\infty K(G, 1) \) - note, example (6)

\[ X - (n - 2) \text{-} \text{connected } (W \text{ complex of dim } a + 1, \text{ s.t.} \]
\[ \tau_n(x) \leq G \text{ as in ex. above and apply Pastukhov tower} \]

Proposition: Homotopy type of a (W complex \( K(G, a) \)) is uniquely det. by \( G \) and \( a \).

Lemma: Let \( X \) be a (W complex \( (U, \Sigma U)) \) for some \( n+1 \). Then for every map \( f: \tau_n(x) \to \tau_n(y) \)

with \( y \) path connected \( \exists \) map \( f: X \to Y \) with \( f|_X = f \)

Proof of Prop.

Suppose \( x, x' \) are \( K(G, n) \) (W complexes).

Since homotopy equivalence is eq. rel. assume \( x = x' \).

Construct \( f: X \to K(G, n) \) from \( x \). By lemma \( f: X \to K(G, n) \)

induce isom on homotopy groups.

Pastukhov tower

\[ \tau_n(x) \leq G \text{ complex } X_n \text{ containing } X \text{ as subcomplex} \]

and \( a_i \tau_i(X_n) = 0 \text{ for } i > n \)

b) \( X \to X_n \) induces isom \( \tau_i \) for \( i < n \)

Attach \( n+2 \) cells to \( X \) using cellular maps \( S^n \to X \)

that generate \( \tau_{n+2}(x) \), etc.
Hurewicz theorem

**Theorem** If a space $X$ is $(n-1)$-connected, $n \geq 2$ then $\tilde{H}_i(X) = 0$ for $i < n$ and $\tilde{H}_n(X) \cong H_n(X)$. If a pair $(X,A)$ is $(n-1)$-connected, $n \geq 2$, with $A$ simply-connected and nonempty, then $H_i(X,A) = 0$ for $i < n$ and $\tilde{H}_n(X,A) \cong H_n(X,A)$.

**Proof**

Extra result 1) (CW aprox.)

A top. space $X$ is CW complex $\tau$ and a weak homotopy eq. $f : \tau \to X$

We assume that $(X,A)$ is a pair

For CW pair rel case, follows from absolute $b/c \tilde{H}_i(X,A) \cong \tilde{H}_i(X/A)$ for $i < n$ and $H_i(X,A) \cong H_i(X/A)$

Extra result 2) If $(X,A)$ is an $n$-conn. $(W$pair $= )$

$\exists$ CW pair $(\tau',A) = (X,A) \text{ rel } A$ s.t. all cells of $\tau' - A$

have dim greater than $n$

Replace $X$ with hom eq. $(CW$ complex with $(n-1)$-skeleton being a pt.) $\Rightarrow H_i(X) = 0$ for $i < n$

To show $\tilde{H}_n(X) \cong H_n(X)$, throw away cells $> n + 2$

\[ X \text{ has the form: } (\vee_{\alpha} C_{\alpha}^n) \cup_{\beta} e^{n+2} \]

$\tilde{H}_{n+1}(X,A) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(A)$, kernel of this map.
Corollary A map $f: X \to Y$ between simply conn. CW complexes is homotopy eq. if $f_*: H_n(X) \to H_n(Y)$ is isom for each $n$

Proof: Replace $Y$ by mapping cylinder $M_f$

$M_f = (([0,1] \times X) \sqcup Y)/\sim \sim (0,x) \sim f(x)$

$\Rightarrow X \to Y \Rightarrow H_n(Y,X) = 0 \Rightarrow$ first non-zero inclusion

$\Rightarrow \forall n \geq 0, H_n(Y,X)$ is isom to $H_n(Y, X)$. All $H_n(Y, X)$ are zero from long exact seq $\Rightarrow \forall n \geq 0, H_n(Y, X)$ also vanish $\Rightarrow$ isom.