Theorem. Suppose $p_1: \tilde{X}_1 \to X$, $p_2: \tilde{X}_2 \to X$ are covering maps, with $\tilde{X}_1$, $\tilde{X}_2$ connected and locally path-connected. Let $x_0 \in \tilde{X}_1$ s.t. $p_1(x_0) = p_2(x_0) = x_0 \in X$. Then

1. Coverings $p_1, p_2$ are isomorphic preserving basepoints iff $p_1 \times \tilde{u}_1(\tilde{X}_1, x_0) = p_2 \times \tilde{u}_1(\tilde{X}_2, \tilde{x}_0)$.

2. Coverings $p_1, p_2$ are isomorphic (forgetting basepoints) iff $p_1 \times \tilde{u}_1(\tilde{X}_1, \tilde{x}_0), p_2 \times \tilde{u}_1(\tilde{X}_2, \tilde{x}_0)$ are conjugate subgroups of $\tilde{u}_1(\tilde{X}_1, \tilde{x})$.

(isomorphic covering spaces, $f: \tilde{X}_1 \to \tilde{X}_2$, s.t. $p_1 = p_2f$) preserving basepoints $f(x_0) = \tilde{x}_0$, $p_1 = p_2h_1$

Proof: 1) From general lifting theorem

\[
\begin{array}{cccc}
\tilde{X}_1 & \xrightarrow{h_2} & \tilde{X}_2 & \xrightarrow{h_1} X, \tilde{x}_0 \\
p_1 & \xrightarrow{p_2} & X, x_0 & h_2 \circ p_1 = p_2h_1 \\
h & \circ h_1 & \tilde{X}_1 & \tilde{x}_1 \substack{\text{by unique lifting property, } h \circ h_1 = h_2 h_1 = \text{id}}
\end{array}
\]

since they fix basepoints.

2) follows from the following lemma:

**Lemma:** Let $p: \tilde{X} \to X$ be a covering map, $p(x_0) = x_0$.

Then a subgroup $H$ at $x = \tilde{u}_1(x, x_0)$ is conjugate to $H_0 = p \times \tilde{u}_1(\tilde{x}, \tilde{x}_0)$ iff $H = p \times \tilde{u}_1(\tilde{x}, \tilde{x})$ for some $\tilde{x} \in p^{-1}(x_0)$.

prove it!
Def. Let \( p : \tilde{X} \to X \) be a covering map.

A deck transformation \( p \) is a homeomorphism \( g : \tilde{X} \to \tilde{X} \) s.t. \( pg = p \).

Ex. The covering transformations form a group.

Def. Covering \( p : \tilde{X} \to X \) is normal regular if for all \( x \in X \) and all \( \tilde{x}_1, \tilde{x}_2 \in \tilde{p}(x) \), \( \exists \) covering transformation \( g : \tilde{X} \to \tilde{X} \) such that \( g(\tilde{x}_2) = \tilde{x}_1 \).

Ex. \( p : S^2 \to \mathbb{RP}^2 \), \( g : S^2 \to S^2 \), \( g(w) = -w \).

\[ \text{Ex.} \quad p : S^2 \to \mathbb{RP}^2, \quad g : S^2 \to S^2, \quad g(w) = -w, \quad g \text{ is a covering transformation.} \]

Lemma. Let \( p : \tilde{X} \to X \) be a covering map with \( \tilde{x}, \tilde{x}' \) path connected, \( \forall \tilde{x}_0 \in \tilde{X} \), \( \exists \tilde{x}_0 \in \tilde{X} \) such that \( p(\tilde{x}_0) = x_0 \in X \). Then the following is equivalent:

1) \( p \) is regular.
2) \( \forall \tilde{x}_1, \tilde{x}_2 \in \tilde{p}(x_0) \) \( \exists \) covering transformation \( g \) s.t. \( g(\tilde{x}_1) = \tilde{x}_2 \).
3) \( \forall \tilde{x}_1, \tilde{x}_2 \in \tilde{p}(x_0) \) \( \exists \) covering transformation \( g \) s.t. \( g(\tilde{x}_1) = \tilde{x}_2 \).

Proof. \( \exists \).
Theorem: Let \( p \colon \tilde{X} \to X \) be a covering with \( \tilde{X} \) connected and locally path connected. Let \( p(\tilde{x}_0) = x_0 \).

Then \( p \) is regular iff \( p_* \pi_1(\tilde{X}, \tilde{x}_0) \) is a normal subgroup of \( \pi_1(X, x_0) \).

Proof: Suppose \( \tilde{x}_0 \in p^{-1}(x_0) \), \( p \) regular iff \( \forall \tilde{x}_2 \in p^{-1}(x_0) \), there is an \( \tilde{g} \) from \( \tilde{x}_2 \) to itself s.t. \( g(\tilde{x}_0) = \tilde{x}_2 \). i.e. iff \( p_* \pi_1(\tilde{X}, \tilde{x}_0) = p_* \pi_1(\tilde{X}, \tilde{x}_2) \) i.e. iff \( p_* \pi_1(\tilde{X}, \tilde{x}_0) \) is normal in \( \pi_1(X, x_0) \).

Theorem: Let \( p \colon \tilde{X} \to X \) be a regular covering with \( \tilde{X} \) connected and locally path connected. Then \( \pi_1 \) group of covering transformations of \( p \) is isomorphic to \( \pi_1(X, x_0)/\pi_* \pi_1(\tilde{X}, \tilde{x}_0) \).

Proof: Define \( \Theta : \pi_1(X, x_0)/\pi_* \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(\tilde{X}, \tilde{x}_0) \) by:

\[
\tilde{x}_0 \mapsto \tilde{x}_0 \quad \tilde{g} \mapsto p_\tilde{g}
\]

where \( \tilde{g} \) path in \( \tilde{X} \) from \( \tilde{x}_0 \) to \( \tilde{g} \tilde{x}_0 \).

If \( u = p \tilde{u} \) another path, \( \tilde{v} \) from \( \tilde{x}_0 \) to \( g \tilde{x}_0 \).

\[
\tilde{v} = p \tilde{v}
\]

Since \( \tilde{u}(1) = \tilde{v}(1) \).

\( u \tilde{v} \) is an element of \( \pi_* \pi_1(\tilde{X}, \tilde{x}_0) \).

\( [u] = [v] \) in \( \pi_1(X, x_0)/\pi_* \pi_1(\tilde{X}, \tilde{x}_0) \).

Ex. Prove that \( \Theta \) is a homomorphism.
**Def.** Covering $p: \tilde{X} \to X$ is universal if $X$, $\tilde{X}$ are path connected and $\tilde{X}$ is simply connected.

**Ex.** $p: S^2 \to \mathbb{RP}^2$, $p: \mathbb{R}^2 \to \mathbb{T}^2$ universal, $\mathbb{Z}$ acts on $\mathbb{R}^2$ by translations.

**Cor.** If $p: \tilde{X} \to X$ is a universal covering, then $p$ is regular and group of covering transforms is isomorphic to $\pi_1(X)$.

**Ex.** Universal covering of $S^1 \vee S^1 = \tilde{X}$.

$$\pi_1(X, x_0) \cong \mathbb{Z} \times \mathbb{Z}$$

$$\Gamma_{x_0}(g) = a_0 b a_0^{-1}$$

**Def.** $X$ is called semilocally simply connected if every point $x \in X$ has an open neighborhood $U$ s.t. $\pi_1(U, x) \to \pi_1(X, x)$ is trivial, i.e. every loop in $U$ (based at $X$) is null homotopic in $X$.

**Theorem.** Suppose $X$ is connected, semi-locally simply connected and $x_0 \in X$. A subgroup $H$ of $\pi_1(X, x_0)$ is covering group if $p: \tilde{X} \to X$, point $\tilde{x}_0 \in p^{-1}(x_0)$ s.t.

$$p_\ast \pi_1(\tilde{X}, \tilde{x}_0) = H \cong \pi_1(X, x_0)$$
Lemma: If $X$ has a universal covering $\Rightarrow X$ is semi-locally simply connected.

Theorem: Any subgroup of a free group is free.

If $H$ is a subgroup of $F_k$ with index $m \Rightarrow H \leq F_m$.

Proof: $F_k = \pi_1(x, x_0) \Rightarrow x = \bigoplus_{k=1}^{n} V_k$.

$H \leq F_k$. There exists a covering $p : \tilde{X} \rightarrow X$.

Let $p_{\pi_2}(\tilde{x}, \tilde{x}_0) = H \leq \pi_2(x, x_0) = F_k$.

$\tilde{X} = \infty$

$\tilde{X}$ is a 1-dim (W complex (i.e. graph)) path connected $\tilde{\pi}_2(\tilde{X}) = \tilde{\pi}_2(\tilde{X}/T)$ is free.

If $|F_k| = m \Rightarrow \tilde{X}$ has $m$ vertices, $T$ has $m^2$ edges.

$\tilde{X}/T$ is a vertex, $\tilde{X} - (m-1)$ edges.