Lecture VIII

Fibrations, fiber bundles

Locally trivial bundles:

\[ \text{Def}(E, B, F, p), \ E, B, F - \text{topological spaces} \]
and \( p: E \to B \) a map, satisfying the following properties

1) \( \forall x \in B \) \exists neighborhood \( U_x \) so that \( p^{-1}(U) \cong U \times F \)

2) Homeomorphism \( U \times F \to p^{-1}(U) \) is compatible with \( p \), i.e. \( U \times F \to p^{-1}(U) \) is commutative.

\( E \) - total space, \( p \) - projection
\( B \) - base
\( F \) - fiber

Examples:

1) \( B \times F \to B \) - trivial bundle

2) Any covering space

3) Möbius band

4) Klein bottle

is a bundle of \( S^1 \) over \( S^2 \)
5) Hopf bundle:
\[ S^3 \rightarrow S^2 \]
\[ S^3 \text{ - set of vectors of unit length in } \mathbb{C}^2. \]
Set of complex lines in \( \mathbb{C}^2 \), passing through the origin \( \mathbb{CP}^1 \). \( \mathbb{CP}^1 \cong S^2 \)
Natural map \( \mathbb{C}^2 \setminus \{ 0 \} \rightarrow \mathbb{CP}^1 \)
Fiber - intersection of a complex line and \( S^3 \), i.e. \( S^1 \).
Explicitly: \( p(z_0, z_1) = (z_0^2, z_0 z_1, z_1^2) \)
\[ 121^2 + x^2 = 1 \]

**Ex.** Show that it is nontrivial.

**Def.** \( p_1 : E_1 \rightarrow B \), \( p_2 : E_2 \rightarrow B \) are equivalent if and only if \( \exists \, \psi : E_1 \rightarrow E_2 \, \) \( p_2 = p_2 \psi \).

**Def.** Trivialization of \( p : E \rightarrow B \) is homeomorphism \( E \rightarrow B \times F \), s.t. \( e \mapsto (p(e), \varphi(e)) \), where \( \varphi(e) \in F \).

Homeomorphism of trivialization is not unique. \( p_1 \neq p_1 \) necessarily

**Ex.** \( T^2 = S^1 \times S^1 \rightarrow S^1 \)

\[ p_1 : S^1 \times S^1 \rightarrow S^1 \]
\[ (\varphi, \psi) \mapsto \varphi \]

\[ p_2 : S^1 \times S^1 \rightarrow S^1 \]
\[ (\varphi, \psi) \mapsto (\varphi + \psi) \]
Proof. \( \mathcal{D} \to \mathcal{D}_{\text{trivial}} \). A locally trivial bundle over \( \mathcal{D} \) is equivalent to the direct product of \( \mathcal{D} \) with a fiber bundle over \( \mathcal{D} \). For a given \( \mathcal{D} \), we have a homeomorphism \( F \), if we take \( \mathcal{D} \times F \). On a point \( \mathcal{D} \times \{ F \} \), we take a family of homeomorphisms \( \phi \). One can look at it as a family of \( \mathcal{D} \) to \( \mathcal{D} \times F \). Therefore we have a homeomorphism \( F \) to \( \mathcal{D} \times F \) and another one, which is the composition of \( \phi \). We need to construct it in such a way that both parts coincide. Let \( \mathcal{D} \) and \( \mathcal{D}_{\text{trivial}} \) be the direct product of \( \mathcal{D} \) with \( \mathcal{D} \) and \( \mathcal{D} \) with \( \mathcal{D} \), respectively. Suppose that there are \( \mathcal{D} \) and \( \mathcal{D}_{\text{trivial}} \). We need to prove that \( \mathcal{D} \) and \( \mathcal{D}_{\text{trivial}} \) are equivalent. We have a homeomorphism \( \phi \) on \( \mathcal{D} \times \{ F \} \). Let \( \phi \) coincide on \( \mathcal{D} \times \{ F \} \).
2) Suppose the locally trivial fiber bundle over $\mathbb{R}^2$ is non-trivial. Let us divide it indefinitely until every point has a neighborhood where it is trivial, therefore we have contradiction.

**Exercise** Is it true that any bundle with fiber $D^n$ is trivial?

**Theorem** Let $p: E \to B$ - locally trivial fiber bundle $(\mathbb{Z}, \mathbb{Z}^*)$ - cell pair. Let $f: \mathbb{Z} \to E$, homotopy $F: \mathbb{Z} \times I \to B$ of a map $pf$ and homotopy $\Phi: \mathbb{Z} \times I \to E$ of a map $f/\mathbb{Z}$, which is the lift of $F$ on $\mathbb{Z} \times I$, i.e. $p \Phi = F$ on $\mathbb{Z} \times I$. Then there is a homotopy $\tilde{\Phi}$ of a map $f$, which is a lift of homotopy $\Phi$ and continuation of $\Phi$, i.e. $p \tilde{\Phi} = F$.

![Diagram](https://via.placeholder.com/150)

This is the most general statement of a homotopy lifting theorem.

Standard homotopy lifting when $\mathbb{Z}' = \emptyset$. 
Proof 1) Assume $E = B \times F$, $p$-projection

(sketch) on $B \Rightarrow$ map $\tau \rightarrow E$ can be considered

as a pair of maps $\tau \rightarrow B$, $\tau \rightarrow F$.

Homotopy in $B$ is given (this is $F_t$)

Need to construct the continuation of the

homotopy in $F$. This is the consequence of

Borsuk's theorem.

(Borsuk's theorem - ability to continue homotopy

from $w$-subcomplex to the whole $w$-complex)

2) let now $E$ be any bundle, but $\tau = D_e$, a closed
disk

We have $F : D_e \times I \rightarrow B$. Using this map

one can construct bundle over $D_e \times I \rightarrow D_e$

This will be a trivial bundle.

By Feldbau's theorem for this induced

bundle $E' \rightarrow B$ the continuation of

homotopy exists because of 1). Composing

homotopy exists because of 2). Composing

it with the map $E' \rightarrow E$ we obtaining

the continuation of homotopy

3) $\tau$-$w$-complex, $\tau \leq \tau$ is $w$-subcomplex

(continue from $w$-$k_i$ to $w$-$k_i+1$ and

use 2) for each of cells which

are not parts of $\tau$!}
Exact sequence of a bundle

let $p : E \to B$ - locally trivial bundle
$\overline{b_0} \in B$ - fixed point $F = p^{-1}(b_0)$ and to $EF$ - fixed pt.
Consider $\pi_i(E, F)$. Projection $p : E \to B$
induces homomorphism $\pi_i(E, F) \to \pi_i(B)$
since $F \to \text{pt under } p$.

Theorem $\pi_i(E, F) \to \pi_i(B)$ is isomorphism

Proof 1) Monomorphism property

Let $a \in \ker$. It is given by the map $a : D^i \to E$
$\sigma(a) \in \ker$. At the same time $p \circ a(D^i) = b_0$
and $p \circ a \to 0$. The corresponding homotopy can
be lifted to homotopy of a map from $a$ to $E$.
The resulting homotopy is therefore "ending".
The fiber $F$. Such maps are $0$ in $\pi_i(E, F)$
(since proj. is mapping to $b_0$).

2) Epimorphism property

Map $(S^i, s_0) \to (B, b_0)$ can be considered as
$f : (D^i, S^{i-1}, s_0) \to (B, b_0, b_0)$. We want
to lift it to the map $(D^i, S^{i-1}, s_0) \to (E, f, f_0)$
Consider the map $\psi : S^i \times I \to D^i$
$\psi(s, t) = t \cdot s$
Instead of $D^i \to E$, we construct map $s^{i-1} \times I \to E$ such that $s^{i-1} \times 103$ maps to $t_0$. This map can be constructed via covering homotopy theorem. Take $S^{i-2}$ as $Z$. Consider the map $Z \to t_0 \in E$ composition with $p$ maps it to $b_0 \in B$.

\[
\begin{array}{c}
S^{i-2} \\
\downarrow p \\
b_0 \in B
\end{array}
\]

This homotopy has a lift, i.e. map $s^{i-2} \times I \to E$ so that $s^{i-1} \times 103$ maps to $t_0$. At the same time $s^{i-2} \times 113$ maps into preimage of $b_0$, i.e. $F$, therefore Q.E.D.

**Exact sequence of a pair:**

\[
\begin{array}{c}
\to \tilde{u}_i(F) \to \tilde{u}_i(E) \to \tilde{u}_i(E, F) \to \tilde{u}_{i+1}(F) \to \ldots
\end{array}
\]

\[
\begin{array}{c}
\to \tilde{u}_i(F) \to \tilde{u}_i(E) \to \tilde{u}_i(B) \to \tilde{u}_{i-1}(F) \to \ldots
\end{array}
\]

**Corollary** If $p: E \to B$ is a covering space

\[
\begin{array}{c}
\tilde{u}_i(B) \cong \tilde{u}_i(E) \text{ if } i \geq 2
\end{array}
\]

**Corollary**

\[
\begin{array}{c}
\tilde{u}_2(S^2) = \emptyset \\
\tilde{u}_2(S^3) \to \tilde{u}_2(S^3) = 0 \\
\tilde{u}_2(S^4) = 0
\end{array}
\]

\[
\begin{array}{c}
\tilde{u}_3(S^2) \to \tilde{u}_3(S^3) \to \tilde{u}_3(S^3) = 0 \\
\tilde{u}_3(S^4) = 0
\end{array}
\]

**Corollary**

\[
\begin{array}{c}
\tilde{u}_2(S^4) \to \tilde{u}_3(S^3) \to \tilde{u}_3(S^3) \to \tilde{u}_2(S^3) \\
\text{we will see later that } \tilde{u}_n(S^4) = \emptyset \text{ for } n \geq 2
\end{array}
\]