Mirror symmetry and UOA

Lecture I

\( \text{loop-coherent sheaves} \)

R - commutative algebra (local ring)

R-loop-module is a vector space \( V = \bigoplus_{\ell \geq 0} V_{\ell} \)

\( \ell_0 \) - grading operator \( \ell[t]: V \to V \)

1. \( \ell[0] = \delta_0 \)
2. R-modules commute with each other
3. \( \ell[0] V_{\ell} \subseteq V_{\ell/2} \)

\[
\left( \sum_{k} \frac{k!}{\ell_2} \ell \right)^{\cdot} \left( \sum_{\ell} \frac{\ell!}{\ell_2} \ell^{\cdot \cdot} \right) = \sum_{\ell} \left( \ell_1 \ell_2 \right)^{\cdot \cdot} [\ell]^{\cdot \cdot -k}
\]

\( ([\ell_1 \ell_2] \ell_0) \neq \ell_1 [\ell_0 \ell_2] \ell_0) \)

**Proposition**
S - multiplicative system in R

Vs - localization w.r.t. \( S \) - loop, generated by \( S[0] \)

Then \( Vs \) has a natural structure of \( R_s \)-loop-module

\( \mathcal{U}: V \to Vs \) - universal morphism

\( \forall \mathcal{S}: V \to V_s \) which is compatible with \( R \to R_s \)

\( \mathcal{S}_2 = \mathcal{S}_1 \mathcal{S}_0 \)
Def. A sheaf $V$ of vector spaces over $C$ is called quasi-loop-coherent if $V$ affine subset $\text{Spec}(R) \times X$ sections, $\mathcal{P}(\text{Spec}(R), V)$ form an $R$-loop-module and restriction maps are exactly the localization maps.

Proposition. Any $R$-loop module, consider filtration $F^eV = \sum \Delta s; [\delta_i] V \leq e, i$, $s_1, s_2, \ldots$,

$F^0V \subseteq F^1V \subseteq \ldots$, $F^{e+1}V/F^eV$ has a nat. structure of $R$-module. (commutes with localizations) $\text{Proof.} (s_1, s_2)^{[0]} - s_1^{[0]} s_2^{[0]} : F^{e+1}V \to F^eV$

Def. A quasi-loop coherent sheaf is called loop-coherent or "loco" if quasi-coherent sheaves $F^{e+1}V/NV_k/F^eV/NV_k$ are coherent $\forall k, e$.

Terminology: "loco" and "quasi loco".

Proposition. Any affine variety $X$ and quasi-loco sheaf $V$ on it, cohomology spaces $H^i(X, V)$ are zero for $i > 1$.

A projective variety $X$ all cohomology spaces are fin. dim for each $i$ and $0$.

Proof. Choose a specific eigenvalue of $\lambda$ of $H_0$ and then do induction one in $F^eV/NV_k$. 

Sheaves of VOA.

Def. $R$ - comm. algebra over $C$.

A graded vertex algebra $V$ is called vertex $R$-algebra if $R$ is mapped to $K_0 = 0$ comp of $V$.

$Y(r_1, t) Y(r_2, t) = Y(r_1 r_2, t)$

do has only non-zero eigenvalues.

Proposition. $S$-multiplicative system, V - vertex $R$-algebra.

$\Rightarrow$ $V_S$ has a natural structure of vertex $K_S$-algebra.

Proof. Borcherds

Def. A (quasi-)laco sheaf $V$ of vector space over $C$ is a (quasi-)laco sheaf of vertex algebras if for $M: Spec(R) \times \mathbb{P}$

$\mathbb{P}(Spec(R), V)$ form a vertex $R$-algebra and restriction maps are loc. maps from above.

Proposition. Cohomology of quasi-laco sheaf of vertex algebras $V$ has a natural structure of $VA$.

If the structure of conf. algebra is compatible with localization maps $\Rightarrow H^*(V)$ has a nat. and structure.

Proof. Use Borcherds definition.
Topological VOA

\[ a \in V \text{ s.t. } a_{(0)} = 0 \quad \{a, b \} = [a, b] \]

Proposition (homology of V w.r.t. a_{(0)} has a structure of VOA.

\[ [a_{(0)}, \mathcal{Y}(b, z)]_z = \mathcal{Y}(a_{(0)} b, z) \]

\[ \Rightarrow \text{If } b \text{ is annih. by } a_{(0)} \Rightarrow \mathcal{Y}(b, z) \text{ commutes with } a_{(0)} \]

and conserves kernel and the image of \( a_{(0)} \)

Chiral de Rham complex as a sheaf of VOA (or smooth variety sheaf) MSV(x) in local coordinates:

2dim x ferm. fields \( \psi(\tau), \bar{\psi}(\bar{\tau}) \)

cond. dim 0

2dim x bosonic fields \( a(\tau), b(\bar{\tau}) \)

\( \psi(\tau) = \sum \psi_i \{ \bar{\tau} \}^{-i} \)

\( a(\tau) = \sum a_i \{ \bar{\tau} \}^{-i} \)

\( b(\bar{\tau}) = \sum b_i \{ \bar{\tau} \}^{-i} \)

\( \{ \bar{\tau} \} = \delta_{\bar{\tau}} \delta^{\bar{\tau}} \)

\( \psi(\tau) = \sum \psi_i \{ \bar{\tau} \}^{-i} \)

\( b(\bar{\tau}) = \sum b_i \{ \bar{\tau} \}^{-i} \)

\( \psi(\tau) = \sum \psi_i \{ \bar{\tau} \}^{-i} \)

\( \langle 0 \rangle = \text{Fock space} \)

\( a_{m}, b_{m} \langle 0 \rangle = 0 \quad m > 0 \)

F \otimes C \sum b_{i} \langle 0 \rangle \) plugged instead of \( \psi_{i} b_{i} \)
A model $X^{(r)}$ also possesses $\mathcal{C}^{(r)}$.

Thus, given a sheaf $\mathbb{F}_X$ over $X$, we define $\mathcal{C}^{(r)}$ in all other fields, being well-defined.

Let $X^{(r)}$ be a smooth algebraic variety over $\mathbb{C}$, the $X^{(r)}$-invariant under coad change. If $X^{(r)}$
Def. If $X$ - CY, Define B-model $TU\alpha$

as follows:

As a vector space it coincides with the A-model $TU\alpha$ at $X$ is related by mirror involution:

$Q_B = G_A, \ G_B = Q_A, \ J_B = -J_A, \ h_B = h_A - 2J_A$

B-model is ill-defined if $X$ is not CY