What this course is about?

OFT: \[ \frac{1}{Z} \int d\Phi \cdots d\Phi e^{-S[\Phi]} = \] fields
partition function = \( \langle f_1 \cdots f_n \rangle \) < correlation function
Computing such integrals is a primary target of QFT.

Topological Field Theories
(exact results, using fixed point formulas and other techniques, correlation functions give various invariants: Donaldson, GW, Jones polynomial, etc)

Conformal Field Theories in 2D: (BPZ)

\[ A_i(x_i) A_j(x_j) = C_{ij} (x_i, x_j) A_{kl}(x_k) \]

Infinite-dimensional symmetry (Virasoro algebra)
lead to constraints on \( C_{ij} (x_i, x_j) \) expressed via diff. equations
Structure of the course

1) Geometric (Segal) and Constructive (BPZ) approach

2) Vertex algebras (simplest CFTs) (minimal models, WZW)

3) Rational conformal field theories, relationships to Quantum Groups

4) Advanced topics:
   1) Chiral de Rham complex
      Applications to Mirror symmetry
   2) SLE approach to CFT
   3) Integrable models and CFT
   4) Applications to Geometric Langlands

Geometric approach, applied to QFT in general

$D+1$-dimensional QFT, among other things is a functor $\Phi : \text{Man}(D) \to \text{Vect}$

- "manifold" - loose notion, correct is $\ast$-manifold
  where $\ast$ is the extra structure

- $\mathfrak{M} \times \mathfrak{M}'$ if there is a diff. preserving $\ast$-structure
Field Theories: topological spin conformal orientation
spin structure complex structure

2 ideas:

i) \( \Sigma \)-closed D-dim manifold \( \rightarrow \) Hilbert space \( \mathcal{H}_\Sigma \)

Physically: space of quantum states obtained by quantizing theory on \( \Sigma \times \mathbb{R} \) spacetime.

ii) "Evolution operator"

\[ \Phi_t : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_\Sigma \]

\[ \Phi_{t+t'} = \Phi_t \circ \Phi_{t'} \]

Evolves from \( \Sigma \) to \( \Sigma' \)

\[ \Phi_m : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\Sigma'} \]

\( \Phi_m \) from physics perspective: local set of fields

\[ \Psi_{\text{out}}(\phi') = \int D\phi \ K_m(\phi', \phi) \Psi_{\text{in}}(\phi) \]

\[ K_m(\phi', \phi) = \int \text{d}\phi' \ K_m(\phi', \phi') K_m(\phi', \phi) \]

Gluing \( \Phi_m \circ \Phi_m' = \Phi_m \circ \Phi_m' \)

\[ K_{m,\circ} \circ (\phi'', \phi) = \int D\phi \ K_m(\phi'', \phi') K_{m,\circ}(\phi', \phi) \]

Axioms:

1) In degenerate case \( \Sigma \) consists of \( \phi \rightarrow \mathcal{H}(\phi) = \mathcal{H} \)

2) If \( \Sigma \) is a disjoint union of several manifolds

\[ \mathcal{H} \Sigma = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\Sigma'} \]
3) The "collapsing" axiom: if $M$ has two boundary components $\Sigma$ and $\Sigma'$, labeled as outgoing and incoming, one can glue the boundaries

\[ M \rightarrow M' \]

\[ \overline{\Phi}_{T^2} M': = Tp_{V_2} \overline{\Phi} M = \sum \overline{\Phi}_M (\nu_i, \nu_i') \frac{v_i, v_i'}{v_i, v_i'} \]

4) $H - \Sigma \cong H^\Sigma$ or if $H^\Sigma$ carries norm structure

\[ H - \Sigma \cong H^\Sigma \]

5) A natural action of perm. group, if several boundary components are isomorphic

Remarks: A closed D+1-manifold $M \Phi_M: C \rightarrow C$ is $Z(M)$-partition function

2) If $M$ has only one single outgoing boundary $\Sigma \rightarrow \overline{\Phi}_M$ is a map $C \rightarrow H^\Sigma \quad \langle \Sigma | M \rangle \in H^\Sigma$

Sueing $M_1, M_2$ into $M$

\[ Z(M) = \langle M_1 | M_2 \rangle = \int D\phi \, \overline{\Psi}_{M_1}(\phi) \Psi_{M_2}(\phi) \]
Topological FT

One can glue $M_1, M_2$ via surgery (applying Diff). In topological theory, $(M) \in \mathcal{H}_\Sigma$ is inv under $\text{Diff}_0(\Sigma)$ since it can always be deformed over a collar $\Sigma \times I \to \text{only } \Gamma_{\Sigma}$ (mapping class group) and

$$1 \to \text{Diff}_0(\Sigma) \to \text{Diff}(\Sigma) \to \Gamma_{\Sigma} \to 1$$

$$\mathcal{Z}(M) = \langle M_2 | U(\Sigma) | M_1 \rangle \quad \Gamma \in \Gamma_{\Sigma}$$

Also $\dim \mathcal{H}_\Sigma = \text{Tr} \mathcal{H}_\Sigma = \mathcal{Z}(\Sigma \times S^1)$ if $\mathcal{H}_\Sigma$ f.d.

2d TFI:

Only one closed manifold: $S^3$. $M = S^3$ or $S^3 \# S^3$

$\Phi M$ ??? They can be defined by considering the sphere with 1, 2, 3 holes.

We call it vacuum vector $1$

1) $O$

$\Phi M \to \mathcal{H}$

$\mathcal{H} \to \text{vac}$

$\mathcal{Z}(S^3) = \langle 1 >$ 0$

2) $\Phi$

$\eta : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$

$\eta_{ij} = \eta(\phi_i, \phi_j)$

$\{\phi_i\}$ - basis

Ex. Show that it is not degenerate
$\eta^{ij}$ - inverse matrix

3. $\phi_i \cdot \phi_j = \sum_k \epsilon^{ijk} \Phi_k$

$c: \mathfrak{gl} \to \mathfrak{gl}$

$d: \beta = c(d, \beta)$

$c: \mathfrak{gl} \to \text{End}(\mathfrak{g})$

$c: \text{Sym}^3 V \to \mathfrak{g}$

$c_{ijk} = c_{ij} \eta_{kk}$

$\eta_{ij} = c_{ij0}$

$\eta(d \cdot \beta, \delta) = \eta(d, \beta \cdot \delta)$

Also:

$\eta(d \cdot \delta) = \langle d \cdot \beta \rangle_0$,

$\eta(d, \beta) = \langle d \cdot \beta \rangle_0$

Duality:

$\phi_i \bigcirc \phi_j = \phi_k \bigodot \phi_e$
\[ \sum_{n} c_{ij} \rho_{n} = \sum_{n} c_{ij} \rho_{n} \]

\[ (\phi_{i})_{j}^{k} = c_{ij}^{k} \quad \text{- symmetric commuting matrices} \]

can be simultaneously diagonalized by \( \psi_{ij} \) that is orthogonal w.r.t. \( \eta_{ij} \)

\[ c_{ij}^{k} = \sum_{n} s_{ij}^{n} (S^{-1})_{n}^{k} \Rightarrow \chi_{i}^{k} = \sum_{n} s_{i}^{n} (S^{-1})_{n}^{k} \]

\[ \delta_{j}^{k} = \sum_{n} s_{j}^{n} (S^{-1})_{n}^{k} \]

\[ \chi_{i}^{k} = \frac{\sum_{n} s_{i}^{n} S_{i}^{n} (S^{-1})_{n}^{k}}{S_{i}^{n}} \quad \text{Verlinde formula} \]

\[ \sum_{n} c_{ij} \rho_{n} \]

Exercises:

1. Calculate \( Z(M) \) for genus \( g \) in terms of \( S \)

\[ Z(M) = \sum_{k} (S_{k}^{g})^{2} \]

\[ \hat{\phi}_{i} \times \hat{\phi}_{j} = \frac{\delta_{ij}}{S_{i}^{n}} \hat{\phi}_{i} \quad (S_{k}^{g}) \text{- dimer} \]

2. Calculate partition functions

\[ \Phi_{M} : \langle \Phi_{i} \cdots \Phi_{i} \rangle \rightarrow \mathbb{C} \]

\[ \langle \Phi_{i} \cdots \Phi_{i} \rangle = \sum_{n} c_{i}^{k} \cdots c_{i}^{n} (S_{i}^{k}) \quad n \text{- point functions} \]