INTRO TO LOCALIZATION AND SMITH-TYPE INEQUALITIES

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Abstract. I’ll start with an introduction to the idea of localization and Smith-type inequalities, discuss the motivation behind studying this phenomenon in various theories, and describe the current landscape in this line of work. Afterwards, we’ll set up a rough schedule of the talks to follow.

1. Welcome

1.1. Tone.
- learning from reading seminars and topics courses; primed to read details
- not an expert, here to learn, share/absorb perspective
- set tone for productive and valuable meetings, steer seminar to match interests of the group
- encourage stupid questions: may have immediately clarifying answers; speaker decide whether to postpone answering

1.2. Goals and Motivation.
- Seidel-Smith paper, Localization for involutions in Floer cohomology [SS10]; link on website, seems valuable to learn well; Hendricks
- conjectures on the Floer side related to my research

2. Ideas of localization

The following discussion follows the exposition for the case $p = 2$ in [LT16]. Consider a topological space $X$ equipped with a $G = \mathbb{Z}/p$ action, with $p$ prime. Let $X^{\text{inv}}$ denote the fixed-point set. (We’ll need some boundedness conditions later on; for example, it’s enough to assume $H^*(X; \mathbb{Z})$ is bounded in homological dimension.)

2.1. Borel equivariant cohomology. Let’s recall some homological algebra. You won’t need to recall in great detail for this seminar; it’s just good to have a sense of where the hands-on constructions we’ll work with actually come from historically and theoretically.

The Borel equivariant cohomology of $X$ is defined to be the singular cohomology of the fiber product $X \times_G EG$:

$$H^*_G(X; \mathbb{Z}) := H^*(X \times_G EG; \mathbb{Z})$$

Recall that $EG$ is the contractible space with free $G$-action; $EG/G = BG$ is the classifying space. For $G = \mathbb{Z}/p$, $EG = S^\infty$, which can be thought of with the equivariant cell structure. (Think of $S^\infty$ as $\lim_{n \to \infty} S^n$; $G$ acts on each $S^n$ by rotation.)

How do we compute this?

1. The $G$-action on $X$ induces a $G$-action on $C_*(X; \mathbb{Z})$, making it a chain complex over $\mathbb{Z}[G]$. Since $C_*(X \times EG; \mathbb{Z})$ is a free resolution of $C_*(X; \mathbb{Z})$ as a $\mathbb{Z}[G]$-module, we can equivalently define $H^*_G(X; \mathbb{Z}) = \text{Ext}_{\mathbb{Z}[G]}(C_*(X); \mathbb{Z})$. Let $\phi$ be a generator of the $G$-action; abuse notation and use this symbol everywhere.
The classical localization theorem says that this isomorphic to $E$ spectral sequences is "easy" to compute and collapses on the more detail at the talk; feel free to talk to me if you want to see more details.) One of the two (row-wise and column-wise) filtrations on the Tate bicomplex. (This was discussed in C is like $\eta$,(1)

This is proven by considering the Tate bicomplex, where we invert $\eta$ before taking homology (this is like $C^*_{Borel}$ except it’s bi-infinite):

The proof of the localization theorem boils down to comparing two spectral sequences from the two (row-wise and column-wise) filtrations on the Tate bicomplex. (This was discussed in more detail at the talk; feel free to talk to me if you want to see more details.) One of the spectral sequences is “easy” to compute and collapses on the $E^2$-page to (2). The other one has $E^1 \cong H_*(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\eta, \eta^{-1}]$ and computes (1). Our boundedness condition ensures that both spectral sequences ultimately compute the same thing: the homology of the total complex.

**Remark 1.** When I say “easy,” I mean “sometimes doable.” Proving a localization theorem usually means computing this spectral sequence, i.e. showing that it collapses on a page that we understand. On the other hand, the other spectral sequence can be arbitrarily complicated.

2.3. **Smith-type inequalities.** An easy corollary of the localization theorem is a rank inequality, due to the fact that taking subquotients can only decrease rank:

$$\dim H^*(X; \mathbb{F}_p) \geq \dim H^*(X^{inv}; \mathbb{F}_p).$$

**Remark 2.** Do beware that this is an inequality in total dimension; because the bigrading in the Tate bicomplex gets collapsed into a single grading in the total complex, we can’t say anything about the individual gradings when we consider the isomorphism between the $E^\infty$-pages of the two spectral sequences. However, if we instead use theories with extra gradings (e.g. Alexander grading in knot Floer, quantum grading in Khovanov, spin$^c$ in Seiberg-Witten Floer), we could check whether our spectral sequence splits along these extra gradings.
There is interest in proving these rank inequalities because they provide obstructions. For example:

- Hendricks used it to obstruct 2-periodicity in some knots using knot Floer homology, and to reprove some classical genus bounds.
- Politarczyk has been looking at obstructing periodicity using Khovanov homology.
- Lidman-Manolescu give strong constraints on L-spaces arising as regular covers.

But you don’t have to care about this to find the world of equivariant methods interesting!

3. Previous results

Let me now give some concrete examples of localization theorems that I’ve encountered. We may decide to explore some of these later in the seminar.

Seidel-Smith \[SS10\]: (\(\mathbb{Z}/2\)-action, \(\mathbb{F}_2\) coefficients)

- \(\dim \tilde{HFK}(\Sigma(K), K') \geq \dim \tilde{HFK}(S^3, K')\)
- \(\dim \tilde{HFK}(S^3, \tilde{K} \cup U) \geq \dim \tilde{HFK}(S^3, K \cup U)\) (The actual results are much finer than this; there is grading information missing here.)

Hendricks \[Hen15\], \[Hen12\], following Seidel-Smith: (\(\mathbb{Z}/2\)-action, \(\mathbb{F}_2\) coefficients)

- \(\dim \tilde{HFK}(\Sigma(K), K) \geq \dim \tilde{HFK}(S^3, K)\)
- \(\dim \tilde{HFK}(S^3, \tilde{K} \cup \tilde{U}) \geq \dim \tilde{HFK}(S^3, K \cup U)\) (The actual results are much finer than this; there is grading information missing here.)

Lidman-Manolescu \[LM18\], using \(SWF\) space: (\(\mathbb{Z}/p\)-action, \(\mathbb{F}_p\) coefficients)

- \(\dim \tilde{H}_*(SWF(\tilde{Y}, \pi^*s)) \geq \dim \tilde{H}_*(SWF(Y, s))\)
- Similarly for \(\tilde{HM}, \tilde{HF}\); also \(HF_{red}\).

Remark 3. This is pretty cool if you care about the L-space conjecture! Suppose \(\pi : \tilde{Y} \to Y\) is a regular \(p^n\)-sheeted covering of orientable 3-manifolds, where \(p\) is prime. Then if \(\tilde{Y}\) is a \(\mathbb{Z}/p\)-L-space, then \(Y\) is also a \(\mathbb{Z}/p\)-L-space. We already know some geometric properties of \(\mathbb{Z}/p\mathbb{Z}\)-L-spaces; see their paper for more info.

4. Organization of seminar

I haven’t read the paper, so this is just an estimate. I’ve also learned to be less optimistic. Near-future talks:

1. (Alex P) Equivariant Morse cohomology (2.1)
2. (Alex P) Invariant Morse functions (2.2)
3. (Alex Z) Localization, part 1 (2.3)
4. (Alex Z) Localization, part 2 (finish 2.3, 2.4)
5. (TBD) Equivariant Floer (3.1, 3.2)
6. (TBD) An example, index discrepancy (3.3, 3.4)
7. (Akram) Equivariant transversality (3.5)
8. (TBD) Localization for Floer!

References


