1. Find the limit \( \lim_{x \to 3} \frac{x^2 + x - 12}{x^2 + 2x - 15} \).

\textit{Solution.} Trying \( x = 3 \) in the top and bottom, they both become zero. So we should try factoring them. We get \( x^2 + x - 12 = (x - 3)(x + 4) \) and \( x^2 + 2x - 15 = (x - 3)(x + 5) \). So

\[
\lim_{x \to 3} \frac{x^2 + x - 12}{x^2 + 2x - 15} = \lim_{x \to 3} \frac{(x - 3)(x + 4)}{(x - 3)(x + 5)} = \lim_{x \to 3} \frac{x + 4}{x + 5} = \frac{3 + 4}{3 + 5} = \frac{7}{8}.
\]

The substitution \( x = 3 \) after the second line was OK because \( \frac{x + 4}{x + 5} \) is nice away from \( x = -5 \).

2. For which value of \( a \) does the limit \( \lim_{x \to 3} \frac{\sqrt{x^2 + 16} - a}{x - 3} \) exist? What is the value of the limit in this instance?

\textit{Solution.} Trying \( x = 3 \) in the bottom, we get zero. If \( a \) is chosen so that the top is not zero when \( x = 3 \), there’s no way the limit can exist, since then the function will
blow up at \( x = 3 \). So we need \( \sqrt{3^2 + 16} - a = 0 \), so \( a = 5 \). Now we have

\[
\lim_{x \to 3} \frac{\sqrt{x^2 + 16} - 5}{x - 3} = \lim_{x \to 3} \frac{\sqrt{x^2 + 16} - 5}{x - 3} \cdot \frac{\sqrt{x^2 + 16} + 5}{\sqrt{x^2 + 16} + 5} \\
= \lim_{x \to 3} \frac{x^2 + 16 - 25}{(x - 3)(\sqrt{x^2 + 16} + 5)} \\
= \lim_{x \to 3} \frac{x^2 - 9}{(x - 3)(\sqrt{x^2 + 16} + 5)} \\
= \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)(\sqrt{x^2 + 16} + 5)} \\
= \lim_{x \to 3} \frac{x + 3}{3 + 3} \\
= \frac{3 + 3}{\sqrt{9 + 16 + 5}} \\
= \frac{6}{5}
\]

3. Consider the function \( f(x) \) defined as follows:

\[
f(x) = \begin{cases} 
\frac{x + \pi}{x + 8} & \text{if } x \leq -\pi \\
\sin x & \text{if } -\pi < x < 0 \\
\frac{x^2 + 1}{x - 1} & \text{if } x > 0
\end{cases}
\]

At which points is \( f(x) \) discontinuous? Justify your answer.

**Solution.** The function \( f(x) \) is discontinuous at three points:

- \( x = -8 \), where it blows up, since \( \lim_{x \to -8^+} f(x) = -\infty \).
- \( x = 0 \), since \( f(0) \) is not defined, and
- \( x = 1 \), where it also blows up, since \( \lim_{x \to 1^+} f(x) = \infty \).

\( f \) is **continuous** at \( x = -\pi \), since \( f(-\pi) = \lim_{x \to -\pi} f(x) = 0 \).

4. Find the limit \( \lim_{x \to \infty} \frac{3x}{3x^2 + 1 + 2x} \).
Let us pull a factor of $3^x$ out of the bottom, so then we have

$$
\lim_{x \to \infty} \frac{3^x}{3^{x+1} + 2^x} = \lim_{x \to \infty} \frac{3^x}{3^x(3 + 2^x/3^x)}
$$

$$
= \lim_{x \to \infty} \frac{1}{3 + 2^x/3^x}
$$

$$
= \lim_{x \to \infty} \frac{1}{3 + (2/3)^x}
$$

$$
= \frac{1}{3 + 0}
$$

$$
= \frac{1}{3},
$$

since $(2/3)^x \to 0$ as $x \to \infty$.

5. Give an example of a function which is discontinuous at infinitely many points.
   Solution. $\tan x$ and $\frac{1}{\sin x}$ are both examples.

6. Find the limit $\lim_{x \to 1} \frac{\sin(x^2 - 1)}{x - 1}$.
   Solution. If the bottom was the same as the thing inside $\sin$, we could use the blah principle. It’s not the same, so we do the trick:

$$
\lim_{x \to 1} \frac{\sin(x^2 - 1)}{x - 1} = \lim_{x \to 1} \frac{\sin(x^2 - 1)}{x - 1} \cdot \frac{x + 1}{x + 1}
$$

$$
= \lim_{x \to 1} \frac{\sin(x^2 - 1)}{x^2 - 1} \cdot (x + 1)
$$

$$
= \lim_{x \to 1} \frac{\sin(x^2 - 1)}{x^2 - 1} \cdot \lim_{x \to 1} (x + 1)
$$

$$
= \lim_{u \to 0} \frac{\sin u}{u} \cdot \lim_{x \to 1} (x + 1)
$$

$$
= 1 \cdot (1 + 1)
$$

$$
= 2.
$$

7. a. Calculate the derivative of $2x^2 + 3$, using the definition of the derivative.
   b. For which value of $b$ is the line $y = 12x + b$ tangent to the curve $y = 2x^2 + 3$?
Solution for a.

\[(2x^2 + 3)’ = \lim_{h \to 0} \frac{2(x + h)^2 + 3 - (2x^2 + 3)}{h} = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 3 - 2x^2 - 3}{h} = \lim_{h \to 0} \frac{4xh + 2h^2}{h} = \lim_{h \to 0} (4x + h) = 4x + 0 = 4x.\]

Solution for b. If \(y = 12x + b\) is tangent to the curve \(y = 2x^2 + 3\), say at the point \((a, 2a^2 + 3)\), then we will have 12 = 4a, since 4a is also the slope of the line tangent to this point. So a = 3, and therefore our line passes through the point \((3, 2 \cdot 3^2 + 3) = (3, 21)\). Putting 3 and 21 for \(x\) and \(y\) in \(y = 12x + b\), we get \(21 = 36 + b\), so \(b = -15\).

8. Suppose the number of badgers in the Central Park Zoo is given by the formula \(b(t) = 40\sqrt{t - 1988} + 3\), where \(t\) denotes time measured in years. Approximately how many badgers did the zoo acquire in the year 2013? (Hint: The slope of the line tangent to the graph of \(y = b(t)\) at any given point \((a, b(a))\) is a good estimate for the rate of badger acquisition at the time \(a\).)
Solution. The slope of the tangent line is given by $b'(2013)$, which by definition is

$$b'(2013) = \lim_{h \to 0} \frac{b(2013 + h) - b(2013)}{h}$$

$$= \lim_{h \to 0} \frac{40\sqrt{25 + h} + 3 - 40\sqrt{25} - 3}{h}$$

$$= 40 \lim_{h \to 0} \frac{\sqrt{25 + h} - 5}{h}$$

$$= 40 \lim_{h \to 0} \frac{\sqrt{25 + h} - 5}{h} \cdot \frac{\sqrt{25 + h} + 5}{\sqrt{25 + h} + 5}$$

$$= 40 \lim_{h \to 0} \frac{25 + h - 25}{h(\sqrt{25 + h} + 5)}$$

$$= 40 \lim_{h \to 0} \frac{1}{\sqrt{25 + h} + 5}$$

$$= 40 \cdot \frac{1}{\sqrt{25 + 0} + 5}$$

$$= 40 \cdot \frac{1}{10} = 4.$$

So they gained about 4 badgers in the year 2013.

**Bonus (5 points):** Is it true that if $\lim_{x \to \infty} f(x) = 0$, then $\lim_{x \to \infty} f'(x)$ as well? Why or why not?

Solution: It is not true. Let $f(x) = \frac{\sin(x^n)}{x}$. Then clearly $f(x) \to 0$ as $x \to \infty$. Let $x_n = \pi^{\frac{1}{n}}$ for $n = 1, 2, 3, 4, \ldots$, so $f(x_n) = \frac{\sin(\pi x_n)}{\pi} = 0$. We can find some $\varepsilon_n \approx \frac{1}{n^2}$ such that $\sin((x_n + \varepsilon_n)^3) = 1$, so $f(x)$ changes from 0 to $\frac{1}{x_n + \varepsilon_n} \approx \frac{1}{n}$, as $x$ moves from $x_n$ to $x_n + \varepsilon_n$. Since

$$\frac{f(x_n + \varepsilon_n) - f(x_n)}{x_n + \varepsilon_n - x_n} \approx \frac{1/n}{1/n^2} \approx n,$$

there must be some point between $x_n$ and $x_n + \varepsilon_n$ where $f'$ is of order of magnitude at least $n$. So $\lim_{x \to \infty} f'(x) \neq 0$.  

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