(1) We used the following group theory fact in class, to show there was a surjective homo-
morphism from the fundamental group of the trefoil complement to $D_3$:

Let $G = \langle g_1, g_2, \ldots, g_k \mid r_1, r_2, \ldots, r_l \rangle$ be a group given in terms of generators and
relations. Write $r_i = g_{i,1}^{n_i,1} g_{i,2}^{n_i,2} \cdots g_{i,j_i}^{n_i,j_i}$.

Let $H$ be any group, and $h_1, \ldots, h_k \in H$. Then there is a group homomorphism
$f: G \to H$ such that $f(g_i) = h_i$ ($i = 1, \ldots, k$) if and only if, for all
$h_{i,1}^{n_i,1} h_{i,2}^{n_i,2} \cdots h_{i,j_i}^{n_i,j_i} = 1_H$
for $i = 1, \ldots, l$.

Prove this. (Hint: one direction is easy. For the other, you’ll use the definition of $G$ as
a quotient group of a free group.)

(2) Let $\mathbb{R}P^2$ denote the space of lines through the origin in $\mathbb{R}^3$. That is, $\mathbb{R}P^2 = (\mathbb{R}^3 \setminus 0)/(p \sim \lambda p)$ (where $\lambda \neq 0$). Compute $\pi_1(\mathbb{R}P^2)$.

(Hint: you can either do this directly using van Kampen’s theorem or by putting a
cell structure on $\mathbb{R}P^2$. To do the former, let $U$ be a neighborhood of the vertical line
($\{(0,0,\lambda)\}$), and $V$ the complement of a smaller neighborhood of this line.)

(3) Let $\text{Homeo}(X)$ denote the group of homeomorphisms $X \to X$ and $\text{Homeo}(X, x_0)$ denote
the subgroup of homeomorphisms $X \to X$ sending $x_0$ to $x_0$.

(a) Prove that $\text{Homeo}(X, x_0)$ acts on $\pi_1(X, x_0)$. Prove that if homeomorphisms $f$ and $g$
are homotopic rel $x_0$ then $f$ and $g$ act in the same way. (This says that the action
descends to an action of the group of path components of $\text{Homeo}(X, x_0)$, where we
endow $\text{Homeo}(X, x_0)$ with the compact-open topology.)

(b) Let $H_1(X) = \pi_1(X, x_0)/[\pi_1, \pi_1]$ denote the abelianization of $\pi_1(X, x_0)$. Prove that
$\text{Homeo}(X)$ acts on $H_1(X)$, and that homotopic homeomorphisms act in the same
way.

(c) Find a homeomorphism $T^2 \to T^2$ which is not homotopic to the identity map, and
prove that it is not homotopic to the identity map.

(4) Optional: recall the definition of $L(5,3)$ from problem set 3. Compute $\pi_1(L(5,3))$.

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