Convergence results for critical points of the one-dimensional Ambrosio-Tortorelli functional with fidelity term

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Abstract

In this paper, we show that critical points of the one-dimensional Ambrosio-Tortorelli functional with fidelity term converge to those of the corresponding Mumford-Shah functional, a famous model for image segmentation. Equi-partition and convergence of the energy-density are also derived.

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1 Introduction

In this paper, we continue our previous study [7] on the convergence of critical points of the Ambrosio-Tortorelli functional [2, 3] to those of the Mumford-Shah functional [11]. Here, as opposed to the Dirichlet case in [7], the functionals we study contain the fidelity term linking the approximate images to the original image. These functionals were proposed as models for image segmentation in computer vision. Because real digital images are two-dimensional, the analysis of these functionals in 2D will be of greater interest and applications. However, carrying out this task is quite a challenge for the time being, and we only limit ourselves to the one-dimensional case. We expect that many statements will eventually carry over to the two-dimensional case.

We now introduce the Ambrosio-Tortorelli and Mumford-Shah functionals used in the paper. Throughout, $C$ stands for a generic positive constant (so that e.g. $C = 2C$) and $L$ is the length of the interval under consideration.
For $\varepsilon > 0$ and $\lambda > 0$, we consider the following $\varepsilon$-indexed one-dimensional Ambrosio-Tortorelli functional with fidelity term $g_\varepsilon$

\begin{equation}
AT_\varepsilon(u, v) = \int_0^L \left( (\eta_\varepsilon + v^2)(u')^2 + \varepsilon(v')^2 + \frac{(1 - v)^2}{\varepsilon} + \lambda(u - g_\varepsilon)^2 \right) \, dx.
\end{equation}

In (1.1), $\eta_\varepsilon$ is a positive number, and $(u, v)$ belongs to the space $Y_\varepsilon$ defined by

$$Y_\varepsilon := \{u, v \in H^1(0, L)\}.$$  

We assume that, as $\varepsilon \searrow 0$,

\begin{equation}
\eta_\varepsilon/\varepsilon \to 0, \text{ i.e., } \eta_\varepsilon \ll \varepsilon.
\end{equation}

We assume that the fidelity term $g_\varepsilon$ satisfies the following

**Assumption.** $g_\varepsilon$ converges weakly in $W^{1,2}((0, L))$ to a function $g \in W^{1,2}((0, L))$.

We also introduce, for $u \in SBV(\mathbb{R})$, the one-dimensional Mumford-Shah functional with fidelity term $g$

\begin{equation}
MS(u, v) = \begin{cases} 
\int_0^L (u')^2 \, dx + 2\#(S(u)) + \lambda(u - g)^2 & \text{if } v \equiv 1 \\
+\infty & \text{otherwise.}
\end{cases}
\end{equation}

In (1.3), $u'$ denotes the approximate derivative of $u$, i.e., the density of the absolutely continuous part of the measure $Du$ with respect to the Lebesgue measure, while $S(u)$ denotes the jump set of $u$, defined as the complement in $\mathbb{R}$ of the set of Lebesgue points of $u$. Finally, $SBV(\mathbb{R})$ stands for the class of special functions of bounded variation (see [1]). They are $L^1(\mathbb{R})$ functions $u$ whose distributional derivatives $Du$ have finite total variation $|Du|$ and have no Cantor parts.

Note that $AT_\varepsilon$ serves as a variational approximation for $MS$ in the sense that $AT_\varepsilon \Gamma$-converges to $MS$. For other approximations of $MS$, see [5]. Therefore, by nature of $\Gamma$-convergence, global minimizers of $AT_\varepsilon$ converge in suitable topology to those of $MS$. The question of interest is whether we have convergence results for intermediate states. This is the purpose of this paper. We propose to study the convergence property of critical points, not necessarily minimizers.

Let $(u_\varepsilon, v_\varepsilon)$ be critical points of the Ambrosio-Tortorelli functional (1.1), i.e., pairs of functions $(u_\varepsilon, v_\varepsilon) \in Y_\varepsilon$ that satisfy the Euler-Lagrange equations

\begin{align}
-\varepsilon v''_\varepsilon + v_\varepsilon(u'_\varepsilon)^2 + \frac{v_\varepsilon - 1}{\varepsilon} &= 0 \\
[u_\varepsilon'(\eta_\varepsilon + v^2)]' &= \lambda(u_\varepsilon - g_\varepsilon) \\
u'_\varepsilon(0) &= u'_\varepsilon(L) = 0 \\
v'_\varepsilon(0) &= v'_\varepsilon(L) = 0.
\end{align}

(1.4)
Our main goal is to study the limit properties of \((u_\varepsilon, v_\varepsilon)\) as \(\varepsilon\) goes to 0, provided additionally that

\[
\mathcal{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq C < \infty. \tag{1.5}
\]

In the framework of \(\Gamma\)-convergence, there have been relatively few attempts in proving that critical points of the approximation functionals converge to those of the limiting functional. One may refer to such convergence results in other settings such as that of the Allen-Cahn functional in phase transitions, see [8], [13], [14], or that of the Ginzburg-Landau functional in superconductivity, see [4], [12], or that in the framework of thin elastic bodies, see [9], [10]. We may also expect those convergence results in the framework of image segmentation.

Our main result here is to show that indeed critical points of the Ambrosio-Tortorelli functional (1.1) converge to corresponding critical points of the Mumford-Shah functional (1.3). Recall, from Chapter 7 in [1], that a critical point of (1.3) is a couple \((u, 1)\) where \(u\) is discontinuous at a finite number of points, and between these points, \(u\) solves the equation \(u'' = \lambda(u - g)\) with homogeneous Neumann boundary conditions.

The paper is organized as follows. We state our main convergence results, Theorem 2.1 and Theorem 2.3, in Sect. 2. As a preparation for the their proofs, we establish essential estimates in Sect. 3. The proof of Theorem 2.1 will be carried out in Sect. 4. Finally, in the final section, we prove Theorem 2.3.

## 2 Main results

In this section, we state our main results on the convergence of critical points of the Ambrosio-Tortorelli functional with fidelity term (1.1).

The first result says that critical points of the Ambrosio-Tortorelli functional (1.1) converge to corresponding critical points of the Mumford-Shah functional (1.3):

**Theorem 2.1** There is a finite number of points \(X_0 < X_1 < \cdots < X_k < X_{k+1} \in [0, L]\) with \(X_0 = 0\) and \(X_{k+1} = L\) such that the limit function \(u\) of \(u_\varepsilon\) solves the following equation on each interval \((X_i, X_{i+1})(0 \leq i \leq k)\):

\[
\begin{aligned}
    u'' &= \lambda(u - g) \\
    u'(X_i) &= u'(X_{i+1}) = 0.
\end{aligned}
\]

Before stating our next result, it is convenient to introduce the following definition whose motivation will be given in Section 3.3.

**Definition 2.2** We call \(x_\varepsilon \in [0, L]\) a \(v\)-jump if \(x_\varepsilon\) is a critical point of \(v_\varepsilon\) with \(v_\varepsilon(x_\varepsilon) \leq C_\varepsilon^{1/2}\).
The second result concerns the convergence of various terms of the Ambrosio-Tortorelli functional (1.1):

**Theorem 2.3** If \( u_\varepsilon \to u \), a critical point for the Mumford-Shah functional, as given in Theorem 2.1, and if \( x_1, \ldots, x_j \) are corresponding limits of \( v \)-jumps associated with \((v_\varepsilon, u_\varepsilon)\) (see also Definition 3.11 and Proposition 3.12), then

- The limit measure of \((\eta_\varepsilon + v^2_\varepsilon)(u'_\varepsilon)^2 dx\) is \((u')^2 dx\) where \( u' \) denotes the approximate gradient of \( u \); in fact, \((\eta_\varepsilon + v^2_\varepsilon)(u'_\varepsilon)^2\) converges weakly in \( L^1((0, L)) \) to \((u')^2\);
- The limit measure of \( \varepsilon(v'_\varepsilon)^2 dx \), which is also that of \((1 - v_\varepsilon)^2 / \varepsilon dx \) (i.e., there is equi-partition of the energy), is a finite sum of Dirac masses at the \( x_i \)'s.

**Remark 2.4** As can be seen from the proof of Theorem 2.1, \( \{X_1, \ldots, X_k\} \subset \{x_1, \ldots, x_j\} \). It follows from the fact that \( u \) can be discontinuous only at limiting \( v \)-jumps, but it does not have to be.

### 2.1 Ingredients of the Proofs

Let us say a few words about the ingredients of the proofs. The difficulties in establishing the convergence result in Theorem 2.1 come from the smallness of \( v_\varepsilon \). In the regions where \( v_\varepsilon \) is small, the limit function \( u \) of \( u_\varepsilon \) can be discontinuous. We need to prove the vanishing of \( u' \) at both sides of the points of discontinuity of \( u \). To do this, we introduce the quantity \( F_\varepsilon \) in (3.5), which is in some sense an approximation for the derivative \( u'_\varepsilon \) of \( u_\varepsilon \). Its limit function \( F \) is smooth and the proof goes about relating the smoothness of \( u \) to the value of \( F \) at each point in the interval \([0, L]\). This will be done in Section 4. In the course of the proof, we also use the gradient estimates for \( v_\varepsilon \) and \( u_\varepsilon \) that can be derived from the balance law (3.3).

For the proof of the equi-partition of energy in Theorem 2.3, we exploit the positivity of the discrepancy measure \((1 - v_\varepsilon)^2 / \varepsilon - \varepsilon \left| v'_\varepsilon \right|^2 \) and the closeness to 1 of \( v_\varepsilon \) away from the \( v \)-jumps. The latter ingredient, to be made precise in Proposition 3.12, is based on the maximum principle and precise estimates on the values of \( v_\varepsilon \) at its critical points established in Lemma 3.8.

### 2.2 Open questions

In this section, we list several open questions left from our study. In the Dirichlet case, we proved in Theorem 2.1 in [7] that critical points of Ambrosio-Tortorelli not only converge to critical points of the corresponding Mumford-Shah functional but also to very special ones: only critical points with jumps that are symmetrically located on the interval of study can be obtained through the limiting process. In other words, in the Dirichlet case, the Ambrosio-Tortorelli approximation acts as a selection mechanism for the Mumford-Shah functional. In our case with the fidelity term, it is not clear whether this is still true. If it is, then the next question is: what is the selection mechanism?
Question 1. Is there any selection mechanism for limiting critical points of the Ambrosio-Tortorelli functional with fidelity term? When $u$ is discontinuous at $X \in [0, L]$, is it true that $\frac{u(X^-) + u(X^+)}{2} = g(X)$?

In Lemma 3.5, we prove that 0 and $L$ are local maxima of $v_\varepsilon$ on $[0, L]$ and the values $v_\varepsilon(0), v_\varepsilon(L)$ are nearly 1. It is not clear if $v$-jumps (critical points of $v_\varepsilon$ at which $v_\varepsilon$ is close to 0) can accumulate to these boundary points. More generally, one can ask: given the limiting fidelity term $g$ and a uniform energy bound $C$, does exist a positive constant $\delta = \delta(g, C)$ such that the distance between any local minimum and local maximum of $v_\varepsilon$ is at least $\delta$? In the Dirichlet case in [7], due to symmetry property of $v_\varepsilon$, the answer is positive.

Question 2. Is there any boundary limiting jump point?

In order to evaluate precisely the energy $\mathcal{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)$, one might also ask

Question 3. Do we have single-multiplicity? i.e., does the weight of $\varepsilon(v_\varepsilon')^2 dx$ at each $x_i$ equal to 1?

Another question, which is related to Remark 2.6 in [7], is whether $u$ is discontinuous at every limiting $v$-jump. In other words, we may ask

Question 4. Is it true that $\{X_1, \cdots, X_k\} = \{x_1, \cdots, x_j\}$?

Finally, in the proof of equi-partition of energy in Theorem 2.3, we used the weak convergence of $g_\varepsilon$ to $g$ in $W^{1,2}((0, L))$. In general, we can ask

Question 5. What are the minimal convergence hypotheses on $g_\varepsilon - g$ that are needed?

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3 Preliminary estimates

3.1 Classical a priori estimates

In this section, we establish a few canonical estimates that will prove instrumental in the proof of Theorems 2.1, 2.3. These estimates are completely standard but we include their proofs for convenience of the reader.

First, it follows from the assumptions on $g_\varepsilon$ and the embedding $W^{1,2}(0, L) \hookrightarrow C([0, L])$ that

$$\|g_\varepsilon\|_{L^2(0, L)} \leq C, \text{ and } \|g_\varepsilon\|_{L^\infty(0, L)} \leq C.$$  (3.1)

For now, we prove some elementary estimates on the critical points $(u_\varepsilon, v_\varepsilon)$ of (1.1), which, by the way, are smooth by elliptic regularity.

A first result is a maximum principle for $v_\varepsilon$ and $u_\varepsilon$, namely,

Lemma 3.1

$$0 \leq v_\varepsilon \leq 1, \ |u_\varepsilon| \leq \|g_\varepsilon\|_\infty \leq C.$$
Proof. The proof of the inequality $0 \leq v_\varepsilon \leq 1$ is exactly the same as that of Lemma 3.1 in [7]. Therefore, we only need to prove the inequality for $u_\varepsilon$.

Multiplying both sides of the second equation of (1.4) by $(u_\varepsilon - \|g_\varepsilon\|_\infty)^+ = \max(0, u_\varepsilon - \|g_\varepsilon\|_\infty)$ and recalling the Neumann boundary conditions on $u_\varepsilon$, we get

$$
\int_0^L -u_\varepsilon'(\varepsilon + v_\varepsilon^2)((u_\varepsilon - \|g_\varepsilon\|_\infty)^+)' = \int_0^L \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon - \|g_\varepsilon\|_\infty)^+. 
$$

This equality can be rewritten as

$$(3.2) \quad \int_0^L -((u_\varepsilon - \|g_\varepsilon\|_\infty)^+)'(\varepsilon + v_\varepsilon^2)((u_\varepsilon - \|g_\varepsilon\|_\infty)^+)' = \int_0^L \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon - \|g_\varepsilon\|_\infty)^+.$$

The left hand side of (3.2) is nonpositive. Thus

$$
\int_0^L \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon - \|g_\varepsilon\|_\infty)^+ \leq 0,
$$

showing that $(u_\varepsilon - \|g_\varepsilon\|_\infty)^+ = 0$.

Multiplication of the second equation of (1.4) by $(u_\varepsilon + \|g_\varepsilon\|_\infty)^- = \max(0, -(u_\varepsilon + \|g_\varepsilon\|_\infty))$ would yield the other inequality for $u_\varepsilon$. □

Next, we establish the convergence properties of the pair $(u_\varepsilon, v_\varepsilon)$.

Lemma 3.2

$v_\varepsilon \to 1$, strongly in $L^2((0, L))$,

and, modulo extraction,

$u_\varepsilon \to u \in BV((0, L))$, strongly in $L^p((0, L))$, for any $p \in [1, \infty)$.

Proof. The energy bound (1.5) immediately implies the first convergence. Together with the bounds obtained in Lemma 3.1, it also implies that $u_\varepsilon(2v_\varepsilon - v_\varepsilon^2)$ is bounded in $BV((0, L))$. Now let $p \in [1, \infty)$. By the compactness of $BV((0, L))$ into $L^p((0, L))$ (see [6]), a subsequence of $u_\varepsilon(2v_\varepsilon - v_\varepsilon^2)$ converges in $L^p((0, L))$ to $u \in BV((0, L))$. Because $u_\varepsilon - u = (u_\varepsilon(2v_\varepsilon - v_\varepsilon^2) - u) + u_\varepsilon(1 - v_\varepsilon)^2$, the strong convergence of $u_\varepsilon$ to $u$ in $L^p((0, L))$ follows.

In the present context, we have the following Noether- type balance law

Proposition 3.3

$$(3.3) \quad \left\{ \frac{1}{2} \left( \frac{(1-v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - \varepsilon(v_\varepsilon')^2 + \lambda(u_\varepsilon - g_\varepsilon)^2 \right) \right\}' = -\lambda g_\varepsilon'(u_\varepsilon - g_\varepsilon).$$
Proof. The left hand side of the previous expression also reads as
\[ A_\varepsilon := \frac{(v_\varepsilon - 1)v_\varepsilon'}{\varepsilon} - \varepsilon v_\varepsilon' + v_\varepsilon v_\varepsilon'(u_\varepsilon'(x))^2 - \frac{(v_\varepsilon'' + \eta)u_\varepsilon' u_\varepsilon'' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon')}{\varepsilon} \]
\[ = v_\varepsilon'(-\varepsilon v_\varepsilon'' - v_\varepsilon'(u_\varepsilon')^2 + \frac{v_\varepsilon - 1}{\varepsilon}) - (v_\varepsilon'' + \eta)u_\varepsilon' u_\varepsilon'' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon'). \]
The first and second equation of (1.4) then imply that
\[ A_\varepsilon = -v_\varepsilon'(2v_\varepsilon(u_\varepsilon')^2) - (v_\varepsilon'' + \eta)u_\varepsilon' u_\varepsilon'' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') \]
\[ = -u_\varepsilon'[u_\varepsilon''(\eta + v_\varepsilon^2) + 2v_\varepsilon v_\varepsilon'] + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') \]
\[ = -\lambda(u_\varepsilon - g_\varepsilon)u_\varepsilon + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') = -\lambda \varepsilon v_\varepsilon'(u_\varepsilon - g_\varepsilon). \]

\[ \square \]

This proposition is key to the following gradient estimates

Lemma 3.4 For \( \varepsilon > 0 \) sufficiently small
\[ \left\| \left\| v_\varepsilon' \right\| \right\|_\infty \leq \frac{C}{\varepsilon}. \]

and the boundary value estimates whose proofs will be given in the next subsection.

Lemma 3.5 For \( \varepsilon \) sufficiently small, we have
\[ v_\varepsilon(0), v_\varepsilon(L) > 1/2. \]
Furthermore, 0 and L are local maxima of \( v_\varepsilon \) on \([0, L]\).

Proof of Lemma 3.4. For any \( x \in (0, L) \), integrating (3.3) from 0 to \( x \), and using the last two equations of (1.4), we get
\[ \frac{1}{2} \left( \frac{(1 - v_\varepsilon(x))^2}{\varepsilon} - (\eta + v_\varepsilon^2(x))(u_\varepsilon'(x))^2 - \varepsilon(v_\varepsilon'(x))^2 + \lambda(u_\varepsilon(x) - g_\varepsilon(x))^2 \right) \]
\[ = \frac{1}{2} \left( \left( \frac{1 - v_\varepsilon(0)}{\varepsilon} + \lambda(u_\varepsilon(0) - g_\varepsilon(0))^2 \right) + \int_0^x -\lambda g_\varepsilon'(t)(u_\varepsilon(t) - g_\varepsilon(t))dt. \]
Rearranging, we obtain
\[ \frac{1}{2} \varepsilon(v_\varepsilon'(x))^2 \leq \frac{1}{2} \left( \frac{(1 - v_\varepsilon(x))^2}{\varepsilon} + \lambda(u_\varepsilon(x) - g_\varepsilon(x))^2 \right) + \int_0^x \lambda g_\varepsilon'(t)(u_\varepsilon(t) - g_\varepsilon(t))dt \]
\[ \leq \frac{1}{2} \left( \frac{1}{\varepsilon} + 4\lambda \|g_\varepsilon\|_{\infty}^2 \right) + \lambda \left\| g_\varepsilon' \right\|_{L^2(0,L)} \|u_\varepsilon - g_\varepsilon\|_{L^2(0,L)} \leq \frac{C}{\varepsilon}, \]
for \( \varepsilon > 0 \) sufficiently small. Therefore, the \( L^\infty \)-estimate of \( v_\varepsilon' \) follows. \( \square \)

Remark 3.6 Note that, on a compact set \( K \) away from the limiting \( v \)-jumps, we have the uniform estimates
\[ \left\| v_\varepsilon' \right\|_{L^\infty(K)} \leq C_K \]
where \( C_K \) is a constant depending only on \( K \). This estimate follows from Proposition 3.12 and Lemma 5.6.
3.2 Approximate slopes

In the sequel, it is convenient to introduce the following

**Notation.** Let

\begin{equation}
(3.5) \quad F_\varepsilon(x) := u'_\varepsilon(x)(\eta_\varepsilon + v_\varepsilon(x))^2 = \int_0^x \lambda(u_\varepsilon(t) - g_\varepsilon(t))dt.
\end{equation}

From (3.1) and the bounds in Lemma 3.1, we have the uniform bound on $F_\varepsilon$

\begin{equation}
(3.6) \quad \|F_\varepsilon\|_{L^\infty} \leq C
\end{equation}

and the convergence results in Lemma 3.2 imply that $F_\varepsilon$ converges uniformly to $F$ on $[0, L]$, where

\begin{equation}
(3.7) \quad F(x) = \int_0^x \lambda(u(t) - g(t))dt.
\end{equation}

**Remark 3.7** $F_\varepsilon$ is a function analogue to the constant $c_\varepsilon$ in our previous paper [7]. It is formally the slope $u'_\varepsilon$ of the function $u_\varepsilon$. The limit values of $F_\varepsilon$ at $v$-jumps are closely related to the limit values of $c_\varepsilon$. On the other hand, if $x$ is a $v$-jump and $u$ is not continuous there, then Theorem 2.1 requires that $u'(x) = 0$, which suggests us to prove that $F(x) = 0$.

**Proof of Lemma 3.5.** Assume that $v_\varepsilon(0) \leq 1/2$. Let $x > 0$ be a critical point of $v_\varepsilon$. This set of $x$’s contains $L$ and is thus not empty. Then, rewriting (3.4) and using the definition of $F_\varepsilon$, we have

\begin{equation}
(3.8) \quad \frac{(1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2}{\varepsilon} - \frac{F_\varepsilon(x)^2}{\eta_\varepsilon + v_\varepsilon^2(x)} = -\lambda(u_\varepsilon(x) - g_\varepsilon(x))^2 + \lambda(u_\varepsilon(0) - g_\varepsilon(0))^2 - \int_0^x 2\lambda g_\varepsilon'(t)(u_\varepsilon(t) - g_\varepsilon(t))dt.
\end{equation}

The bounds (3.1) imply that the left hand side is bounded. This fact, together with the uniform bound on $F_\varepsilon$, shows that for $\varepsilon$ sufficiently small, $v_\varepsilon(x) < 2/3$. This, in turn, implies that

\begin{equation}
(3.9) \quad M_\varepsilon = \max_{x \in [0, L]} v_\varepsilon(x) \leq 2/3.
\end{equation}

However, without any assumption except the energy bound (1.5), we can estimate $M_\varepsilon$ from below via the following inequalities

$$
C \geq \int_0^L \frac{(1 - v_\varepsilon)^2}{\varepsilon} \geq \int_0^L \frac{(1 - M_\varepsilon)^2}{\varepsilon} = \frac{L(1 - M_\varepsilon)^2}{\varepsilon}.
$$

Thus, $M_\varepsilon \geq 1 - C\sqrt{\varepsilon}$. This is a contradiction with (3.9). Therefore $v_\varepsilon(0) > 1/2$. Similarly, $v_\varepsilon(L) > 1/2$.
Because 0 is a critical point of \( v_\varepsilon \) on \([0, L]\) with \( v_\varepsilon''(0) = \frac{v_\varepsilon'(0)}{\varepsilon} \leq 0 \) by the maximum principle from Lemma 3.1, it is a local maximum for \( v_\varepsilon \) on \([0, L]\) and, similarly, so is \( L \). □

The next lemma gives more precise estimates on the values of \( v_\varepsilon \) at its critical points. Though rather simple, they will turn out to be very useful for the proofs of Proposition 3.12 and Theorem 2.3.

**Lemma 3.8** Let \( x \) be a critical point of \( v_\varepsilon \). Then, either

\[
(3.10) \quad v_\varepsilon(x) \leq C\sqrt{\varepsilon}
\]

or

\[
(3.11) \quad v_\varepsilon(x) = v_\varepsilon(0) + O(\sqrt{\varepsilon}) \geq 1 - C\sqrt{\varepsilon}.
\]

**Proof.** Let \( x \) be a critical point of \( v_\varepsilon \). Then (3.8) holds. Multiplying this identity by \((\eta_\varepsilon + (v_\varepsilon(x))^2)\) and using (3.6), we see that \( \eta_\varepsilon + v_\varepsilon(x)^2(1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2 \) is bounded, i.e.,

\[
(3.12) \quad \frac{\eta_\varepsilon + v_\varepsilon(x)^2}{\varepsilon} \left| (1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2 \right| \leq C
\]

If \( v_\varepsilon(x) \leq 1/4 \) then, from the inequality \( v_\varepsilon(0) \geq 1/2 \) in Lemma 3.5, we deduce that \( \eta_\varepsilon + v_\varepsilon(x)^2 \varepsilon \) is bounded. Therefore, \( v_\varepsilon(x) \leq C\sqrt{\varepsilon} \).

On the other hand, if \( v_\varepsilon(x) \geq 1/4 \) then \( (1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2 \) is bounded (by \( C \)). Using the inequality \(|a^2 - b^2| \geq (a - b)^2\) for \( a, b > 0 \), we find that

\[
(v_\varepsilon(x) - v_\varepsilon(0))^2 \leq \left| (1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2 \right| \leq C\varepsilon.
\]

Consequently,

\[
v_\varepsilon(x) = v_\varepsilon(0) + O(\sqrt{\varepsilon}).
\]

This shows, by choosing \( x \) to be the maximum point of \( v_\varepsilon \), that \( v_\varepsilon(0) \) is nearly maximal: it is the maximum value of \( v_\varepsilon \) up to \( C\sqrt{\varepsilon} \). Now, recalling the estimate for the maximum value \( M_\varepsilon \) of \( v_\varepsilon \) in the proof of Lemma 3.5, we have

\[
v_\varepsilon(0) + O(\sqrt{\varepsilon}) = M_\varepsilon \geq 1 - C\sqrt{\varepsilon}.
\]

and the proof of (3.11) is complete. □

The relation (3.11) is crucial in establishing the following result:

**Lemma 3.9** For all \( x \in [0, L] \), we have

\[
(3.13) \quad \left| \frac{(1 - v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - \varepsilon(v_\varepsilon')^2 + \lambda(u_\varepsilon - g_\varepsilon)^2 \right| \leq C
\]

and

\[
(3.14) \quad \left| F_\varepsilon u_\varepsilon' \right| \leq \frac{C}{\varepsilon}.
\]
Proof. From (3.4), it suffices to show that \( \frac{(1-v_\varepsilon(0))^2}{\varepsilon} \) is bounded. From the relation (3.11), we find that \( 1 \geq v_\varepsilon(0) \geq 1 - C\sqrt{\varepsilon} \) and hence the boundedness of \( \frac{(1-v_\varepsilon(0))^2}{\varepsilon} \). From (3.13) and the gradient bound of Lemma 3.4, the inequality (3.14) follows easily. □

Remark 3.10 The inequality (3.13) gives a uniform bound for the generalized discrepancy measure
\[
d_\varepsilon(x) = \frac{(1-v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - \varepsilon(v_\varepsilon')^2 + \lambda(u_\varepsilon - g_\varepsilon)^2.
\]
Its analogue in [7] is a constant, uniformly bounded in \( \varepsilon \).

3.3 Definition of \( v \)-jump

The inequality (3.10) motivates the following

Definition 3.11 We call \( x_\varepsilon \in [0,L] \) a \( v \)-jump if \( x_\varepsilon \) is a critical point of \( v_\varepsilon \) with \( v_\varepsilon(x_\varepsilon) \leq C\varepsilon^{1/2} \).

An important fact in the proofs of the main results is that \( v \)-jumps accumulate at only a finite number of points, as stated in the following

Proposition 3.12 There is a finite number \( j \) of points on \([0,L]\), denoted by \( x_1,\ldots,x_j \), where \( 0 \leq x_1 \leq \cdots \leq x_j \leq L \), that are limits of \( v \)-jumps. For any compact set \( K \subset [0,L]\{x_1,\ldots,x_j\} \), there is a positive constant \( C_K \) depending only on \( K \) such that for all \( \varepsilon \) small, we have \( v_\varepsilon(x) \geq 1 - C_K \varepsilon \) \( \forall x \in K \).

Proof. Let \( x \) be a limit of \( v \)-jumps, i.e., there exists \( x_\varepsilon \) such that \( x_\varepsilon \rightarrow x \) and \( v_\varepsilon(x_\varepsilon) \leq C\varepsilon^{1/2} \). Fix \( \delta > 0 \). Then for \( \varepsilon \) sufficiently small, \( x_\varepsilon \in I_\delta = [x-\delta,x+\delta] \). Let \( T_\varepsilon \) be the maximum value of \( v_\varepsilon \) on \( I_\delta \). Then, the easy estimate
\[
C \geq \int_{I_\delta} \frac{(1-v_\varepsilon)^2}{\varepsilon} \geq \int_{I_\delta} \frac{(1-T_\varepsilon)^2}{\varepsilon} = \frac{2\delta}{\varepsilon}(1-T_\varepsilon)^2
\]
gives
\[
T_\varepsilon \geq 1 - C\sqrt{\varepsilon/\delta}.
\]
Let \( y_\varepsilon \in I_\delta \) be such that \( v_\varepsilon(y_\varepsilon) = T_\varepsilon \) and without loss of generality, assume that \( y_\varepsilon > x_\varepsilon \). We can estimate
\[
\int_{I_\delta} \varepsilon(v_\varepsilon')^2 + \frac{(1-v_\varepsilon)^2}{\varepsilon} \geq \int_{x_\varepsilon}^{y_\varepsilon} 2|v_\varepsilon(1-v_\varepsilon)| \geq \int_{x_\varepsilon}^{y_\varepsilon} (2v_\varepsilon - v_\varepsilon')^2 \geq 2v_\varepsilon(y_\varepsilon) - (v_\varepsilon(y_\varepsilon))^2 - 2v_\varepsilon(x_\varepsilon) - (v_\varepsilon(x_\varepsilon))^2 \geq 1 - C\varepsilon.
\]
This inequality in a small neighborhood of a limiting jump is crucial in proving the finiteness of limiting \( v \)-jumps. Suppose that there is an infinite number of them \( x_1, x_2, \cdots \). Then, given a uniform bound \( E < \infty \) on the energies \( AI_\varepsilon(u_\varepsilon, v_\varepsilon) \), we choose \( N = 2E + 10 \). We
cover each point $x_i$ ($i = 1, \cdots, N$) by a small neighborhood $I_{i,\delta}$ for $\delta$ sufficiently small such that $I_{i,\delta} \cap I_{j,\delta} = \emptyset$ if $i \neq j$. Thus,

$$E \geq A\varepsilon(u_{\varepsilon}, v_{\varepsilon}) \geq \sum_{i=1}^{N} \int_{I_{i,\delta}} \varepsilon(v_{\varepsilon}')^2 + \frac{(1 - v_{\varepsilon})^2}{\varepsilon} \geq N(1 - C\varepsilon) > N/2 > E$$

for $\varepsilon$ sufficiently small. This is a contradiction and hence the proof of the first statement.

Now, we prove the second statement. We start with the following

**Claim.** Fix a positive number $\alpha < 1/2$. Then, for any compact set $K \subset [0, L] \setminus \{x_1, \ldots, x_j\}$, we have $v_{\varepsilon}(x) \geq 1 - \varepsilon^\alpha \forall x \in K$ when $\varepsilon$ is small.

We argue by contradiction. Suppose that there exists a compact set $K \subset [0, L] \setminus \{x_1, \ldots, x_j\}$ such that for all $n \in N$, there exist $\varepsilon_n \leq 1/n$ and $x_n \in K$ with the property that $v_{\varepsilon_n}(x_n) \leq 1 - \varepsilon_n^\alpha$. After extracting a subsequence, we can assume that $x_n \to x \in K$. Let $\delta = \frac{1}{4} \min\{|x-x_i| : 1 \leq i \leq j\}$. Consider the intervals $I_\delta = [x-\delta, x+\delta]$ and $I_{2\delta} = [x-2\delta, x+2\delta]$. Then, $I_{2\delta} \cap \{x_1, \ldots, x_k\} = \emptyset$. For $n$ sufficiently large, $x_n \in I_\delta$. As in the proof of the first statement, there exist $y_n \in [x_n - \delta, x_n)$ and $z_n \in (x_n, x_n + \delta]$ such that $v_{\varepsilon_n}(y_n), v_{\varepsilon_n}(z_n) \geq 1 - C\varepsilon_n/\delta$. We were able to choose $y_n \neq x_n$ because $v_{\varepsilon_n}(x_n) \leq 1 - \varepsilon_n^\alpha \leq 1 - C\varepsilon_n/\delta$, due to the fact that $\alpha < 1/2$. Similar arguments for the inequality $z_n \neq x_n$. For $n$ sufficiently large, $[y_n, z_n] \subset [x_n - \delta, x_n + \delta] \subset I_{2\delta}$. In each interval $I_n = [y_n, z_n]$, there must be a minimum point $t_n$ of $v_{\varepsilon_n}$ such that $t_n \notin \{y_n, z_n\}$ and $v_{\varepsilon_n}(t_n) \leq v_{\varepsilon_n}(x_n) \leq 1 - \varepsilon_n^\alpha < 1 - C\varepsilon_n$ for $n$ sufficiently large. Therefore, by Lemma 3.8, $v_{\varepsilon_n}(t_n) \leq C\varepsilon_n$. In other words, $t_n$ is a v-jump. Because $t_n \in [x-2\delta, x+2\delta] \subset K$, we can assume, after extraction a subsequence, that $t_n \to t \in K$. This means that $K$ contains a limiting jump point. However, this is a contradiction with the definition of $K$ and the $x_i$'s, completing the proof of the Claim.

Now, we employ the maximum principle together with the lower bounds on $v_{\varepsilon}$ established in the Claim to complete the proof of the second statement. Indeed, choose larger compact sets $K_1 \subset K_2 \subset [0, L] \setminus \{x_1, \cdots, x_j\}$ such that $K \subset K_1$, $\text{dist}(K, \partial K_1) > 0$ and $\text{dist}(K_1, \partial K_2) > 0$. Choose $\varepsilon$ sufficiently small such that $v_{\varepsilon}(x) \geq 1 - \varepsilon^{1/4} \forall x \in K$. Now, consider a smooth function $\xi \in C^\infty(K_2)$ such that $1 - \varepsilon^{1/4} \leq \xi \leq 1 - c_1\varepsilon/2$ in $K_2$, $\xi = 1 - c_1\varepsilon/2$ in $K_1$, and $\xi = 1 - \varepsilon^{1/4}$ in $K_2 \setminus K_1$, where $c_1$ is a positive constant to be chosen later. Assume that $\inf_K v_{\varepsilon} \leq 1 - c_1\varepsilon$. Let $h_{\varepsilon}$ be a function defined on $K_2$ by $h_{\varepsilon} = v_{\varepsilon} - \xi$. Then, the function $h_{\varepsilon}$ satisfies $h_{\varepsilon} \geq 0$ on $\partial K_2$, $\inf_K h_{\varepsilon} \leq -c_1\varepsilon/2$. Thus, $h_{\varepsilon}$ has an interior minimum point at, say, $x_0 \in K_2$. At $x_0$, one has $v_{\varepsilon}(x_0) \geq 1 - \varepsilon^{1/4}$,
\[ h_{\varepsilon} = v_{\varepsilon}(x_0) - \xi(x_0) \leq -c_1 \varepsilon / 2 \] and

\[
0 \leq \varepsilon \Delta h_{\varepsilon}(x_0) = \varepsilon \Delta v_{\varepsilon}(x_0) - \varepsilon \Delta \xi(x_0)
= v_{\varepsilon}(x_0)(u_{\varepsilon}'(x_0))^2 + \frac{v_{\varepsilon}(x_0) - 1}{\varepsilon} - \varepsilon \Delta \xi(x_0)
= \frac{v_{\varepsilon}(x_0)}{(\eta_\varepsilon + v_{\varepsilon}^2(x_0))^2} (F_{\varepsilon}(x_0))^2 + \frac{h(x_0) + \xi(x_0) - 1}{\varepsilon} - \varepsilon \Delta \xi(x_0)
\leq C + \frac{-c_1 \varepsilon / 2 + 1 - c_1 \varepsilon / 2}{\varepsilon} + \varepsilon |\Delta \xi(x_0)|
\leq C - c_1 + \varepsilon \sup_{K_2} |\Delta \xi|.
\]

However, the last term is negative provided that \( c_1 \) is large enough. Thus, one must have \( v_{\varepsilon}(x) \geq 1 - c(K) \varepsilon \) \( \forall x \in K \) as desired. \( \square \)

A simple consequence of the bound (3.6) and Proposition 3.12 is the following

**Lemma 3.13** Introduce, if necessary \( x_0 = 0 \) and \( x_{j+1} = L \). Then, on each interval \((x_i, x_{i+1})(0 \leq i \leq j)\), the limit function \( u \) solves the equation

\[
u'' = \lambda(u - g).
\]

If \( v_{\varepsilon}(x) \geq \delta > 0 \) for all \( x \in [0, L] \) and \( \varepsilon > 0 \) then the limit function \( u \) solves the following equation on \((0, L)\)

\[
\begin{align*}
u'' &= \lambda(u - g) \\
u'(0) &= u'(L) = 0.
\end{align*}
\]

**Proof.** We first prove the second statement. Suppose that \( v_{\varepsilon}(x) \geq \delta > 0 \) for all \( x \in [0, L] \) and \( \varepsilon > 0 \). For all \( \varphi \in C^1(0, L) \), one has from the second and third equations of (1.4)

\[
(3.15)
\int_0^L -u_{\varepsilon}'(\eta_\varepsilon + v_{\varepsilon}(x))^2 \varphi' = \int_0^L \lambda(u_{\varepsilon} - g_{\varepsilon}) \varphi.
\]

From (3.6), we deduce that \( u_{\varepsilon}' \) is bounded. Thus \( u_{\varepsilon} \) is bounded in \( W^{1,2}(0, L) \) and, up to extraction, weakly converges to \( u \in W^{1,2}(0, L) \). Now, letting \( \varepsilon \to 0 \) in (3.15) yields

\[
\int_0^L -u' \varphi' = \int_0^L \lambda(u - g) \varphi \ \forall \varphi \in C^1(0, L)
\]

This is equivalent to what is needed to establish.

We now prove the first statement of the lemma. Let \( K \) be any compact set of \((x_i, x_{i+1})\). Then, by Proposition 3.12, there is \( \delta_K > 0 \) such that \( v_{\varepsilon}(x) \geq \delta_K > 0 \) for all \( x \in K \) and \( \varepsilon > 0 \). Now, we argue similarly as above, with \( \varphi \in C^1_0(K) \) to conclude that \( u \) satisfied the equation \( u'' = \lambda(u - g) \) on \((x_i, x_{i+1})\). \( \square \)
4 Proof of Theorem 2.1

In this section, we prove Theorem 2.1. It will follow from the characterization of points of discontinuity of $u$ in Lemma 4.1 and the relation between $u$ and $F$ when $F$ does not vanish, as stated in Lemma 4.2.

Lemma 4.1 If $u$ is discontinuous at $x_0$ then $F(x_0) = 0$.

Lemma 4.2 If $F(x_0) \neq 0$ then $u$ is differentiable at $x_0$ and $u'(x_0) = F(x_0)$.

Proof of Theorem 2.1. Let $X_1 < \cdots < X_k$ be all points of discontinuity of $u$ in $(0, L)$. By Lemma 3.13, we know that $k$ is finite and $X_i \in \{x_1, \cdots, x_l\}$ for all $i \in \{1, \cdots, k\}$.

Now, fix $i \in \{0, \cdots, k\}$. We have to prove that $u'' = \lambda(u - g)$ in $(X_i, X_{i+1})$ with Neumann boundary conditions. First, we claim that $F(X_i) = 0$ for all $i \in \{i, i+1\}$. Indeed, if $X_i$ is a discontinuity point of $u$ in $(0, L)$, then, from Lemma 4.1, we know that $F(x_i) = 0$. Otherwise, $X_j$ is a boundary point of $[0, L]$ and hence $F(X_j) = 0$ because $F(0) = F(L) = 0$.

Consider the case where $F$ does not vanish in $(X_i, X_{i+1})$. Then, by Lemma 4.2, $u' = F$ in $(X_i, X_{i+1})$, but $F' = \lambda(u - g)$. Thus $u'' = \lambda(u - g)$ in $(X_i, X_{i+1})$. The Neumann boundary conditions follow from fact that $F(X_i) = F(x_{i+1}) = 0$ and the continuity up to the boundary of $u'$.

The case $F$ vanishes inside $(X_i, X_{i+1})$ is more involved. It is rather easy in the case where $F$ vanishes at only a finite number of points inside $(X_i, X_{i+1})$. In this case, the argument is as follows. Without loss of generality, we can assume that $F$ only vanishes at $Y$. Then, we can prove that $u$ solves $u'' = \lambda(u - g)$ with Neumann boundary conditions on each interval $(X_i, Y)$ and $(Y, X_{i+1})$. Because $u$ is continuous at $Y$ with zero derivative there, it solves the desired equation on the whole interval $(X_i, X_{i+1})$.

The general argument makes use of Lemma 3.13 and the assumption, which will be justified below, that $F$ is not identically zero on any subinterval in $I = (X_i, X_{i+1})$. There are two cases. Consider the first case where there are no limiting jump points in $(X_i, X_{i+1})$. Then Lemma 3.13 tells us that in $I$, $u'' = \lambda(u - g)$. Thus, in $I$, $u' = F + c$ for some constant $c$. Because $F$ must be different from zero at some point $X$ in $I$, we have by Lemma 4.2, $u'(X) = F(X)$ and hence $c = 0$. It follows that $u' = F$ in $I$ and as above, we must have $u(X_i) = u'(X_{i+1}) = 0$. Now, we consider the second case where there are some limiting jump points in $I$. We only need to consider the case where there is only one limiting jump point $X$ in $I$. Arguing as in the first case, one has $u' = F$ in $(X_i, X)$ and $u' = F$ in $(X, X_{i+1})$. Because $u$ is continuous at $X$ with the same derivative on both sides of $X$, it solve the equation $u'' = \lambda(u - g)$ on the whole interval $(X_i, X_{i+1})$. Now, the Neumann boundary conditions easily follows.

Now, we indicate how to remove the assumption that $F$ is not identically zero on any subinterval in $I = (X_i, X_{i+1})$. Note that in the previous arguments, what is needed is the fact that $u' = F$ between two consecutive limiting jump points $x_i$ and $x_{i+1}$. We show how to get this identity in the case $F$ vanishes identically on $J = (x_i, x_{i+1})$. In this case, we must have $u = g$ on $J$ and by Lemma 3.13, we have $u'' = 0$. Therefore, $u$, and hence $g$, is
a linear function on $J$. On the other hand, on any interval $K = [y, z] \subset J$, we have $v_\varepsilon \geq \frac{1}{2}$, by Proposition 3.12. Because $F_\varepsilon(x) = u'_\varepsilon(x)(\eta_\varepsilon + v_\varepsilon^2(x)) \Rightarrow F(x) = 0$ on $K$, we find that $u'_\varepsilon(x) \Rightarrow 0$ on $K$. Thus, by lower-semicontinuity,

$$0 = \liminf_{\varepsilon \to 0} \int_K |u'_\varepsilon(x)| \, dx \geq \int_K |Du|,$$

implying that $u$ is a constant on $K$. By its linearity on $J$, $u$ is also a constant on $J$. Therefore, on $J$, we have $u' = 0 = F$, as desired.

\[\square\]

**Remark 4.3** The above proof also shows that, for all $x \in [0, L]$, one has $F(x) = u'(x)$.

The remaining part of this section is devoted to the proof of Lemmas 4.1 and 4.2.

**Proof of Lemma 4.1.** Assume by contradiction that $u$ is discontinuous at $x_0$ but $|F(x_0)| = \alpha > 0$. Note that, by Lemma 3.13, $x_0$ must be a limit of $v$-jumps. This means that there are $x'_\varepsilon$’s converging to $x_0$ such that $v_\varepsilon(x'_\varepsilon) \leq C\sqrt{\varepsilon}$. Our proof, like in [7], is then based on the estimate on the size of the set $\{v_\varepsilon \leq M\sqrt{\varepsilon}\}$, for $M$ large enough.

Because $F_\varepsilon$’s are Lipschitz and converge uniformly to $F$, we can choose positive numbers $\varepsilon_0$ and $\delta_0$ sufficiently small so as to have, for all $\delta \leq \delta_0$

$$|F_\varepsilon(x)| \geq \frac{\alpha}{2} \forall \varepsilon \leq \varepsilon_0, \forall x \in I_\delta := (x_0 - \delta, x_0 + \delta).$$

Recalling $u'_\varepsilon = F_\varepsilon/(\eta_\varepsilon + v_\varepsilon^2)$, we rewrite the first equation of (1.4) as

$$-\varepsilon v''_\varepsilon + \frac{v_\varepsilon F_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{\varepsilon} = 0.$$

Integrate this equation over $K_{\varepsilon, \delta} := \{v_\varepsilon \leq M\sqrt{\varepsilon}\} \cap I_\delta$ to obtain

$$\int_{K_{\varepsilon, \delta}} \varepsilon v''_\varepsilon \, dx = \int_{K_{\varepsilon, \delta}} \frac{v_\varepsilon F_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} \, dx + \int_{K_{\varepsilon, \delta}} \frac{v_\varepsilon - 1}{\varepsilon} \, dx.$$

We now claim that

$$\text{(4.3)} \quad \text{The number of connected components of } K_{\varepsilon, \delta} \text{ is bounded by } C\varepsilon^{-1/3}.$$

On each connected component $(a_i, b_i)$ of $K_{\varepsilon, \delta}$, we obtain, by virtue of the gradient bound of Lemma 3.4,

$$\left| \int_{a_i}^{b_i} \varepsilon v''_\varepsilon \right| = \left| \varepsilon v'_\varepsilon(b_i) - \varepsilon v'_\varepsilon(a_i) \right| \leq C.$$

Then, with (4.3), the left hand side of (4.2) is bounded from above by $C\varepsilon^{-1/3}$.

Next, we estimate the size of $K_{\varepsilon, \delta}$. Recall from Lemma 3.9 that $|F_\varepsilon u'_\varepsilon| \leq \frac{C}{\varepsilon}$ on $[0, L]$. This gives

$$\text{(4.4)} \quad \eta_\varepsilon + v_\varepsilon^2(x) = \frac{F_\varepsilon^2(x)}{F_\varepsilon(x)u'_\varepsilon(x)} \geq \frac{(\alpha/2)^2}{C/\varepsilon} = \frac{\alpha^2\varepsilon}{C} \forall x \in K_{\varepsilon, \delta}.$$
It follows that, for $\varepsilon$ sufficiently small, $v_\varepsilon^2(x) \gg \eta_\varepsilon$, $\forall x \in K_{\varepsilon,\delta}$. Thus, by (4.1), the right hand-side of (4.2) is bounded from below by

$$\int_{K_{\varepsilon,\delta}} \frac{C(\alpha/2)^2 v_\varepsilon}{v_\varepsilon^4} dx - \frac{|K_{\varepsilon,\delta}|}{\varepsilon} \geq \frac{C\alpha^2 |K_{\varepsilon,\delta}| - |K_{\varepsilon,\delta}|}{M^3 \varepsilon^{3/2}} \geq \frac{C\alpha^2 |K_{\varepsilon,\delta}|}{M^3 \varepsilon^{3/2}}.$$  

Therefore, for $\varepsilon$ sufficiently small, $C\varepsilon^{-1/3} \geq \frac{C\alpha^2 |K_{\varepsilon,\delta}|}{M^3 \varepsilon^{3/2}}$, implying in turn that

$$(4.5) \quad |K_{\varepsilon,\delta}| \leq \frac{CM^3 \varepsilon^{7/6}}{\alpha^2}.$$  

This estimates combined with (4.1) gives

$$(4.6) \quad \liminf_{\delta \to 0} \liminf_{\varepsilon \to 0} \int_{x_0-\delta}^{x_0+\delta} |u_\varepsilon'| = 0.$$  

Indeed, we split

$$\int_{x_0-\delta}^{x_0+\delta} |u_\varepsilon'| = \int_{K_{\varepsilon,\delta}} |u_\varepsilon'| + \int_{I_\delta \cap \{M \sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}} |u_\varepsilon'| + \int_{I_\delta \cap \{v_\varepsilon \geq 1/2\}} |u_\varepsilon'| := J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon.$$  

We bound each of the terms on the right hand side from above.

**The first term** $J_1^\varepsilon$. From Lemma 3.9 and the inequality (4.1), we note that, on $K_{\varepsilon,\delta}$,

$$|u_\varepsilon'| \leq \frac{C}{\varepsilon F_\varepsilon \leq \frac{C}{\varepsilon^{\alpha/2}} = \frac{C}{\varepsilon^{\alpha}}.$$  

So, by (4.5), the first term is bounded by

$$(4.7) \quad J_1^\varepsilon \leq |K_{\varepsilon,\delta}| \left\| u_\varepsilon' \right\|_{L^\infty(K_{\varepsilon,\delta})} \leq \frac{CM^3 \varepsilon^{7/6}}{\alpha^2} \frac{C}{\varepsilon^{\alpha}} = \frac{CM^3 \varepsilon^{1/6}}{\alpha^3}.$$  

**The second term** $J_2^\varepsilon$. From the energy bound (1.5), it follows that

$$C \geq \int_0^L \frac{(1 - v_\varepsilon)^2}{\varepsilon} dx \geq \int_{\{v_\varepsilon \leq 1/2\}} \frac{(1 - v_\varepsilon)^2}{\varepsilon} dx \geq \int_{\{v_\varepsilon \leq 1/2\}} \frac{1}{4\varepsilon} dx = \frac{1}{4\varepsilon} |\{v_\varepsilon \leq 1/2\}|,$$  

yielding the estimate

$$(4.8) \quad |\{M \sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}| \leq \frac{1}{4\varepsilon} |\{v_\varepsilon \leq 1/2\}| \leq C\varepsilon.$$  

On $\{M \sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}$, we find that

$$(4.9) \quad |u_\varepsilon'| = \frac{|F_\varepsilon|}{\eta_\varepsilon + v_\varepsilon^2(x)} \leq \frac{|F_\varepsilon|}{v_\varepsilon^2} \leq \frac{C}{M^2 \varepsilon}.$$  

$$\text{This gives the desired result.}$$
Inserting inequalities (4.8), (4.9) into the expression for the second term produces the following uniform upper bound:

\[(4.10) \quad J_1^\varepsilon \leq \int_{\{M\sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}} \left| u_\varepsilon' \right| \leq \frac{C}{M^2\varepsilon} C\varepsilon = \frac{C}{M^2}.\]

**The third term** $J_3^\varepsilon$. It is easy to see that the third term is bounded by

\[(4.11) \quad J_3^\varepsilon \leq \int_{I_\varepsilon \cap \{v_\varepsilon \geq 1/2\}} \left| u_\varepsilon' \right| \leq \delta C.\]

Coalescing (4.7), (4.11) and (4.10) and letting $\varepsilon$ tend to 0 finally leads to

\[
\liminf_{\varepsilon \to 0} \int_{x_0}^{x_0+\delta} \left| u_\varepsilon' \right| \leq \liminf_{\delta \to 0} \left( \frac{C M^3 \varepsilon^{1/2}}{\alpha^3} + \frac{C}{M^2} + \delta C \right) = \frac{C}{M^2} + \delta C.
\]

Letting $M$ tend to $\infty$ and $\delta$ tend to 0, we obtain (4.6).

We can now complete our proof. Recall from Lemma 3.2 that $u_\varepsilon \to u$ in $L^4((0,L))$. Thus, by lower semicontinuity, one has

\[
0 = \liminf_{\delta \to 0} \liminf_{\varepsilon \to 0} \int_{x_0}^{x_0+\delta} \left| u_\varepsilon' \right| \geq \liminf_{\delta \to 0} \int_{x_0}^{x_0+\delta} |Du|.
\]

This is, however, a contradiction with the assumption that $u$ is discontinuous at $x_0$. Therefore, we must have $F(x_0) = 0$ as desired.

It remains to prove (4.3). We will use the lower bound on $v_\varepsilon$ in (4.4). Then, for $\varepsilon$ sufficiently small, we have on $K$

\[(4.12) \quad \varepsilon v_\varepsilon'' = \frac{v_\varepsilon F_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{\varepsilon} \geq \frac{C\alpha^2}{v_\varepsilon^3} - \frac{1}{\varepsilon}.\]

Thus $v_\varepsilon''(x) \geq 0$ if $v_\varepsilon(x) \leq C\varepsilon^{1/3}$. Note that for $\varepsilon$ small, $M\sqrt{\varepsilon} \leq C\varepsilon^{1/3}$. By (4.12), $v_\varepsilon$ is convex in each connected component $D_\varepsilon = (a_i, b_i)$ of $K_\varepsilon,\delta$. So, once $v_\varepsilon$ goes above $M\sqrt{\varepsilon}$, it cannot go back down below that value without reaching $C\varepsilon^{1/3}$ first. Thus, the number of such components is certainly no greater than the number of connected component $E_\varepsilon$ of $E_\varepsilon := \{M\sqrt{\varepsilon} \leq v_\varepsilon \leq C\varepsilon^{1/3}\}$. To complete the proof of (4.3), it suffices to show that the number of connected components of $E_\varepsilon$ is bounded by $C\varepsilon^{-1/3}$.

Denote by $s_\varepsilon^i$ the length of $E_\varepsilon^i$. On each $E_\varepsilon^i$, there exist $c_i, d_i \in E_\varepsilon^i$ such that $c_i < d_i$, $v_\varepsilon(c_i) = M\sqrt{\varepsilon}$, and $v_\varepsilon(d_i) = C\varepsilon^{1/3}$. Then, application of the mean value theorem yields, in view of the gradient bound on $v_\varepsilon$ in Lemma 3.4, that, for some $z_i \in (c_i, d_i)$,

\[
v_\varepsilon'(z_i) = \frac{C\varepsilon^{1/3} - M\sqrt{\varepsilon}}{d_i - c_i} \leq \frac{C}{\varepsilon}.
\]

Therefore,

\[(4.13) \quad s_\varepsilon^i \geq d_i - c_i \geq C\varepsilon(\varepsilon^{1/3} - \varepsilon^{1/2}).\]
Next, observe that for $\varepsilon$ sufficiently small, $E_\varepsilon \subset \{v_\varepsilon \leq 1/2\}$. Recalling (4.8), we obtain
\begin{equation}
\sum_i s_i^\varepsilon \leq |\{v_\varepsilon \leq 1/2\}| \leq C\varepsilon.
\end{equation}
Combining (4.13) and (4.14) yields the desired bound on the number of connected components of $E_\varepsilon$. □

**Proof of Lemma 4.2.** We will use the same notations as in Lemma 4.1. Because $F$ is continuous, there is $\gamma > 0$ such that $F$ does not vanish on $(x_0 - \gamma, x_0 + \gamma)$. From Lemma 4.1, we deduce that $u$ is continuous on $(x_0 - \gamma, x_0 + \gamma)$. Because $u_\varepsilon(x) \to u(x)$ a.e. $x \in (0, L)$, we can choose $\delta_i \to 0$ such that $u_\varepsilon(x_0 + \delta_i) \to u(x_0 + \delta_i)$ and $u_\varepsilon(x_0 - \delta_i) \to u(x_0 - \delta_i)$. Denote $I_i = (x_0 - \delta_i, x_0 + \delta_i)$. As above, assuming $|F(x_0)| = \alpha > 0$, we have
\begin{equation}
\int_{x_0 + \delta_i}^{x_0 - \delta_i} u_\varepsilon' = \int_{x_0 - \delta_i}^{x_0 + \delta_i} u_\varepsilon' = \int_{K_{\varepsilon, \delta_i}} u_\varepsilon' + \int_{I_i \cap \{M \sqrt{\varepsilon} \leq v \leq 1/2\}} u_\varepsilon' + \int_{I_i \cap \{v \geq 1/2\}} u_\varepsilon' = O(\frac{\alpha^3}{\alpha^3}) + O(\frac{C}{M^3}) + \int_{I_i} u_\varepsilon' \chi_{\{v \geq 1/2\}}.
\end{equation}
Because $u_\varepsilon'(x) \to F(x)$ a.e $x \in (0, L)$ and $\chi_{\{v \geq 1/2\}}(x) \to \chi_{(0,L)}(x)$ a.e $x \in (0, L)$, it follows that $w_\varepsilon(x) := u_\varepsilon'(x) \chi_{\{v \geq 1/2\}}(x) \to F(x) \chi_{(0,L)}(x)$ a.e. $x \in (0, L)$. On the other hand, for all $x \in (0, L)$,
\begin{equation}
|w_\varepsilon(x)| = \frac{|F_\varepsilon(x)|}{\eta_\varepsilon + \eta_\varepsilon'} \chi_{\{v \geq 1/2\}}(x) \leq 4 |F_\varepsilon(x)| \leq C.
\end{equation}
Hence, by Lebesgue’s dominated convergence theorem,
\begin{equation}
\int_{I_i} u_\varepsilon' \chi_{\{v \geq 1/2\}} = \int_{I_i} w_\varepsilon dx \to \int_{I_i} F \chi_{(0,L)} dx.
\end{equation}
We let $\varepsilon$ tend to 0 and $M$ tend to $\infty$ and obtain
\begin{equation}
u(x_0 + \delta_i) - u(x_0 - \delta_i) = \int_{x_0 - \delta_i}^{x_0 + \delta_i} F
\end{equation}
and the result follows. □

5 Proof of Theorem 2.3

In this section, we prove Theorem 2.3. Its proof is divided into several subsections for the clarity of presentation. In subsection 5.1, we prove the first part of Theorem 2.3. The proof of the second part will follow from subsections 5.2 and 5.4.
5.1 Convergence of the Dirichlet energy

We first prove that the convergence result in Theorem 2.1 implies the nonconcentration of the energy density \((\eta + v^2)(u'_\varepsilon)^2\) in the limit:

**Lemma 5.1** The limit measure of \((\eta + v^2)(u'_\varepsilon)^2\) is \(((u')^2)\) where \(u'\) is the approximate gradient of \(u\).

**Proof.** Let \(\varphi \in C^\infty(0, L).\) Consider

\[
I_\varepsilon(\varphi) = \int_0^L (\eta + v^2)(u'_\varepsilon)^2 \varphi \equiv \int_0^L F_\varepsilon u'_\varepsilon \varphi dx.
\]

Integrating by parts and using the zero boundary conditions on \(F_\varepsilon\), we get

\[
I_\varepsilon(\varphi) = -\int_0^L (F_\varepsilon \varphi)' u_\varepsilon = -\int_0^L (F_\varepsilon' \varphi + F_\varepsilon \varphi') u_\varepsilon dx = -\int_0^L \left(\lambda(u_\varepsilon - g_\varepsilon) \varphi + F_\varepsilon \varphi'\right) u_\varepsilon.
\]

Now, letting \(\varepsilon \to 0\), we have

\[
\lim_{\varepsilon \to 0} I_\varepsilon(\varphi) = -\int_0^L \left(\lambda(u - g) \varphi + F \varphi'\right) u.
\]

Now, using Theorem 2.1, we will show that the right hand side, \(I(\varphi)\), of the above equation is \(\int_0^L ((u')^2) \varphi dx\). Indeed, we decompose

\[
I(\varphi) = -\int_0^L \lambda(u - g) u(x) \varphi(x) dx + \sum_{i=0}^k I_i,
\]

\[
\equiv -\int_0^L \lambda(u - g) u(x) \varphi(x) dx + \sum_{i=0}^k \int_{X_i}^{X_{i+1}} -F u \varphi' dx.
\]

For each \(i \in \{0, 1, \cdots, k\}\), we use integration by parts again, recalling that \(u'' = \lambda(u - g)\), \(u' = F\) on \((X_i, X_{i+1})\), and \(u'(X_i) = u'(X_{i+1}) = F(X_i) = F(X_{i+1}) = 0\). Therefore,

\[
I_i = \int_{X_i}^{X_{i+1}} (F u)' \varphi dx = \int_{X_i}^{X_{i+1}} (F' u + Fu') \varphi dx = \int_{X_i}^{X_{i+1}} \left(\lambda(u - g) u + (u')^2\right) \varphi dx.
\]

Thus,

\[
I(\varphi) = \sum_{i=0}^k \int_{X_i}^{X_{i+1}} ((u')^2) \varphi = \int_0^L (u')^2 \varphi.
\]

Note that \((u')^2\) makes sense in each interval \((X_i, X_{i+1})\) and equals zero on end points \(X_i's\) by Theorem 2.1.

Now, we improve the above convergence result to that of weak \(L^1(\Omega)\). \(\square\)
Lemma 5.2 \((\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2\) converges weakly in \(L^1((0, L))\) to \((u')^2\).

**Proof.** For each positive integer \(k\), set \(E_k = \bigcup_{i=1}^{j}[x_i - \frac{1}{k}, x_i + \frac{1}{k}]\). Then
\[
|E_k| = \frac{2j}{k} \to 0, \text{ as } k \to \infty.
\]

From the convergence result in Lemma 5.1, we deduce that
\[(5.4) \sup_{\varepsilon} \int_{E_k} (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 dx \leq \int_{E_k} (u')^2 dx\]
for each \(k\) fixed (cf. Theorem 1, p. 54, [6]). On the other hand, Proposition 3.12 and Remark 4.3 show that
\[(5.5) (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 = F_\varepsilon^2 \eta_\varepsilon + v_\varepsilon^2 \Rightarrow F_\varepsilon^2 = (u')^2 \text{ on } [0, L] \setminus E_k.\]

Now, let \(\varphi \in L^\infty((0, L)), \|\varphi\|_{L^\infty((0, L))} \leq 1.\) Using (5.4) and (5.5), we have, for each fixed \(k\)
\[
\limsup_{\varepsilon} \left| \int_0^L (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - (u')^2 \right| \varphi dx \leq \limsup_{\varepsilon} \left| \int_{[0, L] \setminus E_k} (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - (u')^2 \right| \varphi dx \leq 2 \int_{E_k} (u')^2 dx.
\]

Now, letting \(k \to \infty\), we see that the right hand side of the above inequality goes to zero and the assertion follows. \(\Box\)

5.2 Nonconcentration of energy away from \(v\)-jumps

We next establish that there is no concentration of energy for the two terms \(\varepsilon(v'_\varepsilon(x))^2\) and \((1 - v_\varepsilon(x))^2/\varepsilon\) away from the limiting jump points \(x_1, \ldots, x_j\).

Lemma 5.3 For any compact subset \(K \subset [0, L] \setminus \cup_{k=1}^{j}\{x_k\}\), we have
\[
\int_K \left( \varepsilon(v'_\varepsilon(x))^2 + (1 - v_\varepsilon(x))^2/\varepsilon \right) dx \leq C_K \varepsilon^{1/4},
\]
where \(C_K\) may depend only on \(K\).

**Proof.** For a given compact subset \(K\) of \([0, L] \setminus \cup_{k=1}^{j}\{x_k\}\), set \(\delta := 1/4 \min_{k=1, \ldots, j} \text{dist}(x_k, K)\). Let \(U_\delta := \bigcup_{k=1}^{j}(x_k - \delta, x_k + \delta)\) and let us denote \(V_\delta = [0, L] \setminus U_\delta\). Consider \(A_\varepsilon := \{x \in [0, L] : v_\varepsilon(x) \leq 1 - \varepsilon^{1/4}\}\). Then, for \(\varepsilon\) sufficiently small, we have by Proposition 3.12
\[(5.6) V_\delta \cap A_\varepsilon = \emptyset.\]
Because $K \subset [0, L] \setminus U_\delta = V_\delta$, it suffices to prove that

\begin{equation}
\int_{V_\delta} \left( \varepsilon (v'_\varepsilon(x))^2 + \frac{(1 - v_\varepsilon(x))^2}{\varepsilon} \right) dx \leq C_K \varepsilon^{1/4}.
\end{equation}

Multiplying both sides of the first equation of (1.4) by $v_\varepsilon - 1$ and integrating over $V_\delta$, we get

\begin{equation}
\int_{V_\delta} -\varepsilon v''_\varepsilon(x)(v_\varepsilon(x) - 1) dx + \int_{V_\delta} v_\varepsilon(x)(u'_\varepsilon(x))^2(v_\varepsilon(x) - 1) dx + \int_{V_\delta} \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} dx = 0.
\end{equation}

Note that $V_\delta$ is a union of a finite $\varepsilon$-independent number $J$ $(\leq j + 1)$ of intervals on $[0, L]$: $V_\delta = \cup_{k=1,\ldots,J} [a^k, b^k]$. Now, integrating by parts the first term of (5.8), and rearranging, one obtains

\begin{equation}
\int_{V_\delta} \left( \varepsilon (v'_\varepsilon(x))^2 + \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} \right) dx = \sum_{k=1}^{J} \varepsilon \left( v'_\varepsilon(b^k)(v_\varepsilon(b^k) - 1) - v'_\varepsilon(a^k)(v_\varepsilon(a^k) - 1) \right) \\
+ \int_{V_\delta} (u'_\varepsilon(x))^2 v_\varepsilon(x)(1 - v_\varepsilon(x)) dx.
\end{equation}

By the definitions of $A_\varepsilon$ and $V_\delta$ and (5.6), we have $|1 - v_\varepsilon| \leq \varepsilon^{1/4}$ on $V_\delta$. Furthermore, from (3.6), we deduce that $u'_\varepsilon$ is bounded on $V_\delta$. Combining this fact with the gradient bound for $v_\varepsilon$ in Lemma 3.4 yields that the right hand side of (5.9) is bounded from above by $C_K \varepsilon^{1/4}$ for some constant $C_K$ which may depend only on $K$. Hence the desired result stated in (5.7) follows.

**Remark 5.4** The previous lemma shows that the measure limits of $\varepsilon (v'_\varepsilon(x))^2 dx$, and of $(v_\varepsilon(x) - 1)^2/\varepsilon dx$ are Dirac masses concentrated at $x_1, \ldots, x_j$.

**Remark 5.5** The above proof is very similar to that of Lemma 6.1 in [7]. If we use the positivity of the discrepancy measure $(1 - v_\varepsilon)^2 - \varepsilon (v'_\varepsilon(x))^2$ in Lemma 5.6, then, also from Proposition 3.12, we get a stronger estimate

$$
\int_K \left( \varepsilon (v'_\varepsilon(x))^2 + \frac{(1 - v_\varepsilon(x))^2}{\varepsilon} \right) dx \leq 2 \int_K (1 - v_\varepsilon(x))^2/\varepsilon dx \leq 2 |K| C_K \varepsilon.
$$

### 5.3 Positivity of the discrepancy measure

We now prove the positivity of the discrepancy measure $\frac{(1 - v_\varepsilon(x))^2}{\varepsilon} - \varepsilon (v'_\varepsilon(x))^2$ which is crucial in the proof of the equi-partition of energy in Lemma 5.7. Its positivity played a central role in establishing the equi-partition of energy in our previous paper [7].

**Lemma 5.6** For all $x \in [0, L]$, $D_\varepsilon(x) := \frac{(1 - v_\varepsilon(x))^2}{\varepsilon} - \varepsilon (v'_\varepsilon(x))^2 \geq 0$. 
**Proof.** If \( x \) is a critical point of \( v_\varepsilon \) then clearly \( D_\varepsilon(x) \geq 0 \). Suppose that \( x \) is not a critical point of \( v_\varepsilon \). Then \( x \) lies between two consecutive critical points \( x^0 \) and \( x^1 \) of \( v_\varepsilon \). Because there is no other critical point between \( x^0 \) and \( x^1 \), \( v_\varepsilon(x) \) must lie between the two values \( v_\varepsilon(x^0) \) and \( v_\varepsilon(x^1) \). We can assume that \( v_\varepsilon(x^0) \geq v_\varepsilon(x) \geq v_\varepsilon(x^1) \). Because

\[
(D_\varepsilon(x))' = 2v_\varepsilon'(x)(\frac{v_\varepsilon(x) - 1}{\varepsilon} - \varepsilon v_\varepsilon''(x)) = -2v_\varepsilon'(x)v_\varepsilon(x)(u_\varepsilon'(x))^2,
\]

we can integrate (3.3) from \( x^0 \) to \( x \) to obtain

\[
D_\varepsilon(x) = D_\varepsilon(x^0) + \int_{x^0}^x (D_\varepsilon(t))' dt = \frac{(1 - v_\varepsilon(x^0))^2}{\varepsilon} + \int_{x^0}^x -2(u_\varepsilon')^2 v_\varepsilon'.
\]

If \( x^0 < x \) then \( v_\varepsilon \) is decreasing in \([x^0, x]\). Thus \( v_\varepsilon'(t) \leq 0 \) for all \( t \in (x^0, x) \) and hence the positivity of the right hand side of the above equation. If \( x^0 > x \) then \( v_\varepsilon \) is increasing in \([x, x^0]\), and because

\[
D_\varepsilon(x) = \frac{(1 - v_\varepsilon(x^0))^2}{\varepsilon} + \int_x^{x^0} 2(u_\varepsilon')^2 v_\varepsilon',
\]

we also have the positivity of \( D_\varepsilon \).

\[
\square
\]

### 5.4 Equi-partition of energy

We are now ready to establish the equi-partition property of energy

**Lemma 5.7**

\[
\lim_{\varepsilon \to 0} \int_0^L |\varepsilon(v_\varepsilon'(x))^2 - (v_\varepsilon(x) - 1)^2/\varepsilon| \, dx = 0.
\]

**Proof.** Let \( K \) be a connected compact subset of \([0, L]\setminus\{x_1, \ldots, x_j\}\) with \(|K| > 0\). Let \( x_0^\varepsilon \) be a global maximum of \( v_\varepsilon \) over \( K \). Then, from Proposition 3.12, one has for \( \varepsilon \) small

\[
v_\varepsilon(x_0^\varepsilon) \geq 1 - \varepsilon^{1/4}.
\]

After extraction a subsequence of \( \varepsilon \)'s, \( x_0^\varepsilon \) converges to \( x_0 \in K \). Set \( \delta := 1/4 \min_{k=1,\ldots,j} \text{dist}(x_k, K) \). Then

\[
K_1 = [x_0 - \delta, x_0 + \delta] \subseteq [0, L]\setminus\{x_1, \ldots, x_j\}.
\]

Using the conservation law (3.3), we find that for all \( x \in (0, L) \)

\[
(D_\varepsilon(x))' = F_\varepsilon(x)u_\varepsilon'(x) - \lambda(u_\varepsilon(x) - g_\varepsilon(x))^2 + \frac{(1 - v_\varepsilon(x_0^\varepsilon))^2}{\varepsilon} - \varepsilon(v_\varepsilon'(x_0^\varepsilon))^2
\]

\[
+ \lambda(v_\varepsilon(x_0^\varepsilon) - g_\varepsilon(x_0^\varepsilon))^2 - F_\varepsilon(x_0^\varepsilon)u_\varepsilon'(x_0^\varepsilon) + \int_{x_0^\varepsilon}^{x} -\lambda g_\varepsilon'(t)|u_\varepsilon(t) - g_\varepsilon(t)| \, dt
\]
Observe that, by Lemma 3.9, we also have

\[ D_{\varepsilon}(x) = F_{\varepsilon}(x)u_{\varepsilon}(x) - \lambda(u_{\varepsilon}(x) - g_{\varepsilon}(x))^2 + d_{\varepsilon}(x) \]

where \( \|d_{\varepsilon}\|_{L^\infty((0,L))} \leq C \). It follows from Lemmata 5.1 and 3.2 that \( D_{\varepsilon}(x) \) converges in the weak sense of Radon measures to \( D(x) \). We are going to evaluate \( D(x) \) by passing to the limit the right hand side of (5.13) in the weak sense of Radon measures.

* The first term \( F_{\varepsilon}(x)u_{\varepsilon}(x) \), by Lemma 5.1, converges to \( (u')^2 \).
* The second term \(-\lambda(u_{\varepsilon}(x) - g_{\varepsilon}(x))^2\), by Lemma 3.2, converges to \(-\lambda(u(x) - g(x))^2\) in \( L^2(0,L) \).
* The third term \( (1 - \varepsilon \frac{v_{\varepsilon}(x_0^\varepsilon)}{x_0^\varepsilon})^2 \), by Proposition 3.12, converges to 0.
* The fifth term \( \lambda(u_{\varepsilon}(x_0^\varepsilon) - g_{\varepsilon}(x_0^\varepsilon))^2 \), by the local maximality of \( v_{\varepsilon} \) at \( x_0^\varepsilon \), converges to \( \lambda(u(x_0) - g(x_0))^2 \). Indeed, we write \( g_{\varepsilon}(x_0^\varepsilon) - g(x_0) = g_{\varepsilon}(x_0^\varepsilon) - g_{\varepsilon}(x_0) + g_{\varepsilon}(x_0) - g(x_0) \). The last term \( g_{\varepsilon}(x_0) - g(x_0) \), by the weak convergence \( g_{\varepsilon} \to g \) in \( W^{1,2}((0,L)) \), converges to 0. So does the first term, because it is bounded by \( \|g_{\varepsilon}'\|_{L^2([x_0^\varepsilon,x_0])}|x_0^\varepsilon - x_0|^{1/2} \leq C|x_0^\varepsilon - x_0|^{1/2} \). On the other hand, for the function \( u \), we also write \( u_{\varepsilon}(x_0^\varepsilon) - u(x_0) = u_{\varepsilon}(x_0^\varepsilon) - u_{\varepsilon}(x_0) + u_{\varepsilon}(x_0) - u(x_0) \). Note that, for \( \varepsilon \) sufficiently small, \([x_0^\varepsilon - \delta, x_0^\varepsilon + \delta] \subset K_1 \subset [0,L]\{x_1, \ldots , x_j\} \) and \( v_{\varepsilon}(x) \geq 1 - \varepsilon^{1/4} \) in \([x_0^\varepsilon - \delta, x_0^\varepsilon + \delta] \) by Proposition 3.12. Thus, from (3.6), we deduce that \( u_{\varepsilon} \) is bounded in \([x_0^\varepsilon - \delta, x_0^\varepsilon + \delta] \). Now the arguments are similar to those for \( g \).
* The sixth term \(-F_{\varepsilon}(x_0^\varepsilon)u_{\varepsilon}'(x_0^\varepsilon) \) converges to \(-(u'(x_0))^2\). Indeed, from (5.11), we get

\[-F_{\varepsilon}(x_0^\varepsilon)u_{\varepsilon}'(x_0^\varepsilon) = \frac{(F_{\varepsilon}(x_0^\varepsilon))^2}{u_{\varepsilon}(x_0^\varepsilon)} - (F(x_0))^2 = -(u'(x_0))^2,\]

by Remark 4.3.
* The final term \( \int_{x_0^\varepsilon}^x -\lambda g_{\varepsilon}'(t)[u_{\varepsilon}(t) - g_{\varepsilon}(t)]dt \) converges to

\[ \int_{x_0^\varepsilon}^x -2\lambda g'(t)(u(t) - g(t)) = \int_{x_0^\varepsilon}^x (\lambda[u - g]^2)' - 2 \int_{x_0^\varepsilon}^x \lambda u'(t)(u(t) - g(t)) = \lambda(u(x) - g(x))^2 - \lambda(u(x_0) - g(x_0))^2 - 2 \int_{x_0^\varepsilon}^x \lambda u'(t)(u(t) - g(t)) \]

Therefore, keeping in mind that the fourth term \(-\varepsilon(v_{\varepsilon}'(x_0^\varepsilon))^2\) of (5.13) is nonpositive, \( D_{\varepsilon} \) converges, in the sense of Radon measures, to

\[ D(x) \leq (u'(x))^2 - (u'(x_0))^2 - 2 \int_{x_0}^x \lambda u'(t)(u(t) - g(t)) = (u'(x))^2 - (u'(x_0))^2 - 2 \int_{x_0}^x u'(t)u''(t) = 0. \]

We indicate how to obtain rigorously the last line of the above equation. It is obvious in the absence of \( X_1 \)'s in the interval \([x_0, x]\). Now, it suffices to consider the case that \( x_0 < x \) and that there is only one, say \( X_1 \), in \([x_0, x]\). Then, \( u'' = \lambda(u - g) \) in \((x_0, X_1)\) and \((X_1, x)\).
Therefore,
\[
2 \int_{x_0}^{x} \lambda u'(t)(u(t) - g(t)) = 2 \int_{x_0}^{X_1} \lambda u'(t)(u(t) - g(t)) + 2 \int_{X_1}^{x} \lambda u'(t)(u(t) - g(t))
= 2 \int_{x_0}^{X_1} u'(t)u''(t) + 2 \int_{X_1}^{x} u'(t)u''(t)
= (u'(X_1))^2 - (u'(x_0))^2 + (u'(x))^2 - (u'(X_1))^2
= u'(x))^2 - (u'(x_0))^2.
\]

Because \(D(x) \leq 0\) and by the positivity of \(D_\epsilon\) from Lemma 5.6, we must have \(D(x) = 0\). This completes the proof of equi-partition of energy. \(\square\)

References


