On topological degree for potential operators of class \((S)_+\)

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Abstract

We extend some results of Amann about the topological degree for compact potential vector fields to potential operators of class \((S)_+\). Using these results, we obtain the existence of a third buckled state of a thin elastic shell in \(\mathbb{R}^3\) even in the case when the gradient of the potential energy of the shell might not be compact.

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1. Introduction

In [9] Rabinowitz established some results about the topological degree for compact potential vector fields and applied them to prove the existence of the third buckled state possessed by a thin elastic shell. However, the above results were obtained under the assumption that the functional whose gradient is a compact vector field was of class \(C^2\). Later, Amann [1] showed that the degree results of Rabinowitz hold for \(C^1\)-mappings.

In the present paper, we extend some results of [1] to a class of continuous potential operators of class \((S)_+\) and use them to relax conditions on the compactness of operators and the shallowness of shells in [9]. We also give an explicit example illustrating the relaxation of the compactness of operators (see (iii) of Lemma 3.3).

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The remainder of our paper consists of two sections. In Section 2, we improve results in [1]. Then we apply them to study the number of buckled states possessed by a thin elastic shell in the last section.

2. Topological degree for potential operators of class \((S)_+\)

In this section, we extend the results about the topological degree in [1] to potential operators of class \((S)_+\). Throughout our paper, \(H\) is a Hilbert space with the scalar product \(\langle \cdot, \cdot \rangle\) and its dual space \(H^*\) is always identified with \(H\). We denote by \(B(x_0, r)\) the set \(\{x \in H \mid \|x - x_0\| < r\}\) for any \(x_0\) in \(H\). Now, we recall the class \((S)_+\) introduced by Browder (see [2,3]).

**Definition 2.1.** Let \(A\) be a subset of \(H\) and \(h\) be a mapping of \(A\) into \(H\). We say:

(i) \(h\) is demicontinuous if the sequence \(\{h(x_n)\}\) converges weakly to \(h(x)\) in \(H\) for any sequence \(\{x_n\}\) converging strongly to \(x\) in \(H\).

(ii) \(h\) is of class \((S)_+\) if \(h\) is demicontinuous and has the following property:

Let \(\{x_n\}\) be a sequence in \(A\) such that \(\{x_n\}\) converges weakly to \(x\) in \(H\). Then \(\{x_n\}\) converges strongly to \(x\) in \(H\) if \(\limsup_{n \to \infty} \langle h(x_n), x_n - x \rangle \leq 0\).

Let \(D\) be a bounded open subset in \(H\) with boundary \(\partial D\) and closure \(\bar{D}\). Let \(h\) be a mapping of class \((S)_+\) on \(\bar{D}\) and \(p\) be in \(H \setminus h(\partial D)\). By Theorems 4 and 5 in [2], the topological degree of \(h\) on \(D\) at \(p\) is defined as a family of integers and is denoted by \(\text{deg}(h; D; p)\). In [10] Skrypnik showed that this topological degree is single valued (see also [3]). Combining the Galerkin approximation technique in the proof of Theorem 4 in [3] and results in [1] for continuous mappings on finite-dimensional spaces, we have the following result.

**Theorem 2.1.** Let \(U\) be an open subset of \(H\) and \(G\) be a real \(C^1\)-function on \(U\) such that its gradient \(\nabla G\) is of class \((S)_+\) on \(U\). Denote \(\nabla G\) by \(g\) and assume that there are \(x_0\) in \(H\) and real numbers \(\alpha, \beta\) and \(r > 0\) and

(i) \(V \equiv G^{-1}(\langle -\infty, \beta \rangle)\) is bounded and \(\bar{V} \subset U\),

(ii) \(G^{-1}(\langle -\infty, \alpha \rangle) \subset \bar{B}(x_0, r) \subset V\) and

(iii) \(g(x) \neq 0\) for every \(x\) in \(G^{-1}([\alpha, \beta])\).

Then \(G^{-1}(\langle -\infty, \beta \rangle)\) is bounded and \(\text{deg}(g, V, 0) = 1\).

**Proof.** First, since \(V \equiv G^{-1}(\langle -\infty, \beta \rangle)\) is bounded, \(V\) is contained in a ball \(B(0, R)\) for some positive number \(R\). If there is \(z\) in \(G^{-1}(\{\beta\}) \setminus B(0, 2R)\) then \(G\) has a local minimum at \(z\) since \(G(x) \geq \beta\) for all \(x \in U \setminus B(0, R)\). Thus \(g(z) = 0\), which contradicts (iii). Hence

\[ G^{-1}(\{\beta\}) \subset B(0, 2R). \]

This proves the first statement of the theorem.
By the continuity of $G$, $V$ is an open subset and
\[ \partial V \subset G^{-1}((-\infty, \beta]) \backslash G^{-1}((\infty, \beta)) = G^{-1}(\{\beta\}). \]

Therefore, $0 \not\in g(\partial V)$ and the topological degree $\text{deg}(g, V, 0)$ is well defined in the sense of topological degree for $(S)_+$ mappings. Denote by $\{X_\lambda\}_{\lambda \in A}$ the family of all finite-dimensional subspaces of $H$ such that $V_\lambda \equiv V \cap X_\lambda$ is nonempty. We define
\[ \lambda \succ \mu \iff X_\mu \subset X_\lambda \quad \forall \lambda, \mu \in A. \]

For each $\lambda$ in $A$, let $\pi_\lambda$ be the orthogonal projection of $H$ onto $X_\lambda$. For $x \in V_\lambda$, put
\[ G_\lambda(x) = G(x), \]
\[ g_\lambda(x) = \pi_\lambda(g(x)). \]

Then, it is easy to see that, for any $y \in V_\lambda$ and $x \in X_\lambda$,
\[ \langle \nabla G_\lambda(y), x \rangle = \langle \nabla G(y), x \rangle = \langle \pi_\lambda(\nabla G(y)), x \rangle = \langle \pi_\lambda(g(y)), x \rangle = \langle g_\lambda(y), x \rangle, \]
i.e.,
\[ \nabla G_\lambda(y) = g_\lambda(y) \quad \forall y \in V_\lambda. \]

We shall show that there exists $\lambda_0$ in $A$ such that
\[ g_\lambda(x) \neq 0 \quad \forall x \in G_\lambda^{-1}([x, \beta]), \quad \lambda \succ \lambda_0. \quad (2.1) \]

Suppose by contradiction that, for each $\lambda$, there exist a $\mu \succ \lambda$ and an $x_\mu$ in $G_\mu^{-1}([x, \beta])$ such that $g_\mu(x_\mu) = 0$. Thus,
\[ \langle g(x_\mu), x_\mu \rangle = \langle g_\mu(x_\mu), x_\mu \rangle = 0, \quad \langle g(x_\mu), v \rangle = \langle g_\mu(x_\mu), v \rangle = 0 \quad \forall v \in X_\mu. \quad (2.2) \]

We define
\[ W_\lambda = \{ u \in G^{-1}([x, \beta]) \mid \langle g(u), u \rangle = 0 \text{ and } \langle g(u), v \rangle = 0 \quad \forall v \in X_\mu \} \quad \forall \lambda \in A. \]

Note that $W_\lambda \subset W_\beta$ if $x \succ \beta$. Therefore, by $(2.2)$, $\{W_\lambda\}_{\lambda \in A}$ is a family of nonempty sets and has the finite intersection property. Denote by $K_\lambda$ the closure of $W_\lambda$ in the weak topology of $H$ for any $\lambda$ in $A$. It is clear that $\{K_\lambda\}_{\lambda \in A}$ is a family having the finite intersection property. Since $W_\lambda$ is contained in the bounded set $G^{-1}([x, \beta])$, $K_\lambda$ is a weakly compact subset of $H$ for any $\lambda$ in $A$. Thus
\[ K \equiv \bigcap_{\lambda \in A} K_\lambda \neq \emptyset. \]

Let $x$ be in $K$ and $v$ be in $H$. Choose $\lambda$ in $A$ such that $x$ and $v$ are in $X_\lambda$. Since $x$ lies in $K_\lambda$ and $H$ is a Hilbert space, there exists a sequence $\{x_j\}$ in $W_\lambda$ which converges weakly to $x$ in $H$. By the definition of $W_\lambda$, $x_j$ belongs to $G^{-1}([x, \beta])$ and
\[ \langle g(x_j), x_j \rangle = 0, \quad \langle g(x_j), x \rangle = 0, \quad \langle g(x_j), w \rangle = 0 \quad \forall j \in \mathbb{N}, \ w \in X_\lambda. \]

It implies that
\[ \lim_{j \to \infty} \langle g(x_j), x_j - x \rangle = 0. \]
Since \( g \) is of class \((S)_+\) on \( G^{-1}([x, \beta])\), \( \{x_j\} \) converges strongly to \( x \) and hence \( x \) lies in \( G^{-1}([x, \beta])\). Using the continuity of \( g \), we obtain
\[
\langle g(x), v \rangle = \lim_{j \to \infty} \langle g(x_j), v \rangle = 0.
\]
Since \( v \) is an arbitrary element of \( H \), it follows that \( g(x) = 0 \), which is a contradiction to the assumption on \( g \). Therefore, we obtain (2.1).

Now, let \( \lambda \) be in \( A \) such that \( \lambda \succ \lambda_0 \) and \( x_0 \in X_{\lambda} \). Then
\[
G_{\lambda}^{-1}((-\infty, x]) = G^{-1}((-\infty, x]) \cap X_{\lambda} \subset \overline{B}(x_0, r) \cap X_{\lambda}
\]
\[
\equiv \overline{B}(x_0, r) \subset V_{\lambda} = G_{\lambda}^{-1}((-\infty, \beta))
\]
and
\[
\overline{V}_{\lambda} \subset \overline{V} \cap X_{\lambda} \subset U \cap X_{\lambda} \equiv U_{\lambda}.
\]
Therefore, the mapping \( G_{\lambda} \) satisfies all conditions of the Theorem in [1] for the finite-dimensional Hilbert space \( X \). Thus, we obtain
\[
\deg(g_{\lambda}, V_{\lambda}, 0) = \deg(\nabla G_{\lambda}, V_{\lambda}, 0) = 1 \quad \forall \lambda \succ \lambda_0.
\]
Since \( g \) is of class \((S)_+\), by definition (see [3], p. 75), the theorem follows.

Theorem 2.1 was proved in [1] for compact potential operators. A result similar to Theorem 2.1 was proved by Klimov [6] (see also [11]).

We have the following corollaries of Theorem 2.1.

**Corollary 2.1.** Let \( G \) be a real \( C^1 \)-function on \( H \) such that its gradient \( g \equiv \nabla G \) is of class \((S)_+\) on \( H \). Suppose that \( G(x) \to \infty \) when \( \|x\| \to \infty \) and \( G \) maps bounded sets into bounded sets. Moreover, suppose that \( g(x) \neq 0 \) for all \( \|x\| \geq r_0 \) and some \( r_0 > 0 \). Then there is a number \( r_1 \geq r_0 \) such that
\[
\deg(g, B(0, r), 0) = 1 \quad \text{for all } r \geq r_1.
\]

The proof of this corollary is similar to that of Corollary 1 in [1] and is omitted.

**Corollary 2.2.** Let \( W \) be an open convex subset of \( H \) and \( G \) be a real \( C^1 \)-function on \( W \) such that its gradient \( g \equiv \nabla G \) is of class \((S)_+\) on \( W \). Let \( w_0 \) be an isolated critical point of \( G \) in \( W \). Assume that \( G \) has a local minimum at \( w_0 \). Then
\[
i(g, w_0) = \lim_{s \to 0} \deg(g, B(w_0, s), 0) = 1.
\]

**Proof.** First, we prove that \( G \) is weakly sequentially lower semicontinuous on \( W \). Suppose by contradiction that there exists a sequence \( \{w_k\} \) in \( W \) converging weakly to \( w \) in \( W \) such that \( G(w_k) > \liminf_{k \to \infty} G(w_k) \). Choose a subsequence \( \{w_{km}\} \) of \( \{w_k\} \) such that
\[
\lim_{m \to \infty} G(w_{km}) = \liminf_{k \to \infty} G(w_k).
\]
Then
\[ \lim_{m \to \infty} (G(w_{k_n}) - G(w)) < 0. \]

On the other hand, for each \( m \), there exists a \( t_m \in (0, 1) \) such that
\[ G(w_{k_n}) - G(w) = \langle g(w + t_m(w_{k_n} - w)), w_{k_n} - w \rangle. \]

Hence
\[ \limsup_{m \to \infty} t_m[G(w_{k_n}) - G(w)] = \limsup_{m \to \infty} \langle g(w + t_m(w_{k_n} - w)), w + t_m(w_{k_n} - w) - w \rangle \leq 0. \]

Since \( g \) is of class \( (S)_+ \) on \( W \) and the sequence \( \{w + t_m(w_{k_n} - w)\} \) converges weakly to \( w \), the above inequality shows that \( \{w + t_m(w_{k_n} - w)\} \) converges strongly to \( w \). Since \( g \) is continuous, we have
\[ \lim_{m \to \infty} [G(w_{k_n}) - G(w)] = \lim_{m \to \infty} \langle g(w + t_m(w_{k_n} - w)), w_{k_n} - w \rangle \]
\[ = \lim_{m \to \infty} \langle g(w), w_{k_n} - w \rangle = 0, \]
which contradicts the inequality \( \lim_{m \to \infty} (G(w_{k_n}) - G(w)) < 0 \). Hence, \( G \) is weakly sequentially lower semicontinuous on \( W \).

Without loss of generality, we can assume that \( w_0 = 0 \), \( G(0) = 0 \) and there is a positive real number \( r_0 \) such that \( W = B(0, r_0) \) and \( 0 \) is the unique critical point of \( G \) in \( W \). Let \( r_1 \) and \( r_2 \) be two positive real numbers such that \( r_1 < r_2 < r_0 \). We claim that
\[ \beta \equiv \inf G(\bar{B}(0, r_2) \setminus B(0, r_1)) > 0. \] (2.3)

Suppose by contradiction that there is a sequence \( \{w_k\} \) in \( \bar{B}(0, r_2) \setminus B(0, r_1) \) such that \( \{G(w_k)\} \) converges to \( 0 \). We can (and shall) assume that \( \{w_k\} \) converges weakly to \( w \) in \( \bar{B}(0, r_2) \). Since \( G \) is weakly sequentially lower semicontinuous on \( W \), we have
\[ 0 \leq G(w) = \liminf_{k \to \infty} G(w_k) = 0, \] (2.4)
which implies \( w = 0 \). On the other hand, for any integer \( n \), there is an \( s_n \) in \((0, 1)\) such that
\[ G(w_n) - G \left( \frac{w_n}{2} \right) = \langle g \left( \frac{w_n}{2} + s_n \frac{w_n}{2}, \frac{w_n}{2} \right), \frac{w_n}{2} \rangle. \]

Since \( G(w_n/2) \geq 0 \) and \( \lim_{n \to \infty} G(w_n) = 0 \),
\[ \limsup_{n \to \infty} \langle g \left( \frac{w_n}{2} + s_n \frac{w_n}{2}, (1 + s_n) \frac{w_n}{2} \right), (1 + s_n) \frac{w_n}{2} \rangle \leq 0. \]

As \( \{(1 + s_n)w_n/2 \} \) converges weakly to \( 0 \) and \( g \) is of class \( (S)_+ \) on \( W \), it follows that \( \lim_{n \to \infty} w_n = 0 \), which contradicts the fact that \( \|w_n\| \geq r_1 > 0 \) for all \( n \). Therefore, (2.3) holds.

Choose \( r > 0 \) such that \( \bar{B}(0, r) \subset G^{-1}((-\infty, \beta)) \) and put
\[ \alpha = (1/2) \inf G(\bar{B}(0, r_2) \setminus B(0, r)). \]

By definitions and (2.3), we see that \( r < r_1 \), \( 0 < \alpha < \beta \), \( V \equiv G^{-1}((-\infty, \beta)) \subset B(0, r_1) \), \( \hat{V} \subset B(0, r_2) \), and \( G^{-1}((-\infty, \alpha]) \subset \bar{B}(0, r) \).
Now, applying Theorem 2.1 for \( U = B(0,r_2) \), we obtain
\[
i(g,0) = \text{deg}(g,V,0) = 1. \quad \square
\]

We have the following theorem, which will be useful for the application in Section 3.

**Theorem 2.2.** Let \( g \) be a mapping of class \((S)_+\) on \( H \) and \( x_0 \) be an isolated zero of \( g \) in \( H \). Suppose that \( g \) is Fréchet differentiable at \( x_0 \), \( \nabla g(x_0) \) is of class \((S)_+\) and \( \nabla g(x_0)x \neq 0 \) for all nonzero vector \( x \) in \( H \). Then
\[
i(g,x_0) \equiv \lim_{s \to 0} \text{deg}(g,B(x_0,s),0) \in \{-1,1\}.
\]

**Proof.** First, we prove that there exists a positive real number \( a \) such that
\[
\|\nabla g(x_0)x\| \geq a\|x\| \quad \forall x \in H. \tag{2.5}
\]

Suppose by contradiction that there exist a sequence \( \{a_n\} \) in \( \mathbb{R} \) and a sequence \( \{x_n\} \) in \( H \setminus \{0\} \) such that \( \lim_{n \to \infty} a_n = 0 \) and
\[
\|\nabla g(x_0)x_n\| \leq a_n\|x_n\|. \tag{2.6}
\]

For any positive integer \( n \), put \( x^*_n = x_n/\|x_n\| \).

Then \( \|x^*_n\| = 1 \) and we can assume that the sequence \( \{x^*_n\} \) converges weakly in \( H \) to \( x^* \) with \( \|x^*\| \leq 1 \). On the other hand, from (2.6) we have
\[
|\langle \nabla g(x_0)x_n^*, x_n^* - x^* \rangle| \leq \left\| \nabla g(x_0) \frac{x_n}{\|x_n\|} \right\| \|x_n^* - x^*\| \leq 2a_n \to 0.
\]

Since \( \nabla g(x_0) \) is of class \((S)_+\), it follows that \( \{x_n^*\} \) converges strongly to \( x^* \). Hence \( \|x^*\| = 1 \), and \( \nabla g(x_0)x^* = 0 \), which is a contradiction to our assumption on \( \nabla g(x_0) \). Thus we have (2.5).

Since \( g \) is Fréchet differentiable at \( x_0 \), there is a mapping \( \varphi \) from a ball \( B(0,r) \) into \( H \) such that \( \lim_{h \to 0} \varphi(h) = 0 \) and
\[
g(x) = g(x_0) + \nabla g(x_0)(x - x_0) + \|x - x_0\|\varphi(x - x_0)
\]
\[
= \nabla g(x_0)(x - x_0) + \|x - x_0\|\varphi(x - x_0).
\]

For any \((t,x)\) in \([0,1] \times B(x_0,r)\), put
\[
\tilde{g}(x) = \nabla g(x_0)(x - x_0),
\]
\[
h(t,x) = tg(x) + (1 - t)\tilde{g}(x) = \nabla g(x_0)(x - x_0) + t\|x - x_0\|\varphi(x - x_0).
\]

Then \( \tilde{g} \) is of class \((S)_+\) as \( \nabla g(x_0) \) is linear and of class \((S)_+\). By (2.5), \( h(t,x) \neq 0 \) for all \((t,x)\) in \([0,1] \times \partial B(x_0,\rho)\) when \( \rho \) is a sufficiently small positive number. Since \( x_0 \)
is an isolated zero of the continuous mappings $g$ and $\tilde{g}$, by the homotopy invariance of the degree, we obtain

$$i(g, x_0) = \text{deg}(g, B(x_0, \rho), 0) = \text{deg}(\tilde{g}, B(x_0, \rho), 0).$$  \hfill (2.7)

Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the family of all finite-dimensional subspaces of $H$ as in the proof of Theorem 2.1 such that $x_0 \in X_\lambda$. Denote by $B_\lambda(x_0, \rho)$ the set $B(x_0, \rho) \cap X_\lambda$ and $\tilde{g}_\lambda$ the restriction of $\pi_\lambda \circ \tilde{g}$ to $B_\lambda(x_0, \rho)$. Using the fact that $\tilde{g}$ is of class $(S)_+$ and $0 \not\in \tilde{g}(\partial B(x_0, \rho))$, and arguing as in the proof of Theorem 2.1, we can find a $\lambda_0$ in $\Lambda$ such that $0 \not\in \tilde{g}_\lambda(\partial B_\lambda(x_0, \rho))$ and

$$\text{deg}(\tilde{g}_\lambda, B(x_0, \rho), 0) = \text{deg}(\tilde{g}_\lambda, B_\lambda(x_0, \rho), 0) \quad \forall \lambda \succ \lambda_0.$$  \hfill (2.8)

Since $\tilde{g}_\lambda(x) \neq 0$ for all $x \in \partial B_\lambda(x_0, \rho)$, it is obvious that $\tilde{g}_\lambda(x) \neq 0$ for all $x \neq x_0$. Furthermore, we have

$$\tilde{g}_\lambda(x_0 + h) - \tilde{g}_\lambda(x_0) = \pi_\lambda(\tilde{g}(x_0 + h) - \tilde{g}(x_0)) = \pi_\lambda(\nabla g(x_0)(h)) \quad \forall h \in X_\lambda.$$

It implies that

$$\nabla \tilde{g}_\lambda(x_0)(h) = \pi_\lambda(\nabla g(x_0)(h)) \quad \forall h \in X_\lambda.$$

Since $\tilde{g}_\lambda(x) = \pi_\lambda(\tilde{g}(x)) = \pi_\lambda[\nabla g(x_0)(x - x_0)] \neq 0$ for all $x \in X_\lambda \setminus \{x_0\}$, it follows that

$$\nabla \tilde{g}_\lambda(x_0)(h) \neq 0 \quad \forall h \in X_\lambda \setminus \{0\}.$$

Thus, by the Leray–Schauder index formula [7, Theorem 4.7, pp. 136–137]

$$\text{deg}(\tilde{g}_\lambda, B_\lambda(x_0, \rho), 0) \in \{-1, 1\}. \hfill (2.9)$$

By (2.7)–(2.9), we obtain the theorem. $\square$

3. An application to shell buckling

Let $S$ be a thin elastic shell in $\mathbb{R}^3$ which has a single-valued projection $\Omega$ on the $xy$ plane. $S$ is assumed to be in equilibrium under the effects of edge stressing and of an applied force. In [9] Rabinowitz used the results of the topological degree for compact vector fields to prove the existence of three buckled states of $S$. In this section, we apply our results in Section 2 to obtain a similar result and show that we can relax some conditions on the compactness of mappings and the shallowness of the shells.

First, we need some definitions. Let $H$ be the usual Sobolev space $W^{2,2}_0(\Omega)$ with the scalar product and norm

$$\langle u, v \rangle_H = \int_\Omega \Delta u \Delta v \, dx \, dy,$$

$$\|u\|_H = \left\{ \int_\Omega |\Delta u|^2 \, dx \, dy \right\}^{1/2} \quad \forall u, v \in H,$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. 

In this section, let \((k_{ij})_{1 \leq i,j \leq 2}\) be a symmetric matrix whose elements are given measurable functions on \(\Omega\). We put
\[
[u,v] = \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y},
\]
\[
N(u) = \frac{\partial}{\partial x} \left( k_{11} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} \left( k_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( k_{21} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_{22} \frac{\partial u}{\partial y} \right) \quad \forall u,v \in H.
\]
Using the Cauchy–Schwarz inequality and Corollary 9.10 in [5], we obtain
\[
\int_{\Omega} ||[u,v]| | \, dx \, dy \leq \|u\|_H \|v\|_H \quad \forall u,v \in H.
\] (3.1)
Integrating by parts, we see that
\[
\int_{\Omega} [w, \psi] \phi \, dx \, dy = \int_{\Omega} [w, \phi] \psi \, dx \, dy,
\] (3.2)
\[
\int_{\Omega} N(w) \phi \, dx \, dy = \int_{\Omega} N(\phi) w \, dx \, dy \quad \forall w, \psi, \phi \in H.
\] (3.3)
We assume that there is a positive real number \(C\) such that
\[
\left| \int_{\Omega} N(w) \phi \, dx \, dy \right| \leq C \|\omega\|_H \|\phi\|_H \quad \forall w, \phi \in H.
\] (3.4)
Hereafter, by \(C\) we mean a general constant.
Let \(w, f, Z(x,y), \Psi_1, \Psi_2\) and \(\lambda\) be, respectively, the deflection of the shell from its initial state, the Airy stress function, the applied force, the edge stresses, and a measure of the magnitude of \(\Psi_1\) and \(\Psi_2\). After appropriate dimensional scalings have been made, \(w\) and \(f\) satisfy the von Kármán equations:
\[
\Delta^2 f = -\frac{1}{\tau} [w,w] - N(w),
\] (3.5)
\[
\Delta^2 w = [f,w] + N(f) + Z,
\] (3.6)
\[
w = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0; \quad \frac{\partial^2 f}{\partial n \partial \tau} = \lambda \Psi_1, \quad \frac{\partial^2 f}{\partial \tau \partial \tau} = \lambda \Psi_2 \quad \text{on} \ \partial \Omega,
\] (3.7)
where \(n(x,y)\) and \(\tau(x,y)\) are the normal and tangent to \(\partial \Omega\), respectively.
Assume that \(\Omega, \Psi_1\) and \(\Psi_2\) are sufficiently smooth so that there is a solution \(F_0\) in \(C^4(\Omega)\) to the following boundary problem:
\[
\begin{cases}
\Delta^2 F_0 = 0 \quad \text{in} \ \Omega, \\
\frac{\partial^2 F_0}{\partial n \partial \tau} = \Psi_1, \quad \frac{\partial^2 F_0}{\partial \tau \partial \tau} = \Psi_2 \quad \text{on} \ \partial \Omega.
\end{cases}
\]
We further require that \(Z = -\lambda N(F_0)\), i.e., the external force just balances the edge stresses so that \(w = 0, \ f = \lambda F_0\) is an equilibrium state of the shell and satisfies
(3.5)–(3.7). Putting $F = f - \lambda F_0$, we can reduce the systems (3.5)–(3.7) to the following system:

$$
\Delta^2 F = -\frac{1}{2} [w, w] - N(w),
$$

(3.8)

$$
\Delta^2 w = [F + \lambda F_0, w] + N(F),
$$

(3.9)

$$
w = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \quad \text{on } \partial \Omega.
$$

(3.10)

Put

$$
l_w(\varphi) = -\int_{\Omega} \left\{ \frac{1}{2} [w, w] + N(w) \right\} \varphi \, dx \, dy \quad \forall w, \varphi \in H.
$$

(3.11)

By (3.1), (3.4) and the Sobolev embedding theorem, we have

$$
|l_w(\varphi)| \leq \frac{1}{2} \|w\|^2_H \|\varphi\|_{C(\bar{\Omega})} + C \|w\|_H \|\varphi\|_H
\leq C(\|w\|^2_H + \|w\|_H) \|\varphi\|_H \quad \forall w, \varphi \in H.
$$

(3.12)

Thus, by the Riesz representation theorem, for each $w$ in $H$, there exists a unique $F(w)$ in $H$ such that

$$
l_w(\varphi) = \langle F(w), \varphi \rangle_H \quad \forall \varphi \in H.
$$

(3.13)

Choosing $F$ as $F(w)$, by (3.13), we see that system (3.9) becomes

$$
\Delta^2 w - [F(w) + \lambda F_0, w] - N(F(w)) = 0,
$$

or, in the weak form

$$
\int_{\Omega} \{ \Delta w \Delta \varphi - [F(w) + \lambda F_0, w] \varphi - N(F(w)) \varphi \} \, dx \, dy = 0 \quad \forall \varphi \in H.
$$

(3.14)

In order to solve (3.14), we consider the following potential energy of possible shell states:

$$
G(\lambda, w) = \frac{1}{2} \int_{\Omega} (|\Delta w|^2 + |\Delta F(w)|^2 - \lambda [F_0, w] w) \, dx \, dy.
$$

The following lemmas contain properties of $G(\lambda, \cdot)$ and its derivatives. First, let us prove

**Lemma 3.1.** Let $A, B$ be two weakly continuous operators of $H$ into $H$, that is, for any sequence $\{w_n\}$ converging weakly to $w$ in $H$, the sequences $\{A(w_n)\}$ and $\{B(w_n)\}$ converge weakly to $A(w)$ and $B(w)$, respectively. Define an operator $T$ on $H$ as follows:

$$
\langle T(w), \varphi \rangle_H = \int_{\Omega} [A(w), B(w)] \varphi \, dx \, dy \quad \forall \varphi \in H.
$$

Then $T$ is a compact operator on $H$. 
Lemma 3.2. Fix a \( w \) in \( H \). By (3.1) and the Sobolev embedding theorem, we have

\[
\left| \int_{\Omega} [A(w), B(w)] \varphi \, dx \, dy \right| \leq \|A(w)\|_H \|B(w)\|_H \|\varphi\|_{C(\tilde{\Omega})} \\
\leq C \|A(w)\|_H \|B(w)\|_H \|\varphi\|_H \quad \forall \varphi \in H.
\]

Therefore, \( T \) is well defined by the Riesz representation theorem. Let \( \{w_n\} \) be a sequence in \( H \) converging weakly to \( w \) in \( H \). Then the sequences \( \{A(w_n)\} \) and \( \{B(w_n)\} \) converge weakly to \( A(w) \) and \( B(w) \) in \( H \), respectively. By the Sobolev embedding theorem, the sequences \( \{A(w_n)\} \) and \( \{B(w_n)\} \) converge strongly to \( A(w) \) and \( B(w) \) in \( C(\tilde{\Omega}) \), respectively. Moreover, by (3.1), we have

\[
\|T(w_n) - T(w)\|_H^2 = \langle T(w_n) - T(w), T(w_n) - T(w) \rangle_H \\
= \int_{\Omega} \left( [A(w_n), B(w_n)] - [A(w), B(w)] \right) (T(w_n) - T(w)) \, dx \, dy \\
\leq \left| \int_{\Omega} [A(w_n) - A(w), B(w_n)](T(w_n) - T(w)) \, dx \, dy \right| \\
+ \left| \int_{\Omega} [A(w), B(w_n) - B(w)](T(w_n) - T(w)) \, dx \, dy \right| \\
= \left| \int_{\Omega} [T(w_n) - T(w), B(w_n)](A(w_n) - A(w)) \, dx \, dy \right| \\
+ \left| \int_{\Omega} [A(w), T(w_n) - T(w)](B(w_n) - B(w)) \, dx \, dy \right| \\
\leq \|T(w_n) - T(w)\|_H \|B(w_n)\|_H \|A(w_n) - A(w)\|_{C(\tilde{\Omega})} \\
+ \|T(w_n) - T(w)\|_H \|A(w)\|_H \|B(w_n) - B(w)\|_{C(\tilde{\Omega})}.
\]

Hence

\[
\|T(w_n) - T(w)\|_H \leq (\|A(w_n)\|_H + \|B(w_n)\|_H)(\|A(w_n) - A(w)\|_{C(\tilde{\Omega})} \\
+ \|B(w_n) - B(w)\|_{C(\tilde{\Omega})}).
\]

Therefore, the sequence \( \{T(w_n)\} \) converges strongly to \( T(w) \) in \( H \) and we obtain the lemma. \( \Box \)

Lemma 3.2. (i) For any fixed \( \lambda \), \( G(\lambda, .) \) is Fréchet differentiable on \( H \) and its gradient \( g(\lambda, .) \equiv \nabla G(\lambda, .) \) is given by the following formula:

\[
g(\lambda, w) = w - \lambda L(w) - L_1(M_1(w)) - M(w) + L_1^2(w) \quad \forall w \in H,
\]
where \( L, L_1, M_1 \) and \( M \) are operators on \( H \) defined as follows:

\[
\langle L(w), \varphi \rangle_H = \int_{\Omega} [F_0, w] \varphi \, dx \, dy,
\]

\[
\langle L_1(w), \varphi \rangle_H = \int_{\Omega} N(w) \varphi \, dx \, dy,
\]

\[
\langle M_1(w), \varphi \rangle_H = -\frac{1}{2} \int_{\Omega} [w, w] \varphi \, dx \, dy,
\]

\[
\langle M(w), \varphi \rangle_H = \int_{\Omega} [F(w), w] \varphi \, dx \, dy \quad \forall w, \varphi \in H.
\]

Furthermore, the operator \( L_1 \) is continuous; the operators \( L, M_1 \) and \( M \) are compact.

(ii) \( G(\lambda, \cdot) \) is a \( C^1 \)-function on \( H \).

(iii) \( g(\lambda, \cdot) \) is of class \((S)_+\).

(iv) \( \nabla g(\lambda, 0) \) is given by the formula

\[
\nabla g(\lambda, 0)(w) = w - \lambda L(w) + L_2^1(w) \quad \forall w \in H
\]

and is of class \((S)_+\).

**Proof.** (i) Consider functionals \( G_a(\lambda, \cdot) \) and \( G_b(\cdot) \) defined as follows:

\[
G_a(\lambda, w) = \frac{1}{2} \int_{\Omega} \{ |\Delta w|^2 - \lambda |F_0, w| \} \, dx \, dy,
\]

\[
G_b(w) = \frac{1}{2} \int_{\Omega} |\nabla F(w)|^2 \, dx \, dy = \frac{1}{2} \|F(w)\|^2_H \quad \forall w, \varphi \in H.
\]

Then

\[
G(\lambda, w) = G_a(\lambda, w) + G_b(w) \quad \forall w \in H.
\]

It is clear that \( G_a(\lambda, \cdot) \) is Fréchet differentiable on \( H \) and its gradient \( g_a(\lambda, \cdot) \equiv \nabla G_a(\lambda, \cdot) \) is defined as follows:

\[
\langle g_a(\lambda, w), \psi \rangle_H = \langle w, \psi \rangle_H - \lambda \int_{\Omega} [F_0, w] \psi \, dx \, dy \quad \forall w, \psi \in H.
\] (3.15)

Fix \( w, \psi \in H \) and put

\[
S(w, \psi) = F(w + \psi) - F(w).
\]

By (3.11) and (3.13), we have

\[
\langle S(w, \psi), \varphi \rangle_H = \langle F(w + \psi), \varphi \rangle_H - \langle F(w), \varphi \rangle_H = l_{w+\psi}(\varphi) - l_w(\varphi)
\]

\[
= -\int_{\Omega} \left\{ \frac{1}{2} [w + \psi, w + \psi] \varphi + N(w + \psi) \varphi \right\} \, dx \, dy
\]
\[+ \int_\Omega \left\{ \frac{1}{2} [w,w] \phi + N(w) \phi \right\} \, dx \, dy = \int_\Omega \left\{ -[w,\psi] \phi - \frac{1}{2} [\psi, \psi] \phi - N(\psi) \phi \right\} \, dx \, dy \quad \forall \phi \in H. \quad (3.16)\]

It follows that
\[
\|S(w,\psi)\|_H^2 \leq \left| \int_\Omega \left\{ [w,\psi] S(w,\psi) + \frac{1}{2} [\psi, \psi] S(w,\psi) + N(\psi) S(w,\psi) \right\} \, dx \, dy \right|
\leq C \left( \|w\|_H \|\psi\|_H + \|\psi\|_H^2 + \|\psi\|_H \right) \|S(w,\psi)\|_H.
\]

Hence
\[
\lim_{\psi \to 0} \|S(w,\psi)\|_H = 0. \quad (3.17)
\]

Using the definition of \( G_b(\cdot) \) and substituting \( \phi = F(w) + \frac{1}{2} S(w,\psi) \) into (3.16) yield
\[
G_b(w + \psi) - G_b(w) = \frac{1}{2} \|F(w) + S(w,\psi)\|_H^2 - \frac{1}{2} \|F(w)\|_H^2
= \langle S(w,\psi), F(w) + \frac{1}{2} S(w,\psi) \rangle_H
= \int_\Omega \left\{ -[w,\psi] F(w) - N(\psi) F(w) \right\} \, dx \, dy
+ \frac{1}{2} \int_\Omega \left\{ -[w,\psi] S(w,\psi) - \frac{1}{2} [\psi, \psi] (2F(w)
+ S(w,\psi)) - N(\psi) S(w,\psi) \right\} \, dx \, dy. \quad (3.18)
\]

By (3.1), (3.4) and the Sobolev embedding theorem, we have
\[
\left| \int_\Omega \left\{ -[w,\psi] F(w) - N(\psi) F(w) \right\} \, dx \, dy \right|
\leq \|w\|_H \|\psi\|_H \|F(w)\|_{C(\Omega)} + C \|\psi\|_H \|F(w)\|_H
\leq C \{ \|w\|_H \|F(w)\|_H + \|F(w)\|_H \} \|\psi\|_H, \quad (3.19)
\]
\[
\left| \int_\Omega \left\{ -[w,\psi] S(w,\psi) - \frac{1}{2} [\psi, \psi] (2F(w) + S(w,\psi)) - N(\psi) S(w,\psi) \right\} \, dx \, dy \right|
\leq \|w\|_H \|\psi\|_H \|S(w,\psi)\|_{C(\Omega)} + \frac{1}{2} \|\psi\|_H^2 \|2F(w)
+ S(w,\psi)\|_{C(\Delta)} + C \|\psi\|_H \|S(w,\psi)\|_H
\leq C \|\psi\|_H \{ \|w\|_H \|S(w,\psi)\|_H + \|2F(w)
+ S(w,\psi)\|_H \} \|\psi\|_H. \quad (3.20)
\]
Combining (3.2), (3.3) and (3.17)–(3.20) and the equation

\[ F(w) = M_1(w) - L_1(w), \]

we see that \( G_b \) is Fréchet differentiable at \( w \) and its gradient \( g_b \equiv \nabla G_b \) is defined as follows:

\[
\langle g_b(w), \psi \rangle_H = \int \Omega \left\{ -[w, \psi]F(w) - N(F(w))\psi \right\} \, dx \, dy
\]

\[
= \int \Omega \left\{ -[F(w), w]\psi - N(F(w))\psi \right\} \, dx \, dy
\]

\[
= \int \Omega \left\{ -[F(w), w]\psi - N(M_1(w))\psi
\right. \\
\left. + N(L_1(w))\psi \right\} \, dx \, dy \quad \forall w, \psi \in H.
\] (3.21)

Therefore, the formula of \( g(\lambda, \cdot) \) follows from (3.15) and (3.21).

By (3.4), \( L_1 \) is continuous on \( H \). Now, let \( \{w_n\} \) be a sequence converging weakly to \( w \) in \( H \). By Lemma 3.1, the sequence \( \{M_1(w_n)\} \) converges strongly to \( M_1(w) \) in \( H \). Now, fix a \( \varphi \) in \( H \). By (3.4) and the weak convergence of \( \{w_n\} \) to \( w \), we have

\[
\left| \langle L_1(w_n) - L_1(w), \varphi \rangle_H \right| = \left| \int \Omega N(\varphi)(w_n - w) \right| \rightarrow 0.
\]

Hence, the sequence \( \{L_1(\omega_n)\} \) converges weakly to \( L_1(\omega) \). Thus, the operators \( L_1, M_1, M \) are weakly continuous. Hence, by Lemma 3.1, the operators \( L, M_1, M \) are compact. The proof of (i) is complete.

(ii) It is easy to see that (ii) is derived from (i).

(iii) Let \( \{w_n\} \) be a sequence converging weakly to \( w \) in \( H \) such that

\[
\limsup_{n \rightarrow \infty} \langle g(\lambda, w_n), w_n - w \rangle_H \leq 0.
\] (3.22)

We shall prove that \( \{w_n\} \) converges strongly to \( w \). Note that

\[
\langle g(\lambda, w_n), w_n - w \rangle_H = \langle w_n, w_n - w \rangle_H - \langle \lambda L(w_n) + M(w_n), w_n - w \rangle_H
\]

\[
- \langle L_1(M_1(w_n)), w_n - w \rangle_H + \langle L_1^2(w_n), w_n - w \rangle_H. \]

(3.23)

By (i), the sequences \( \{\lambda L(w_n) + M(w_n)\} \) and \( \{M_1(w_n)\} \) converge strongly in \( H \) and \( \{L_1(w_n)\} \) converges weakly to \( L_1(w) \) in \( H \). Thus,

\[
\lim_{n \rightarrow \infty} - \langle \lambda L(w_n) + M(w_n), w_n - w \rangle_H = 0,
\]

\[
\lim_{n \rightarrow \infty} - \langle L_1(M_1(w_n)), w_n - w \rangle_H = \lim_{n \rightarrow \infty} - \langle L_1(w_n) - L_1(w), M_1(w_n) \rangle_H = 0,
\]

\[
\limsup_{n \rightarrow \infty} \langle L_1^2(w_n), w_n - w \rangle_H = \limsup_{n \rightarrow \infty} \|L_1(w_n) - L_1(w)\|^2_H \geq 0.
\]
Therefore, by (3.22) and (3.23), we have
\[
\limsup_{n \to \infty} \| w_n - w \|_H^2 = \limsup_{n \to \infty} \langle w_n, w_n - w \rangle_H \leq 0.
\]
Thus, \( \{ w_n \} \) converges in \( H \) to \( w \), and we obtain (iii).

(iv) Using the formula of \( g(\lambda, w) \) and arguing as in the proof of (i) and (iii), we find that \( \nabla g(\lambda, 0) \) has the desired properties. \( \square \)

Lemma 3.3. Let \( L_1 \) be as in Lemma 3.2. Then

(i) The operator \( I + L_1^2 \) is an isomorphism from \( H \) onto \( H \).

(ii) If \( L_1^2 \) is compact then \( L_1 \) is also compact.

(iii) If \( t L_1 w = B(0, 1) \) and
\[
k_{ij}(z) = \begin{cases} 0 & \text{when } i \neq j, \\ (1 - \| z \|)^{-2} & \text{when } i = j,
\end{cases}
\]
where \( z = (x, y) \in \Omega \) and \( \| z \| = ((x^2 + y^2)^{1/2}, \) then \( g(\lambda, \cdot) \) is a continuous but not compact vector field on \( H \) for any \( \lambda \) in \( (0, \infty) \).

Proof. (i) Note that
\[
\langle (I + L_1^2)(w), w \rangle_H = \| w \|_H^2 + \| L_1(w) \|_H^2 \geq \| w \|_H^2.
\]
Therefore, by (3.4) and the Lax–Milgram theorem, we obtain (i).

(ii) Let \( \{ w_n \} \) be a sequence in \( H \) converging weakly to \( w \) in \( H \) such that \( \{ L_1^2(w_n) \} \) converges strongly in \( H \). Since
\[
\| L_1(w_n - w) \|_H^2 = \langle L_1^2(w_n - w), w_n - w \rangle_H = \langle L_1^2(w_n) - L_1^2(w), w_n - w \rangle_H,
\]
the sequence \( \{ L_1(w_n) \} \) converges strongly to \( L_1(w) \) in \( H \). So, (ii) is proved.

(iii) By the Cauchy–Schwarz inequality and the weighted Sobolev inequality [8, Theorem 8.4], we have
\[
\left| \int_{\Omega} N(w) \varphi \, dx \, dy \right| = \left| \int_{\Omega} (1 - \| z \|)^{-2} \nabla w \cdot \nabla \varphi \, dx \, dy \right| \\
\leq \left\{ \int_{\Omega} (1 - \| z \|)^{-2} | \nabla w |^2 \, dx \, dy \right\}^{1/2} \\
\times \left\{ \int_{\Omega} (1 - \| z \|)^{-2} | \nabla \varphi |^2 \, dx \, dy \right\}^{1/2} \\
\leq C \| w \|_H \| \varphi \|_H.
\]
Consequently, (3.4) holds and \( L_1 \) is a linear continuous operator on \( H \). For each positive integer \( i \), let \( r_i \) and \( a_i \) be, respectively, the number \( 2^{-i-2} \) and the point \((1 - 2^{-i}, 0)\) in \( \Omega \). Then \( \{ B(a_i, r_i) \} \) is a family of disjoint subsets of \( \Omega \). Choose \( \varphi \) in \( C_c^\infty(B(0, 1)) \) such that
\[
\left\{ \int_{\Omega} | \nabla^m \varphi |^2 \, dx \, dy \right\}^{1/2} = A_m > 0 \quad \forall m = 1, 2.
\]
We define
\[ \phi_i(z) = r_i \phi \left( \frac{z - a_i}{r_i} \right) \quad \forall i \in \mathbb{N}. \]

Then, it is easy to see that \( \phi_i \) is in \( C^\infty_c(B(a_i, r_i)) \) and
\[ \left\{ \int_{\Omega} |\nabla^m \phi_i|^2 \, dx \, dy \right\}^{1/2} = r_i^{2-m} A_m > 0 \quad \forall m = 1, 2, \ i \in \mathbb{N}. \]

Since the supports of \( \phi_i \) are disjoint and \( \| \phi_i \|_H = A_2 \), \( \{ \phi_i \} \) is an orthogonal system in \( H \). Therefore, \( \{ \phi_i \} \) converges weakly to 0 in \( H \). Hence, \( \{ L_1(\phi_i) \} \) converges weakly to \( L_1(0) = 0 \) by the linearity of \( L_1 \). Moreover, we note that
\[ (1 - \|z\|)^{-2} \geq ((1 - \|a_i\|) + \|a_i - z\|)^{-2} \geq 5^{-2} r_i^{-2} \quad \forall z \in B(a_i, r_i), \ i \in \mathbb{N}, \]
\[ |\langle L_1(\phi_i), \phi_i \rangle_H| = \left| \int_{\Omega} N(\phi_i) \phi_i \, dx \, dy \right| = \int_{\Omega} (1 - \|z\|)^{-2} |\nabla \phi_i|^2 \, dx \, dy \]
\[ = \int_{B(a_i, r_i)} (1 - \|z\|)^{-2} |\nabla \phi_i|^2 \, dx \, dy \]
\[ \geq \frac{1}{25} r_i^{-2} \int_{\Omega} |\nabla \phi_i|^2 \, dx \, dy = \frac{A_1^2}{25} \]
and
\[ |\langle L_1(\phi_i), \phi_i \rangle_H| \leq \|L_1(\phi_i)\|_H \|\phi_i\|_H = \|L_1(\phi_i)\|_H A_2. \]

Thus, \( \{ L_1(\phi_i) \} \) does not converge strongly to 0 in \( H \) since \( \|L_1(\phi_i)\|_H \geq A_1^2/25 A_2 > 0 \) for all \( i \in \mathbb{N} \), and the operator \( L_1 \) is not compact. By (i) of Lemma 3.2 and (ii) of Lemma 3.3, we obtain (iii). \( \square \)

Arguing as in the proof of Lemma 2.1 in [9], we obtain the following result.

**Lemma 3.4.** Let \( \lambda \) and \( \beta \) be two real numbers. Then the set \( \{ w \in H \mid G(\lambda, w) \leq \beta \} \) is bounded.

By Lemmas 3.2 and 3.3, the operator \( (I + L_1^2)^{-1} \circ L \) is compact. We shall suppose that the linear characteristic value problem
\[ \psi + L_1^2 \psi = \lambda L \psi \tag{3.24} \]
has infinitely many positive characteristic values \( \lambda_1 \leq \cdots \leq \lambda_n \to \infty \) as \( n \to \infty \).

**Lemma 3.5.** Let \( \lambda \) be in the interval \((\lambda_1, \infty)\) and \( w \) be an eigenvector of (3.24) corresponding to \( \lambda_1 \). Then \( G(\lambda, w) < 0 \) when \( \|w\|_H \) is sufficiently small.

**Proof.** Let \( w \) be an arbitrary eigenvector of (3.24) corresponding to \( \lambda_1 \). By the definitions of operators \( L, L_1, F \) and (3.24), we have
\[ \int_{\Omega} [F_0, w] w \, dx \, dy = \langle L(w), w \rangle_H = \frac{1}{\lambda_1} \|w\|_H^2 + \frac{1}{\lambda_1} \langle L_1^2(w), w \rangle_H, \tag{3.25} \]
Combining (3.25)–(3.27), we obtain

\[
\left\| \int_{\Omega} [w, w](L_1(w) - F(w)) \, dx \, dy \right\| \leq \|L_1(w) - F(w)\|_{C(\tilde{\Omega})} \int_{\Omega} \|w\| \, dx \, dy \\
\leq C \|w\|^2_H (\|L_1(w)\|_{C(\tilde{\Omega})} + \|F(w)\|_{C(\tilde{\Omega})}) \\
\leq C \|w\|^2_H (\|L_1(w)\|_H + \|F(w)\|_H) \\
\leq C \|w\|^2_H (\|w\|_H + \|w\|^2_H). \tag{3.27}
\]

Combining (3.25)–(3.27), we obtain

\[
2G(\lambda, w) = \int_{\Omega} (|\Delta w|^2 + |\Delta F(w)|^2 - \lambda [F_0, w]w) \, dx \, dy \\
= \|w\|^2_H + \|F(w)\|^2_H - \lambda \int_{\Omega} [F_0, w]w \, dx \, dy \\
= \|w\|^2_H + \|F(w)\|^2_H - \frac{\lambda}{\lambda_1} \|w\|^2_H - \frac{\lambda}{\lambda_1} \langle L_1^2(w), w \rangle_H \\
= \left(1 - \frac{\lambda}{\lambda_1}\right) (\|w\|^2_H + \|F(w)\|^2_H) \\
+ \frac{\lambda}{2\lambda_1} \int_{\Omega} [w, w](L_1(w) - F(w)) \, dx \, dy \\
\leq \left(1 - \frac{\lambda}{\lambda_1}\right) (\|w\|^2_H + \|F(w)\|^2_H) + C \|w\|^3_H (\|w\|_H + 1).
\]

Since \(\lambda > \lambda_1 > 0\), we have \(1 - \lambda/\lambda_1 < 0\) and thus \(G(\lambda, w) < 0\) if \(\|w\|_H\) is sufficiently small. \(\square\)

**Lemma 3.6.** Let \(\lambda\) be in \((\lambda_1, \infty)\). Then there exists \(w_1\) in \(H\) such that \(G(\lambda, w_1)\) is a negative minimum of \(G(\lambda, \cdot)\) in \(H\).

**Proof.** By Lemma 3.4, \(G(\lambda, \cdot)\) is bounded from below. By Lemma 3.2, \(G(\lambda, \cdot)\) is a \(C^1\)-function and its gradient \(g(\lambda, \cdot)\) is of class \((S)_+\). Hence, arguing as in the proof
of Corollary 2.2, we see that $G(\lambda, \cdot)$ is weakly sequentially lower semicontinuous. Therefore, by Theorem 9.2 in [12], $G(\lambda, \cdot)$ has a minimum in $H$. By Lemma 3.5, this minimum is negative.

By (3.14), Lemma 3.2 and standard arguments, we see that $(w, F(w))$ is a solution of (3.8) and (3.9) if and only if $(w, F(w))$ satisfies

\[
\begin{align*}
I_w(\varphi) &= \langle F(w), \varphi \rangle_H \quad \forall \varphi \in H, \\
g(\lambda, w) &= 0.
\end{align*}
\]

Since $F(w)$ is uniquely determined in terms of $w$ from Eq. (3.13), it remains to solve the latter equation of the above system. Clearly, this equation implies that $w$ is a critical point of $G(\lambda, \cdot)$. Consequently, the problem of finding solutions to the system (3.8)–(3.10) now turns to the problem of seeking critical points of $G(\lambda, \cdot)$.

We have the main result of this section as follows.

**Theorem 3.1.** Let $\lambda$ be in $(\lambda_1, \infty) \setminus \{\lambda_n : n \in \mathbb{N}\}$. Then the system (3.8)–(3.10) possess at least three distinct solutions.

**Proof.** A first solution is $(w_0, F(w_0))=(0, 0)$ and a second solution $(w_1, F(w_1)) \neq (0, 0)$ is obtained from Lemma 3.6. If $G(\lambda, \cdot)$ has infinitely many critical points then there is nothing more to prove. Therefore, we only consider the case in which all critical points of $G(\lambda, \cdot)$ are isolated and there exists $r_0 > 0$ such that $g(\lambda, w) \neq 0$ whenever $\|w\| \geq r_0$. Since $\lambda$ is not a characteristic value of (3.24), $\nabla g(\lambda, 0)(w) \neq 0$ if $w \neq 0$. By Lemma 3.2, parts (iii) and (iv), the conditions of Theorem 2.2 are satisfied. Hence

\[i(g(\lambda, \cdot), w_0) \in \{-1, 1\}.\]  \hspace{1cm} (3.28)

Lemma 3.4 implies that $\lim_{\|w\| \to \infty} G(\lambda, w) = \infty$. By Lemma 3.2 and Corollary 2.1, there exists a number $r_1 \geq r_0$ such that

\[
\deg(g(\lambda, \cdot), B(0, r), 0) = 1 \quad \text{for all } r \geq r_1.
\]

By Lemma 3.2, part (ii), Lemma 3.6 and Corollary 2.2,

\[i(g(\lambda, \cdot), w_1) = 1.\]  \hspace{1cm} (3.29)

Suppose by contradiction that $G(\lambda, \cdot)$ has only critical points $w_0$ and $w_1$ in $B(0, r)$. Then by (3.28) and (3.29) and the additivity of the degree, we have

\[
1 = \deg(g(\lambda, \cdot), B(0, r), 0) = i(g(\lambda, \cdot), w_0) + i(g(\lambda, \cdot), w_1) \in \{0, 2\},
\]

which is a contradiction. This proves our theorem. ∎

**Remark 3.1.** Theorem 3.1 was showed by Rabinowitz in [9] if the shell is sufficiently shallow and the vector field $g(\lambda, \cdot)$ is compact. Here we relax both conditions. Lemma 3.3 gives us an example of the relaxation of the compactness. We can also get other examples by using weighted inequalities in [4].
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References