Generalized Formal Degree

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Let $G$ be a reductive group over a local field of characteristic zero or a finite central cover of such a group. We present a conjecture that enables one to define formal degree for all unitary representations of $G$. The conjecture is proved for $GL_n$ and $\widetilde{SL}_2$ over real and $p$-adic fields, together with a formal degree relation concerning the local theta correspondence between $\widetilde{SL}_2$ and $SO_3$.

1 Introduction

For a real or $p$-adic connected reductive group $G$, Harish-Chandra introduced the Plancherel measure on the tempered dual $\hat{G}^{\text{temp}}$ and founded the Plancherel formula [7, 18] relating functions on $G$ to functions on $\hat{G}^{\text{temp}}$. While the Plancherel measure is equal to the formal degree for square-integrable representations, there has been no similar interpretation for tempered but nonsquare-integrable representations because formal degree is not defined for them yet. This paper intends to generalize formal degree to all unitary representations.

The traditional theory of formal degree deals only with square-integrable representations of $G$ on Hilbert spaces because it involves an integral of matrix coefficients that diverges for nonsquare-integrable representations. Let $Z$ be the center of $G$, and $\pi$ an irreducible unitary action of $G$ on a Hilbert space $V_{\pi}$. When $\pi$ is square-integrable,
the following integral is absolutely convergent and defines a functional $I(-)$ on $V_\pi \otimes V_\pi \otimes V_\pi \otimes V_\pi$ that is linear in the first and fourth factors, and anti-linear in the second and third factors:

$$I(v_1, v_2, v_3, v_4) := \int_{G/Z} (\pi(g)v_1, v_2)(\pi(g)v_3, v_4) \, dg, \quad v_i \in V_\pi.$$ 

In this situation, $I(-)$ is $G \times G$-invariant in the sense of

$$I(g \cdot v_1, v_2, g \cdot v_3, v_4) = I(v_1, g \cdot v_2, v_3, g \cdot v_4) = I(v_1, v_2, v_3, v_4).$$

Let $I_{\text{st}}(-)$ denote the standard $G \times G$-invariant functional, that is, $I_{\text{st}}(v_1, v_2, v_3, v_4) := (v_1, v_3)(v_2, v_4)$, then the formal degree $d(\pi)$ is the constant such that $I_{\text{st}} = d(\pi)I$.

To extend this to all unitary representations, we need to make the integral defining $I(-)$ meaningful for nonsquare-integrable representations; our idea is to locate the integral in a family of weighted integrals and then evaluate it via meromorphic continuation. For this purpose, we make three innovations.

First, we choose to work in the algebraic category of $G$-representations and use the following terms. (The algebraic setting may allow one to carry the theory over to nonunitary representations, but we keep to unitary representations in this paper.)

(a) When $G$ is a real reductive group over $\mathbb{R}$, let $\mathfrak{g}$ be its Lie algebra, and $K$ a maximal compact subgroup of $G$; an admissible $G$-representation means an admissible $(\mathfrak{g}, K)$-module; a unitary $G$-representation means an admissible $(\mathfrak{g}, K)$-module that is unitarizable.

(b) When $G$ is a $p$-adic reductive group, a unitary $G$-representation means a unitarizable admissible $G$-module.

(c) In cases (a) and (b), irreducibility means algebraic irreducibility.

Second, we introduce certain height function $\Delta(g)$ on $G/Z$ (cf. Section 2). For an irreducible unitary $G$-representation, it is used to formally define a family of weighted integrals, namely

$$I(s; v_1, v_2, v_3, v_4) := \int_{G/Z} (\pi(g)v_1, v_2)(\pi(g)v_3, v_4)\Delta(g)^s \, dg, \quad v_i \in V_\pi.$$ 

The weight factor $\Delta(g)^s$ makes the above weighted integral absolutely convergent when $\text{Re}(s) \gg 0$. Let $X_\pi$ be the space of functionals on $V_\pi \otimes V_\pi \otimes V_\pi \otimes V_\pi$ that is linear in the first and fourth factors, and anti-linear in the second and third factors; then $I(s)$ is an $X$-valued function well defined on a right half-plane.
Third, we use the notion of pointwise holomorphicity for one-variable functions valued in $X_\pi$ since $X_\pi$ is without topology. Let $F : \mathbb{C} \to X_\pi$ be a function; $F(s)$ is called holomorphic if $F(s; v_1, v_2, v_3, v_4)$ is holomorphic for all $v_i \in V_\pi$; $F(s)$ is called meromorphic if there exists a meromorphic function $f(s)$ such that $f(s)F(s)$ is holomorphic; thus $I(s)$ is said to have meromorphic continuation to $\mathbb{C}$ if there exists a $\mathbb{C}$-valued meromorphic function $f(s)$ such that $f(s)I(s)$ has holomorphic continuation to $\mathbb{C}$. For an $X_\pi$-valued meromorphic function $F(s)$, we define its order of zero at a point $s_0$ by

$$\text{ord}_{s=s_0} F(s) := \min \{ \text{ord}_{s=s_0} F(s; v_1, v_2, v_3, v_4) : v_i \in V_\pi \}.$$ 

If $n_0$ is the order of zero at $s = s_0$, then we define the leading coefficient of $F(s)$ at $s = s_0$ to be $(s - s_0)^{-n_0} F(s)|_{s=s_0}$. (It is clear that $\text{ord}_{s=s_0} F(s)$ is nonnegative for holomorphic $F(s)$ and finite for meromorphic $F(s)$.)

Our study of $I(s)$ for representations of classical groups guides us to propose the following conjecture and definition.

**Conjecture 1.1.** Let $F$ be $\mathbb{R}$ or a $p$-adic field, $G$ a finite central cover of a connected reductive group over $F$, and $\pi$ an irreducible unitary $G$-representation.

(i) $I(s)$ has meromorphic continuation to $\mathbb{C}$;
(ii) The leading coefficient of $I(s)$ at $s = 0$ is $G \times G$-invariant;
(iii) When $G$ is a connected reductive group, let $\gamma(s, \pi, \text{ad}, \psi) = \epsilon(s, \pi, \text{ad}, \psi) \cdot \frac{L(1-s, \pi', \text{ad})}{L(s, \pi, \text{ad})}$ be the adjoint $\gamma$-factor of $\pi$ with respect to a nontrivial character $\psi$ of $F$. Then $\text{ord}_{s=0} I(s) = -\text{ord}_{s=0} \gamma(s, \pi, \text{ad}, \psi)$. □

**Definition 1.2.** Suppose that Conjecture 1.1 is true for a unitary $G$-representation $\pi$. If $d_\pi(s)$ is a meromorphic function and satisfies $I_{st} = \lim_{s \to 0} d_\pi(s)I(s)$, then we call it a formal degree factor of $\pi$. The leading coefficient of a formal degree factor at $s = 0$ is called the (generalized) formal degree of $\pi$ and denoted by $d(\pi)$. □

Here we do not explicitly deal with complex reductive groups because reductive groups over $\mathbb{C}$ can be naturally considered as reductive groups over $\mathbb{R}$. Before presenting the main results of this paper, we make several conceptual remarks.

**Remark 1.3.** When $\pi$ is square-integrable, the generalized formal degree coincides with the usual formal degree (cf. Lemma 2.5); when $\pi$ is nonsquare-integrable, $s = 0$ may be a pole of $I(s)$ and its order measures the divergence of the integral in the formal...
functional \( I(-) \). If \( s = 0 \) is indeed a pole of \( I(s) \), then the value of the generalized formal degree depends on the concrete choice of the height function; such a dependence should be elementary in nature and we are still in search of a law or a canonical choice of the height function.

\[ \square \]

**Remark 1.4.** When \( G \) is a connected reductive group, part (iii) of Conjecture 1.1 implies the existence of a constant \( d_0(\pi) \) such that \( d_0(\pi) \gamma(s, \pi, \text{ad}, \psi) \) is a formal degree factor. \( d_0(\pi) \) should be of arithmetic nature; the Hiraga–Ichino–Ikeda conjecture (cf. [8, Conjecture 1.4]) is essentially a formula of \( d_0(\pi) \) for square-integrable representations and we expect a general but similar formula working for all unitary representations.

\[ \square \]

**Remark 1.5.** The generalized formal degree has the potential to relate to the Plancherel measure. Suppose that the center of \( G \) is anisotropic and \( (\pi, V_\pi) \) is a nonsquare-integrable tempered unitary representation of \( G \), then \( \pi \) belongs to a family \( \{\pi_t\}_{t \in T} \), where the index set \( T \) is a compact real torus and the representations \( \pi_t \) are actions on \( V_\pi \). Let \( dt \) be a Haar measure on \( T \) and \( \lambda(t) \, dt \) be the Plancherel measure on \( T \); let \( v \) be a unit vector in \( V_\pi \) and define the \( v \)-Fourier transform for continuous functions on \( T \) by

\[ \varphi(t) \mapsto \hat{\varphi}(g) = \int_T \varphi(t) (\pi_t(g)v, v) \lambda(t) \, dt; \]

let \( K(s) \) be a family of functions on \( T \times T \) defined by

\[ K(s; t_1, t_2) := \int_G (\pi_{t_1}(g)v, v)(\pi_{t_2}(g)v, v) \Delta(g)^s \, dg, \quad \text{Re}(s) \gg 0. \]

If \( K(s) \) has meromorphic continuation, then the isometry of the \( v \)-Fourier transform implies

\[ \int_T \varphi_1(t)\overline{\varphi_2(t)} \lambda(t) \, dt = \int_G \varphi_1(g)\overline{\varphi_2(g)} \, dg = \lim_{s \to 0} \int_{T \times T} \varphi_1(t_1)\overline{\varphi_2(t_2)} \lambda(t_1)\lambda(t_2) K(s; t_1, t_2) \, dt_1 \, dt_2. \]

In other words, when \( s \to 0 \), one has \( K(s) \to \lambda(t)^{-1} 1_{\Delta T} \) in the sense of distribution, where \( \Delta T \) is the diagonal of \( T \times T \). A closer look at the limiting behavior of \( K(s)|_{\Delta T} \), that is, \( I_{\pi_t}(s; v, v, v, v) \), may connect the generalized formal degree to the Plancherel measure. Particularly, for \( G = \text{PGL}_2(F) \), where \( F \) is \( p \)-adic, one can verify that the Plancherel measure at a continuous series representation is a constant multiple of the product of the generalized formal degree and a canonical Haar measure.

\[ \square \]

In this paper, we have verified the meromorphic continuation of \( I(s) \) for split classical groups over \( p \)-adic fields (cf. Corollary 2.4) and obtained three precise
Generalized Formal Degree results concerning $\text{GL}_n, \widetilde{\text{SL}}_2$. When necessary, the symbol $I_\pi(s)$ is used to emphasize its association with $\pi$.

**Theorem A.** Let $F$ be $\mathbb{R}$ or a $p$-adic field and $G = \text{GL}_n(F)$.

1. Parts (i) and (ii) of Conjecture 1.1 are true.
2. Part (iii) of Conjecture 1.1 is true for all $\pi$ when $F = \mathbb{R}$, and for generic $\pi$ when $F$ is $p$-adic. □

Theorem A is proved by applying the normalization theory of intertwining operators [1, 13] to certain induced representations of $\text{GL}_{2n}$. When $n = 2$, we compute the main part of $I_\pi(s; -)$ using asymptotic formulas of matrix coefficients and explicitly determine the formal degree factor, namely, we show that $d_0(\pi) = \frac{\pi(-1)}{2}$ when $\pi$ is square-integrable, $\frac{1}{2}$ when $\pi$ is a continuous or complementary series, and $-\frac{1}{2}$ when $\pi$ is one dimensional (cf. Proposition 3.5).

**Theorem B.** Let $F$ be $\mathbb{R}$ or a $p$-adic field, then Conjecture 1.1 is true for all genuine irreducible unitary representations $\tilde{\pi}$ of $G = \text{SL}_2(F)$. □

The proof of Theorem B is based on several asymptotic formulas of matrix coefficients (cf. Lemmas 4.12, 4.13, 4.19, and 4.20), which help us determine the main part of $I_{\tilde{\pi}}(s; -)$.

Concerning the reductive dual pair $(\text{SL}_2, \text{SO}_3)$, we prove the following interesting formal degree relation for representations that are in local theta correspondence.

**Theorem C.** Let $F$ be $\mathbb{R}$ or a $p$-adic field and the measures on $\text{SL}_2(F)$ and $\text{SO}_3(F)$ be determined by the same nontrivial additive character of $F$. If $\tilde{\pi}$ and $\pi$ are irreducible unitary representations of $\text{SL}_2(F)$ and $\text{SO}_3(F)$ that are in local theta correspondence (with respect to certain nontrivial character $\psi'$ of $F$), then $d_\pi(s) = |2|^{-1} d_{\tilde{\pi}}(s)$. □

There are two isomorphism classes of $\text{SO}_3$ over $F$: one compact and the other noncompact. Theorem C is first proved for the compact $\text{SO}_3(F)$ and then extended to square-integrable representations of the noncompact $\text{SO}_3(F)$ by Jacquet–Langlands correspondence. When the $\text{SO}_3(F)$ is noncompact and $\pi$ is nonsquare-integrable, the equality is proved by a direct computation.
2 Preliminary

Let $F$ be a field that is either $\mathbb{R}$ or $p$-adic. When $F$ is $p$-adic, $\mathcal{O}_F$ denotes the ring of integers in $F$, $\omega$ a uniformizer of $F$, $\kappa = \mathcal{O}_F/\omega \mathcal{O}_F$ the residue field, $v(\cdot)$ the valuation, $q$ the number of elements in $\kappa$, and $|\cdot| = q^{-v(\cdot)}$ the norm function on $F$; when $F$ is $\mathbb{R}$, then $|\cdot|$ denotes the usual absolute value function on $\mathbb{R}$. The zeta function of $F$ is denoted by $\zeta_F(s)$, whose value is $\frac{1}{1 - |\omega|^s}$ when $F$ is $p$-adic and $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$ when $F$ is real.

We fix a nontrivial character $\psi$ of $F$: when $F$ is $p$-adic, we require the conductor of $\psi$ to be $\mathcal{O}_F$; when $F$ is real, we choose $\psi(x) = e^{2\pi i x}$. For $a \in F^\times$, let $\psi_a$ denote the $a$-twist of $\psi$, namely $\psi_a(x) = \psi(ax)$.

For a connected reductive group $G$ over $F$, let $\mu_{G, \psi}$ denote the Haar measure on $G$ determined by $\psi$ as in [6] (when $F = \mathbb{R}$) and [5] (when $F$ is $p$-adic); any finite central cover of $G$ is then given an induced measure. For a subset $X$ of $G$, we write $\mu_{G, \psi}(X)$ or $m(X)$ for its measure. For example, when $F$ is $p$-adic and $G$ is unramified over $F$, a maximal compact subgroup $K$ of $G$ has measure $m(K) = q^{-\dim G}|G(K_F)|$. As another example, the Haar measure of $\text{GL}_n(F)$ at $g = (g_{ij})$ ($1 \leq i, j \leq n$) is $dg = \frac{dg_{11} \cdots dg_{nn}}{|\det g|^n}$, where $dg_{ij}$ is the additive Haar measure on $F$ such that $\mathcal{O}_F$ has measure 1.

2.1 The height function $\Delta(g)$

When $G$ is a split connected reductive group over $F$, we choose a maximal $F$-split torus $T$, a Borel subgroup containing $T$, and a maximal compact subgroup $K$ of $G$ that contains the maximal compact subgroup of $T$; $(T, B)$ determines a set of simple roots $\Delta := \{\alpha_1, \ldots, \alpha_n\}$ and $(T, K)$ determines a $KTK$ decomposition of $G$. Consider the following subset of $T$:

$$T^- = \{t \in T : |\alpha_i(t)| \leq 1, \ 1 \leq i \leq n\},$$

then every element $g \in G$ can be written in the form $g = k_1 t k_2$, where $k_1, k_2 \in K$ and $t \in T^-; \ t$ the element $t$ is unique up to multiplication by $T \cap K$. We require the height functions $\Delta(g)$ to be $K$-biinvariant and depends only on $|\alpha(t)|$ ($\alpha \in D$); our experience further suggests us to consider only the height functions of the form $\Delta(g) = \Delta(t) = h(\alpha_1(t), \ldots, |\alpha_n(t)|)$, where $h(\cdot) = h_0(\cdot) h_1(\cdot)$ with $h_0(\cdot)$ being a bounded nonzero rational function and $h_1(\cdot)$ a monomial in $n$-variables.

When $G$ is a general connected reductive group over $F$, we choose a finite extension $E$ of $F$ such that $G_E := G \otimes_F E$ is split over $E$ and denote by $i : G \rightarrow G_E$ the canonical embedding. We then consider height functions on $G$ of the form $\Delta(g) = \Delta_E(i(g))$, where $g \in G$ and $\Delta_E$ is a height function on $G_E$ as specified in the previous paragraph. If $G$ is
a finite central cover of a connected reductive group $G_0$ and $p : G \to G_0$ is the covering map, then we consider height functions on $G$ of the form $\Delta_0 \circ p$, where $\Delta_0$ is a height function on $G_0$.

The main theorems in this paper are about $GL_n(F)$ and $SL_2(F)$ and we fix an explicit choice of height function for them as below.

(i) $G = GL_n(F)$.

$$\Delta(g) := \begin{cases} 
\frac{|\det(g)|}{|g|^{n \cdot \min\{ v(g_{ij}) \}}}, & F \text{ is } p\text{-adic}, \\
\frac{|\det(g)|}{\text{Tr}(gg^t)^{n/2}}, & F \text{ is } \mathbb{R}.
\end{cases}$$

(ii) $G = SL_2(F)$. Let $T$ be the diagonal subgroup, $B$ be the upper triangular subgroup, $K$ be $SL_2(\mathcal{O}_F)$ or $SO(2, \mathbb{R})$, and $\Delta(g)$ be the height function determined by

$$h(x) := \begin{cases} 
|x|^{1/2}, & F \text{ is } p\text{-adic}, \\
1/|x|^{1/2} + |x|^{-1/2}, & F \text{ is } \mathbb{R}.
\end{cases}$$

Here $\Delta_{SL_2}(\cdot)$ is $K$-biinvariant and satisfies $\Delta\left((\frac{\sigma^n}{\sigma^{-n}})\right) = |\sigma|^n$ when $n \geq 0$ and $\Delta\left((y_{y^{-1}})\right) = \frac{1}{|y^{1/2}|} + \frac{1}{|y^{-1/2}|}$ for $y \in \mathbb{R}^\times$.

2.2 Meromorphic continuation

This section concerns the meromorphic continuation of $I(s)$ when $F$ is $p$-adic and $G$ is an $F$-split connected reductive group.

Let the notation $T, B, K$, and $D$ be as in Section 2.1. Let $N$ denote the unipotent radical of $B$ and $\delta$ the modulus character of $B$, that is, $\Delta(b) = |\det \text{Ad}_n(b)|$, where $n$ is the Lie algebra of $N$. For a subset $\Theta \subseteq D$, let $M_{\Theta}$ denote the centralizer in $G$ of $\{ t \in T : \alpha(t) = 1, \alpha \in \Theta \}$ and write $P_{\Theta} = M_{\Theta} \cdot N$ for the standard parabolic subgroup associated to $\Theta$; let $N_{\Theta}$ denote the unipotent radical of $P_{\Theta}$, $N_{\Theta}^-$ the unipotent subgroup of $G$ opposite to $N_{\Theta}$, and $\delta_{P_{\Theta}}$ the modulus character of $P_{\Theta}$. We further introduce the following subsets of $T^-$:

$$T_{\Theta}^- := \{ t \in T^- : |\alpha(t)| = 1 \text{ for } \alpha \in \Theta \text{ and } |\alpha(t)| < 1 \text{ for } \alpha \in D - \Theta \}.$$

$$T_{\Theta}^-(\epsilon) := \{ t \in T^- : \epsilon \leq |\alpha(t)| \leq 1 \text{ for } \alpha \in \Theta \text{ and } |\alpha(t)| < \epsilon \text{ for } \alpha \in D - \Theta \}, \quad \epsilon \in (0, 1].$$

They give disjoint decompositions of $T^-$, that is, $T^- = \sqcup_{\Theta} T_{\Theta}^- = \sqcup_{\Theta} T_{\Theta}^-(\epsilon)$. 
2.2.1 Volume hypothesis

The following hypothesis on the volume of $K$-double cosets are supposed to hold in general, but we verify it here only for split classical groups.

**Volume Hypothesis.** For each subset $\Theta$ of $D$, there exists a constant $C_{\Theta}$ such that $m(KtK) = C_{\Theta}|\delta(t)|^{-1}$ for $t \in T_{\Theta}$.

The split classical groups can easily be described with matrices. $GL_n$ is the group of $n \times n$ matrices with nonzero determinant and $SL_n$ is the subgroup of elements with determinant 1; let $J_n$ be the $n \times n$-matrix with 1’s on the minor diagonal, then the split symplectic-type and orthogonal-type groups are

\[ GSp_n = \{ g \in GL_{2n} : g \begin{pmatrix} -J_n & J_n \\ J_n & -J_n \end{pmatrix} t g = v(g) \begin{pmatrix} -J_n & J_n \\ J_n & -J_n \end{pmatrix} \}, \quad Sp_n = \{ g \in GSp_n : v(g) = 1 \}, \]

\[ SO(n, n) = \{ g \in GL_{2n} : gJ_{2n}t g = J_{2n}, \det(g) = 1 \}. \]

\[ SO(n + 1, n) = \{ g \in GL_{2n+1} : gJ_{2n+1}^t g = J_{2n+1}, \det(g) = 1 \}. \]

When $G$ is one of them, that is, $GL_{n+1}$, $SL_{n+1}$, $GSp_n$, $Sp_n$, $SO(n, n)$, and $SO(n + 1, n)$, we let $T$ be the diagonal subgroup, $B$ the upper-triangular subgroup, $D$ the corresponding set of simple roots, and $K$ the subgroup of elements in $G$ whose matrix entries are all in $O_F$.

**Lemma 2.1.** $GL_{n+1}$, $SL_{n+1}$, $GSp_n$, $Sp_n$, $SO(n, n)$, and $SO(n + 1, n)$ satisfy the volume hypothesis.

**Proof.** Suppose that $\Theta \subseteq D$ and $t \in T_{\Theta}$. Write $K_t = K \cap tKt^{-1}$, then $m(KtK) = m(K)^2/m(K_t)$ and the volume hypothesis is equal to, say, that when $t \in T_{\Theta}$, the measure $m(K_t)$ is proportional to $|\delta(t)|$ by a constant depending only on $\Theta$.

Now suppose that $G$ is one of the listed groups and $T, B, K,$ and $D$ be as chosen in the paragraph before Lemma 2.1. When $\Theta = D$, it is obvious that $T_{\Theta}$ is in the center of $G$ and hence $K_t = K$ and $\delta(t) = 1$ for $t \in T_{\Theta}$; the statement is obviously true. Now we deal with the case $\Theta \subset D$; in this case, for $t \in T_{\Theta}$, it is straightforward to check that $K_t = N_{\Theta}(O_F)M_{\Theta}(O_F)(tN_{\Theta}(O_F)t^{-1})$, so

\[ m(K_t) = m(N_{\Theta}(O_F))m(M_{\Theta}(O_F))m(tN_{\Theta}(O_F)t^{-1}). \]
Because \( m(tN_\Theta(O_F)t^{-1}) = m(N_\Theta(O_F))|\delta_{P_\Theta}(t)| = m(N_\Theta(O_F))|\delta(t)| \), we see that \( m(K_t) \) is indeed proportional to \( |\delta(t)| \) by a constant depending only on \( \Theta \).

\[ \Box \]

2.2.2 Asymptotic behavior of matrix coefficients

For an irreducible unitary representation \((\pi, V_\pi)\) of \(G\) and a subset \(\Theta \subset D\), let \(\pi_{N_\Theta}\) denote the Jacquet module of \(\pi\) with respect to the parabolic subgroup \(P_\Theta\), that is,

\[ V_{\pi_{N_\Theta}} = V_\pi / \text{span}\{ v - \pi(n)v : v \in V_\pi, \ n \in N_\Theta \}. \]

Then there is a canonical nondegenerate Hermitian pairing \((\cdot, \cdot)_{N_\Theta}\) on \(V_{\pi_{N_\Theta}} \times V_{\pi_{N_\Theta}}\) fitting into the following lemma (cf. [2, Theorem 4.3.4]).

Lemma 2.2 ([2]). Let \( v_1, v_2 \in V_\pi \) be given and \( u_{1,\Theta} \) and \( u_{2,\Theta} \) be their images in \(V_{\pi_{N_\Theta}}\). There exists \( \epsilon > 0 \) such that whenever \( t \in T^- \) satisfies \(|\alpha(t)| < \epsilon\) for all \( \alpha \in D - \Theta \), one has

\[ (\pi(t)v_1, v_2) = (\pi_{N_\Theta}(t)u_{1,\Theta}, u_{2,\Theta})_{N_\Theta}. \]

\[ \Box \]

2.2.3 The meromorphic continuation of \(I(s)\)

Proposition 2.3. Let \(F\) be a \(p\)-adic field, \(G\) an \(F\)-split connected reductive group over \(F\), and \(\pi\) an irreducible unitary representation of \(G\). If \(G\) satisfies the volume hypothesis and the height function is associated to a monomial, then \(I(s)\) has meromorphic continuation.

\[ \Box \]

Proof. Let \(n\) be the semi-simple rank of \(G\) and \(T, B, K,\) and \(D\) be as in Section 2.1. We shall show that for all \(v_i \in V_\pi (1 \leq i \leq 4)\), the function

\[ F(s) := I(s; v_1, v_2, v_3, v_4) = \int_{G/Z} (\pi(g)v_1, v_2)\overline{(\pi(g)v_3, v_4)} \Delta(g)^s \, dg \]

has meromorphic continuation to \(\mathbb{C}\) and that there exists a numeric holomorphic function \(f(s)\) such that \(f(s)F(s)\) is holomorphic for all such \(F(s)\). For this purpose, we need to analyze different parts of the integral in \(F(s)\).

1) The height function. Suppose that \(\Delta(\cdot)\) is associated to the monomial

\[ h(y_1, \ldots, y_n) = \prod_{i=1}^{n} y_i^{q_i}, \text{ where } q_i \in \mathbb{Z}_{>0}. \]

Then \(\Delta(t) = \prod_{i=1}^{n} |\alpha_i(t)|^{q_i}\).

2) Estimate the matrix coefficients. For simplicity, write \(g \circ v\) for \(\pi(g)v\); for \(\Theta \subset D\), denote by \(p_\Theta\) the projection from \(V\) to \(V_{\pi_{N_\Theta}}\). Lemma 2.2 provides us \(\epsilon \in (0, 1)\) such that
for all \( \Theta \subseteq D \), all \( k_1, k_2, k_3, k_4 \in K \), and all \( t \in T^- \) satisfying \( |\alpha(t)| < \epsilon (\alpha \in D - \Theta) \), one has

\[
(\pi(t) \circ v_1, k_2 \circ v_2) = (\pi_N(t) \circ p_\Theta(k_1 \circ v_1), p_\Theta(k_2 \circ v_2))_{N_\Theta},
\]

\[
(\pi(t) \circ v_3, k_4 \circ v_4) = (\pi_N(t) \circ p_\Theta(k_3 \circ v_3), p_\Theta(k_4 \circ v_4))_{N_\Theta}.
\]

Because each \( \pi_N \) is an admissible \( M_\Theta \)-module of finite length, we can further decompose it as a finite direct sum of the generalized eigenspaces of the center of \( M_\Theta \), namely

\[ \pi_N = \bigoplus_{r_\Theta} W_{\Theta, r_\Theta}; \]

for each \( r_\Theta \), there is a quasi-character \( \omega_{\Theta, r_\Theta} \) of the center of \( M_\Theta \) and a positive integer \( n_{r_\Theta} \) such that \( (\pi_N(x) - \omega_{\Theta, r_\Theta}(x))^{n_{r_\Theta}} \) annihilates \( W_{\Theta, r_\Theta} \) for all \( x \) in the center of \( M_\Theta \). Write \( p_{\Theta, r_\Theta} \) for the projection map from \( \pi_N \) to \( W_{\Theta, r_\Theta} \) and put \( n_{\Theta} = \max\{n_{r_\Theta} \} \); if \( x \) is in the center of \( M_\Theta \), then \( (\pi_N(x) - \omega_{\Theta, r_\Theta}(x))^{n_{\Theta}} \) annihilates \( W_{\Theta, r_\Theta} \) for each \( r_\Theta \).

(3) Decompose the integral domain. Write \( \tilde{G} = G/Z \) and \( \tilde{K} = K/Z \cap K \); there is a bijective map \( T^- / ZT(O_F) \to \tilde{K} \) \( t \to km \), and an injective map

\[ 7\psi : T^- / ZT(O_F) \hookrightarrow Z_{\geq 0}^n, \quad t \to (v(\alpha_1(t)), \ldots, v(\alpha_n(t))). \]

For each \( 1 \leq i \leq n \), choose \( t_i \in T^- \) such that \( \alpha_j(t_i) = 1 \) for \( j \neq i \) and that \( |\alpha_i(t_i)| < \epsilon \); write

\[ S = \{(m_1v(\alpha_1(t_1)), \ldots, m_nv(\alpha_n(t_n)) : m_i \in \mathbb{Z}_{\geq 0} \}; \]

there are finitely many elements \( t'_1, \ldots, t'_p \in T^- \) such that \( \text{Im}(\psi) \) is the disjoint union of \( S = \{(\alpha_j(t'_j) + S, 1 \leq j \leq p \). For \( \tilde{m} := (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n \), write \( \tilde{t} = t_{m_1} \cdots t_{m_n} \) (the product of \( t_{m_1}, \ldots, t_{m_n} \)), then \( \{t_j \tilde{m} : 1 \leq j \leq p, \tilde{m} \in \mathbb{Z}_{\geq 0}^n \} \) is a set of representatives of \( T^- / ZT(O_F) \).

Now we start proving the proposition. The idea is to decompose \( F(s) \) when \( \text{Re}(s) \gg 0 \) and show that it is a rational function of \( |\sigma|^s \). First,

\[ F(s) = I(s : v_1, v_2, v_3, v_4) \]

\[ = \sum_{[t] \in T^- / ZT(O_F)} \left[ \int_{\tilde{K} \times \tilde{K}} \frac{(\pi(t) \circ v_1, k_2 \circ v_2)(\pi(t) \circ v_3, k_4 \circ v_4) \Delta(t)^s}{\Delta(t)^s} dt \right] \]

\[ = \sum_{[t] \in T^- / ZT(O_F)} \frac{m(\tilde{K} \times \tilde{K})}{m(K)^2} \left[ \int_{\tilde{K} \times \tilde{K}} (t_{k_1} \circ v_1, k_2 \circ v_2)(t_{k_1} \circ v_3, k_2 \circ v_4) \Delta(t)^s dt \right]. \]

Noticing the representatives of \( T^- / ZT(O_F) \), one sees that the outer summation is actually on \( 1 \leq j \leq p \) and \( \tilde{m} \in \mathbb{Z}_{\geq 0}^n \).

Secondly, \( \mathbb{Z}_{\geq 0}^n \) is the disjoint union of \( X_\Theta (\Theta \subseteq D) \), where

\[ X_\Theta = \{(m_1, \ldots, m_n) : m_i = 0 \text{ when } \alpha_i \in \Theta \text{ and } m_j > 0 \text{ when } \alpha_j \notin \Theta \}. \]
Hence the outer summation can further be decomposed into \( \sum_{t \in T/T(O)} = \sum_{1 \leq j \leq p} \sum_{\Theta \subseteq D} \sum_{m \in X_\Theta} \).

Put

\[
F_{j, \Theta}(s) := \sum_{m \in X_\Theta} m(\bar{K} t_j^m K) \int_{\bar{K} \times \bar{K}} (t_j^m k_1 \circ v_1, k_2 \circ v_2) \Delta(t_j^m)^s \, dk_1 k_2.
\]

Then \( F(s) = \sum_{j, \Theta} \frac{1}{m(\bar{K})^s} F_{j, \Theta}(s) \).

Thirdly, we decompose \( F_{j, \Theta}(s) \) using the decomposition of \( \pi_{N_\Theta} \) into \( W_{\Theta, r_\Theta} \). The following two facts are needed:

(i) By the volume hypothesis, there exists a constant \( C_{j, \Theta} \) such that

\[
m(\bar{K} t_j^m) = C_{j, \Theta} |\delta(t_j^m)| \quad \text{for } m \in X_\Theta.
\]

(ii) By the choice of \( \epsilon, t_i (1 \leq i \leq n) \), one has

\[
(t_j^m k_1 \circ v_1, k_2 \circ v_2) = (\pi_{N_\Theta}(t_j^m) p_{\Theta}(t_j^m k_1 \circ v_1), p_{\Theta}(k_2 \circ v_2))_{N_\Theta} = \sum_{r_\Theta} (\pi_{N_\Theta}(t_j^m) p_{\Theta, r_\Theta} p_{\Theta}(t_j^m k_1 \circ v_1), p_{\Theta}(k_2 \circ v_2))_{N_\Theta}, \quad m \in X_\Theta.
\]

Similarly,

\[
(t_j^m k_1 \circ v_3, k_2 \circ v_4) = \sum_{r_\Theta'} (\pi_{N_\Theta}(t_j^m) p_{\Theta, r_\Theta'} p_{\Theta}(t_j^m k_1 \circ v_3), p_{\Theta}(k_2 \circ v_4))_{N_\Theta}, \quad m \in X_\Theta.
\]

Put

\[
F_{j, \Theta, r_\Theta, r_\Theta'}(s) = \sum_{m \in X_\Theta} |\delta(t_j^m)| \Delta(t_j^m)^s \int_{\bar{K} \times \bar{K}} (\pi_{N_\Theta}(t_j^m) p_{\Theta, r_\Theta} p_{\Theta}(t_j^m k_1 \circ v_1), k_2 \circ v_2)
\]

\[
\cdot (\pi_{N_\Theta}(t_j^m) p_{\Theta, r_\Theta'} p_{\Theta}(t_j^m k_1 \circ v_3), k_2 \circ v_4) \, dk_1 k_2.
\]

Then \( F_{j, \Theta}(s) = \sum_{r_\Theta, r_\Theta'} C_{j, \Theta} |\delta(t_j^m)| F_{j, \Theta, r_\Theta, r_\Theta'}(s) \).

Fourth, we estimate \( F_{j, \Theta, r_\Theta, r_\Theta'}(s) \) by determining how \( \pi_{N_\Theta}(t_j^m) \) acts on \( W_{\Theta, r_\Theta} \). Suppose that \( \alpha_i \in D - \Theta \) and \( m_i > 0 \), then \( (\pi_{N_\Theta}(t_i) - \omega_{\Theta, r_\Theta}(t_i))^{m_i} \) annihilates \( W_{\Theta, r_\Theta} \); hence for
\[ u \in W_{\Theta, r_0}, \text{ one has} \]
\[ \pi_{N_m}(t^{m_i}(u)) = [\pi_{N_m}(t_i) - \omega_{\Theta, r_0}(t_i) + \omega_{\Theta, r_0}(t_i)]^{m_i}(u) \]
\[ = \sum_{0 \leq m_i' < n_0} C_{m_i}^{m_i} \omega_{\Theta, r_0}(t_i)^{m_i-m_i'}(\pi_{N_m}(t_i) - \omega_{\Theta, r_0}(t_i))^{m_i'}(u). \]

Here \( C_a^b \) is the binomial coefficient; for convenience, set \( C_a^b = 1 \) if \( a \leq 0 \) or when \( a < b \).

For \( \tilde{m} = (m_1, \ldots, m_n), \tilde{m}' = (m_1', \ldots, m'_{n'}) \in \mathbb{Z}_{\geq 0}^n \), we write \( C_{\tilde{m}}^{m_i} = C_{m_1}^{m_1} \cdots C_{m_n}^{m_n} \); we also write \( \tilde{m}' \leq \tilde{m} \) if \( m_i' \leq m_i \) for all \( 1 \leq i \leq n \); for an integer \( n \), the notion \( \tilde{m} \geq n \) means that \( m_i \geq n \) for each \( i \); similar interpretation is given to the notion \( \tilde{m} \leq n \). Then for \( \tilde{m} \in X_{\Theta} \) and \( u \in W_{\Theta, r_0} \), one has
\[ \pi_{N_m}(t^{\tilde{m}}(u)) = \pi_{N_m}(t^{m_1}) \cdots \pi_{N_m}(t^{m_n})(u) = \sum_{0 \leq m_i' < n_0} C_{\tilde{m}}^{m_i} \omega_{\Theta, r_0}(t^{\tilde{m}-m_i'})u^{\tilde{m}'}, \]

where \( u^{\tilde{m}'} = (\pi_{N_m}(t_1) - \omega_{\Theta, r_0}(t_1))^{m_1} \cdots (\pi_{N_m}(t_n) - \omega_{\Theta, r_0}(t_n))^{m_n}(u) \). Recall that for \( \tilde{m} \in X_{\Theta} \), if \( \alpha_i \in \Theta \), then \( m_i = 0 \) and hence \( C_{m_i}^{m_i} \) in the above expression is 0.

The above formulas lead to the following expression for \( F_{j, \Theta, r_0}(s) \):
\[ F_{j, \Theta, r_0}(s) = \sum_{0 \leq m < n_0} \sum_{0 \leq m' < n_0} \left( \sum_{\tilde{m} \in X_{\Theta}} |\delta(t^{\tilde{m}})| \Delta(t^{\tilde{m}}) \nabla(t^{\tilde{m}})C_{\tilde{m}}^{m_i} \omega_{\Theta, r_0}(t^{\tilde{m}-m_i'}) \omega_{\Theta, r_0}(t^{\tilde{m}-m_i'}) \right) \]
\[ \cdot \left( \int_{K \times K} ([p_{\Theta, r_0} p_{\Theta}(t_j k_1 \circ v_1)]^{m_i}, k_2 \circ v_2) ([p_{\Theta, r_0} p_{\Theta}(t_j k_1 \circ v_3)]^{m_i}, k_4 \circ v_4) d k_1 k_2 \right) \]
\[ = \sum_{0 \leq m < n_0} \sum_{0 \leq m' < n_0} C_{j, \Theta, r_0}(v_1, v_2, v_3, v_4) f_{j, \Theta, r_0}(v_1, v_2, v_3, v_4). \]

Here
\[ f_{j, \Theta, r_0}(s) = \sum_{\tilde{m} \in X_{\Theta}} |\delta(t^{\tilde{m}})| \Delta(t^{\tilde{m}}) \nabla(t^{\tilde{m}})C_{\tilde{m}}^{m_i} \omega_{\Theta, r_0}(t^{\tilde{m}-m_i'}) \omega_{\Theta, r_0}(t^{\tilde{m}-m_i'}), \]
\[ C_{j, \Theta, r_0}(v_1, v_2, v_3, v_4) = \int_{K \times K} ([p_{\Theta, r_0} p_{\Theta}(t_j k_1 \circ v_1)]^{m_i}, k_2 \circ v_2) \]
\[ \times ([p_{\Theta, r_0} p_{\Theta}(t_j k_1 \circ v_3)]^{m_i}, k_4 \circ v_4) d k_1 k_2. \]
Finally, combining the above decompositions, we obtain a formula for $F(s)$ when $\text{Re}(s) \gg 0$, namely

$$F(s) = \sum_{j, \theta} \sum_{n, r, r'} \sum_{0 \leq m' < n_0} \sum_{0 \leq m'' < n_0} \frac{C_{j, \theta} |\delta(t_j)|}{m(K)^2} C_{j, \theta, r, r', \tilde{m}, \tilde{m}'}(v_1, v_2, v_3, v_4) f_{j, \theta, r, r', \tilde{m}, \tilde{m}'}(s).$$

The above summation is a finite summation; the functions $f_{j, \theta, r, r', \tilde{m}, \tilde{m}'}(s)$ are independent of $v_i$ and their series summation expression shows that they are rational functions of $|\omega|^2$; the numbers $C_{j, \theta, r, r', \tilde{m}, \tilde{m}'}(v_1, v_2, v_3, v_4)$ depend on $v_i$.

At this moment, it is clear that $F(s)$ is a rational function of $|\omega|^2$ on a right half-plane and hence has meromorphic continuation to $\mathbb{C}$; let $f(s)$ be a polynomial that annihilates the denominators of all $f_{j, \theta, r, r', \tilde{m}, \tilde{m}'}(s)$, then $f(s)F(s)$ is holomorphic for all $v_i$. \hfill \Box

**Corollary 2.4.** When $F$ is $p$-adic and the height function is associated to a monomial, part (i) of Conjecture 1.1 is true for $GL_{n+1}$, $SL_{n+1}$, $GSp_n$, $Sp_n$, $SO(n, n)$, and $SO(n+1, n)$, where $n \geq 1$. \hfill \Box

### 2.3 The case of square-integrable representations

Here, we explain the relation between the generalized formal degree and the usual formal degree.

**Lemma 2.5.** Suppose that $\pi$ is square-integrable and that the functional-valued function $I(s)$ has meromorphic continuation to $\mathbb{C}$, then $I(0)$ is $G \times G$-invariant and the generalized formal degree of $\pi$ is equal to its usual formal degree. \hfill \Box

**Proof.** Note that the height function $\Delta(g)$ is bounded. So when $\pi$ is square-integrable, the following integral defining $I(s)$ is absolutely convergent when $\text{Re}(s) \geq 0$:

$$I(s; v_1, v_2, v_3, v_4) = \int_{G/Z} (\pi(g)v_1, v_2)(\pi(g)v_3, v_4) \Delta(g)^s \, dg, \quad v_i \in V_\pi.$$

Hence 0 is not a pole of $I(s)$ and $I(0; v_1, v_2, v_3, v_4) = \int_{G/Z} (\pi(g)v_1, v_2)(\pi(g)v_3, v_4) \, dg$. Obviously, $I(0)$ is $G \times G$-invariant; by Definition 1.2, the generalized formal degree is a constant $d(\pi)$ such that $d(\pi)I(0) = I_{st}$, that is, $d(\pi)$ is such that $\int_{G/Z} (\pi(g)v_1, v_2)(\pi(g)v_3, v_4) = d(\pi)^{-1}(v_1, v_3)(v_2, v_4)$ for all $v_i \in V_\pi$. Therefore, $d(\pi)$ is the usual formal degree. \hfill \Box
Now let the height function on $GL_n(F)$ be the explicit one given in Section 2.1. We shall prove Theorem A, and when $n = 2$, explicitly compute the formal degree factor.

3.1 $n \geq 2$

Theorem A is the consequence of the following two propositions.

**Proposition 3.1.** $I(s)$ has meromorphic continuation to $\mathbb{C}$ and its leading coefficient at $s = 0$ is $G \times G$-invariant.

**Proposition 3.2.** $F$ is $\mathbb{R}$ or a $p$-adic field; we further assume that $\pi$ is generic when $F$ is $p$-adic. Then

(i) $\text{ord}_{s=0} I(s) = -\text{ord}_{s=0} \gamma(s, \pi, \text{ad}, \psi)$;

(ii) $\lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) I(s) = d_0(\pi)^{-1} I_{st}$, where $d_0(\pi)$ is $\frac{1}{n}$ or $-\frac{1}{n}$.

The idea to prove Propositions 3.1 and 3.2 is to interpret the weighted period $I(s; v_1, v_2, v_3, v_4)$ in terms of the coefficient of a $GL_n \times GL_n$-induced representation of $GL_{2n}$. Write $\tilde{G} = GL_{2n}$ and $\tilde{P} = \tilde{M} \cdot \tilde{N}$ with

$$\tilde{M} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} \middle| a, a' \in GL_n \right\},$$

$$\tilde{N} = \left\{ \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \middle| x \in M_{n \times n} \right\},$$

then $\tilde{P}$ is a maximal parabolic subgroup of $\tilde{G}$ and its Levi subgroup is $\tilde{M} \cong GL_n \times GL_n$. We consider the representations $\pi|\det|^{s/2} \otimes \pi|\det|^{-s/2}$ of $\tilde{M}$ and their unitarily normalized induction to $\tilde{G}$, namely $I(s, \pi \otimes \pi) := \text{Ind}^{\tilde{G}}_{\tilde{P}}(\pi|\det|^{s/2} \otimes \pi|\det|^{-s/2})$.

Put $w = (\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}) \in \tilde{G}$ and denote by $M(s, w, \pi \otimes \pi) : I(s, \pi \otimes \pi) \to I(-s, w(\pi \otimes \pi))$ the associated intertwining operator. When Re$(s) \gg 0$, $M(s, w, \pi \otimes \pi)$ is defined by an integral, that is, for $g \in \tilde{G}, \phi_s \in I(s, \pi \otimes \pi)$, one has

$$M(s, w, \pi \otimes \pi)\phi_s(\tilde{g}) = \int_{M_{n \times n}} \phi_s \left( w^{-1} \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \tilde{g} \right) dx.$$

$M(s, w, \pi \otimes \pi)$ is then extended to the whole $s$-plane by meromorphic continuation.
For given $v_1, v_4 \in V_\pi$, we define a section $\phi_s \in I(s, \pi \otimes \pi)$ supported in $\tilde{P}_w \tilde{N}_w^{-1}$ by putting

$$\phi_s \left( \begin{array}{cc} 1_n & 0 \\ x & 1_n \end{array} \right) = \varphi(x) v_1 \otimes v_4,$$

where $\varphi(x)$ is given by

$$\varphi(x) = \begin{cases} 1_{M \otimes \otimes (\mathbb{Q}_p)}(x) & \text{if } F \text{ is p-adic,} \\ e^{-\pi Tr(xx')} & \text{if } F = \mathbb{R}. \end{cases}$$

Then the following lemma is the key to proof of Propositions 3.1 and 3.2.

**Lemma 3.3.** $(M(s, w, \pi \otimes \pi)\phi_s(1), v_2 \otimes v_3) = \frac{\xi_F(1)}{\xi(x)} I(s; v_1, v_2, v_3, v_4)$ when $Re(s) \gg 0$. □

**Proof.** Here $(M(s, w, \pi \otimes \pi)\phi_s(1))$ is a function on $\tilde{G}$ belonging to $I(s, w(\pi \otimes \pi))$; when evaluated at 1, $M(s, w, \pi \otimes \pi)\phi_s(1)$ is a vector in $V_\pi \otimes V_\pi$. The left-hand side of the above formula is then the pairing between two vectors of $V_\pi \otimes V_\pi$.

For $x \in \text{GL}_n(F)$, one has

$$\left( \begin{array}{cc} 1_n & x^{-1} \\ 0 & 1_n \end{array} \right) w^{-1} \left( \begin{array}{cc} 1_n & x \\ 0 & 1_n \end{array} \right) = \left( \begin{array}{cc} x^{-1} & 0 \\ 0 & x \end{array} \right) \left( \begin{array}{cc} 1_n & 0 \\ x^{-1} & 1_n \end{array} \right).$$

Hence

$$(M(s, w, \pi \otimes \pi)\phi_s(1), v_2 \otimes v_3) = \int_{\text{GL}_n(F)} \phi \left( w^{-1} \left( \begin{array}{cc} 1_n & x \\ 0 & 1_n \end{array} \right), v_2 \otimes v_3 \right) dx$$

$$= \int_{\text{GL}_n(F)} \phi \left( \left( \begin{array}{cc} 1_n & x^{-1} \\ 0 & 1_n \end{array} \right) w^{-1} \left( \begin{array}{cc} 1_n & x \\ 0 & 1_n \end{array} \right), v_2 \otimes v_3 \right) dx$$

$$= \int_{\text{GL}_n(F)} \phi \left( \left( \begin{array}{cc} x^{-1} & 0 \\ 0 & x \end{array} \right), v_2 \otimes v_3 \right) dx$$

$$= \int_{\text{GL}_n(F)} \varphi(x^{-1}) |\det(x)|^{-s-n}(\pi(x^{-1})v_1, v_2)(\pi(x)v_4, v_3) dx$$
\[ \int_{\text{GL}_n(F)} \varphi(x) |\det(x)|^s (\pi(x)v_1, v_2)(\pi(x^{-1})v_4, v_3) \, d^\times x \]

\[ \int_{\text{GL}_n(F) / F^\times} \varphi_s(x)(\pi(x)v_1, v_2)(\pi(x)v_3, v_4) \, d^\times x, \]

where \( d \times x = |\det(x)|^{-n} \, dx \) is the multiplicative Haar measure on \( \text{GL}_n(F) \) and \( \varphi_s(x) = \int_{F^\times} \varphi(zx) \, |\det(zx)|^s \, dx \). Now the lemma follows from the fact \( \varphi_s(x) = \frac{\zeta_F(ns)}{\zeta_F(1)} \Delta(x)^s \) as shown below.

When \( F \) is \( p \)-adic, a straightforward calculation shows that \( \varphi_s(x) = (1 - |\sigma|)(1 - |\sigma|^n)^{-1} |\det(x)|^s \) \( \frac{\zeta_F(ns)}{\zeta_F(1)} \Delta(x)^s \). When \( F \) is \( \mathbb{R} \), \( \varphi_s(x) \) is \( O_n(\mathbb{R}) \)-biinvariant; for \( x = \text{diag}(y_1, \ldots, y_n) \) with \( y_i > 0 \), we have

\[ \varphi_s(x) = (y_1 \ldots y_n)^s \int_{\mathbb{R}^n} e^{-\pi x^2(y_1^2 + \cdots + y_n^2)} |z|^n \, dx \]

\[ = \frac{(y_1 \ldots y_n)^s}{(y_1^2 + \cdots + y_n^2)^{n/2}} \pi^{-\frac{n}{2}} I\left( \frac{ns}{2} \right) \]

\[ = \frac{\zeta_F(ns)}{\zeta_F(1)} \Delta(x)^s. \]

Hence \( \varphi_s(x) \) is also equal to \( \frac{\zeta_F(ns)}{\zeta_F(1)} \Delta(x)^s \) when \( F = \mathbb{R} \).

Now we prove Propositions 3.1 and 3.2. For convenience, we introduce the swap isomorphism \( \text{sw} : V_\pi \otimes V_\pi \to V_\pi \otimes V_\pi, \ u \otimes v \to v \otimes u \); \( \text{sw} \) induces an \( \tilde{M} \)-isomorphism between \( I(0, \pi \otimes \pi) \) and \( I(0, \pi \otimes \pi) \), which we denote by the same notation.

**Proof of Proposition 3.1.** By Lemma 3.3, we have

\[ I(s; v_1, v_2, v_3, v_4) = \frac{\zeta_F(1)}{\zeta_F(ns)} (M(s, w, \pi \otimes \pi) \phi_3(1), v_2 \otimes v_3). \]

Because \( M(s, w, \pi \otimes \pi) \) has meromorphic continuation to \( \mathbb{C} \), so does \( I(s; -) \).

Furthermore, write \( m' = -\text{ord}_{s=0} M(s, w, \pi \otimes \pi) \) and \( M' = \lim_{s \to 0} s^{m'} M(s, w, \pi \otimes \pi) \), then \( \text{sw} \circ M' \) is a nonzero homomorphism from \( I(0, \pi \otimes \pi) \) to itself. By Tadic’s work [14, 15] on the unitary dual of general linear groups, we know that “\( \pi \) is irreducible unitary \( \Rightarrow \) \( I(0, \pi \otimes \pi) \) is irreducible unitary”. Hence \( \text{sw} \circ M' \) is a scalar multiplication and therefore \( M' = d \cdot \text{sw} \) for certain nonzero constant \( d \). It follows that

\[ \lim_{s \to 0} s^{m'-1} I(s; v_1, v_2, v_3, v_4) = \lim_{s \to 0} \frac{s^{m'-1} \zeta_F(1)}{\zeta_F(ns)} (M(s, w, \pi \otimes \pi) \phi_3(1), v_2 \otimes v_3) \]

\[ = \frac{\zeta_F(1)}{\lim_{s \to 0} s \zeta_F(ns)} \left( \lim_{s \to 0} s^{m'} M(s, w, \pi \otimes \pi) \phi_3(1), v_2 \otimes v_3 \right) \]
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\[ \lim_{s \to 0} \zeta_F(ns) (v_2 \otimes v_3) = \zeta_F(1) \]

\[ \lim_{s \to 0} \frac{\zeta_F(1)}{s} (dsw(v_1 \otimes v_4), v_2 \otimes v_3) = \frac{d\zeta_F(1)}{\lim_{s \to 0} s\zeta_F(ns)} (v_1, v_3)(v_2, v_4). \]

In other words, \( \text{ord}_{s=0} I(s) = m' - 1 \) and the leading coefficient of \( I(s) \) at \( s = 0 \) is \( d(\pi)^{-1} I_{st} \) with \( d(\pi) = \frac{\lim_{s \to 0} \zeta_F(ns)}{d\zeta_F(1)}. \)

\[ \text{Proof of Proposition 3.2.} \quad \text{The intertwining operator } M(s, w, \pi \otimes \pi) \text{ can be normalized by the } \gamma \text{ factor of } \pi \otimes \pi^\vee. \text{ According to [1, 13], when } F = \mathbb{R}, \gamma(s, \pi \times \pi^\vee, \psi) M(s, w, \pi \otimes \pi) \text{ is holomorphic at } s = 0 \text{ and } \gamma(0, \pi \times \pi, \psi) M(0, w, \pi \otimes \pi) \text{ is unitary. Hence, } sw \circ (\gamma(0, \pi \times \pi^\vee, \psi) M(0, w, \pi \otimes \pi)) \text{ is a nontrivial unitary endomorphism of the irreducible } GL_{2n} \text{-representation } I(0, \pi \otimes \pi). \text{ Because } I(0, \pi \otimes \pi) \text{ is irreducible, it is forced that } sw \circ (\gamma(0, \pi \times \pi^\vee, \psi) M(0, w, \pi \otimes \pi)) = d' \text{Id, where } d' \text{ is a number of norm 1.} \]

On the other hand, the normalization by root numbers satisfies the property

\[ \gamma(s, \pi \otimes \pi^\vee, \psi) M(s, w, \pi \otimes \pi) \cdot \gamma(-s, \pi \otimes \pi^\vee, \psi) M(-s, w^{-1}, \pi \otimes \pi) = \text{Id}. \]

Setting \( s = 0 \), we get \( d'^2 = 1 \) and hence \( d' = \pm 1 \).

Now we proceed to prove Proposition 3.2. By Lemma 3.3, we have

\[ (\gamma(s, \pi \times \pi^\vee, \psi) M(s, w, \pi \otimes \pi) \phi_s(1), v_2 \otimes v_3)) = \frac{\zeta_F(1 - s)\zeta_F(ns)}{\zeta_F(1)\zeta_F(s)} \gamma(s, \pi, \text{ad}, \psi) I(s; v_1, v_2, v_3, v_4). \]

It follows that

\[ \lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) I(s; v_1, v_2, v_3, v_4) = \lim_{s \to 0} \frac{\zeta_F(1)\zeta_F(s)}{\zeta_F(1 - s)\zeta_F(ns)} (\gamma(s, \pi \times \pi^\vee, \psi) M(s, w, \pi \otimes \pi) \phi_s(1), v_2 \otimes v_3) = nd'(v_1, v_3)(v_2, v_4). \]
So \( \lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) I(s) = nd'' I_{st} \), which implies the two assertions in Proposition 3.2.

Propositions 3.1 and 3.2, when combined, give the following theorem, which is exactly Theorem A in the introduction.

**Theorem 3.4.**

1. Parts (i) and (ii) of Conjecture 1.1 are true.
2. \( F \) is \( \mathbb{R} \) or a \( p \)-adic field; we further assume that \( \pi \) is generic when \( F \) is \( p \)-adic. Then part (iii) of Conjecture 1.1 is true and the formal degree factor of \( \pi \) is either \( \frac{1}{n} \gamma(s, \pi, \text{ad}, \psi) \) or \( -\frac{1}{n} \gamma(s, \pi, \text{ad}, \psi) \).

3.2 \( n=2 \)

We compute here the formal degree factor of an arbitrary irreducible unitary representations of \( GL_2(F) \). As is well known, if \( \pi \) is such a representation, then it is

1. either a square-integrable representation,
2. or a principal series representations \( \rho(\chi_1, \chi_2) \) (as in [10]), where \( \chi_1 \) and \( \chi_2 \) are unitary or \( \chi_1 = \chi | \cdot |^{\lambda}, \chi_2 = \chi | \cdot |^{-\lambda} \) with \( \chi \) unitary and \( \lambda \in (0, \frac{1}{2}) \),
3. or a one-dimensional representation \( \chi(\det) \) with \( \chi \) unitary.

Conjecture 1.1 holds in all three cases: cases (1) and (2) are covered by Theorem 3.4 and case (3) can be verified directly.

**Proposition 3.5.** If \( n=2 \), the formal degree factor of \( \pi \) is \( d_\pi(s) = d_0(\pi) \gamma(s, \pi, \text{ad}, \psi) \) with

\[
d_0(\pi) = \left\{ \begin{array}{ll}
\frac{\pi(-1)}{2}, & \text{if } \pi \text{ is square-integrable,} \\
1, & \text{if } \pi \text{ is a principal series that is unitary (continuous series or complement series),} \\
\frac{-1}{2}, & \text{if } \pi \text{ is a one-dimensional character.}
\end{array} \right.
\]

Proposition 3.5 is the union of three lemmas, with each for one of the three cases of \( \pi \). For convenience, we fix the following notations in this subsection: \( K = GL_2(O_F) \text{ or } O(2, \mathbb{R}) \), \( \overline{K} = K/\{\pm I_2\} \), \( N = \left\{ \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \right\} \), \( w_1 = \left( \begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix} \right) \) and \( w_0 = I_2 \).
Lemma 3.6. \( d_0(\pi) = \frac{\pi(-1)}{2} \) when \( \pi \) is square-integrable. □

Proof. When \( \pi \) is square-integrable, it is generic. By Proposition 3.2, we have \( d(\pi) = d_0(\pi)\gamma(0, \pi, \text{ad}, \psi) > 0 \) with \( d_0(\pi) = \pm \frac{1}{2} \). So \( \gamma(0, \pi, \text{ad}, \psi) \) is a real number and the sign of \( d_0(\pi) \) is the same as the sign of \( \gamma(0, \pi, \text{ad}, \psi) \). Because the adjoint \( L \)-factor of \( \pi \) can be made explicit (cf. [3, 9]), one sees that the sign of \( \gamma(0, \pi, \text{ad}, \psi) \) is the sign of \( \epsilon(1/2, \pi, \text{ad}, \psi) \). By [11, Theorem 1], one has \( \epsilon(1/2, \pi, \text{ad}, \psi) = \pi(-1) \). Hence the sign of \( d_0(\pi) \) is \( \pi(-1) \) and \( d_0(\pi) = \frac{\pi(-1)}{2} \). ([11, Theorem 1] applies to standard \( L \)-functions of \( \text{SL}(2) \)-representations and the resulting formula is then interpreted in terms of adjoint \( L \)-functions of \( \text{GL}(2) \)-representations.) □

Lemma 3.7. If \( \pi = \chi \circ \text{det} \) with \( \chi \) unitary, then \( \text{ord}_{s=0} I(s) = 0 \) and \( d_0(\pi) = -\frac{1}{2} \). □

Proof. According to [3], we have \( \gamma(0, \pi, \text{ad}, \psi) = -\frac{\zeta_F(2)}{\zeta_F(-1)} \) when \( \pi = \chi(\text{det}) \). Since \( V_\pi \) is one dimensional, it suffices to pick a vector \( v \in V_\pi \) of norm 1 and prove that \( I(s; v, v, v, v)|_{s=0} = \frac{2\zeta_F(-1)}{\zeta_F(2)} \). Actually, it is straightforward to compute that

\[
I(s; v, v, v, v) = \begin{cases} 
\frac{\zeta_F(s-1)\zeta_F(s)}{\zeta_F(2)\zeta_F(2s)}, & \text{if } F \text{ is } p\text{-adic}, \\
-2\pi^2 \frac{2^{1-s}}{1-s}, & \text{if } F \text{ is } \mathbb{R}.
\end{cases}
\]

Hence we have \( I(s; v, v, v, v)|_{s=0} = \frac{2\zeta_F(-1)}{\zeta_F(2)} \) either when \( F \) is \( p\)-adic or \( \mathbb{R} \). □

Lemma 3.8. If \( \pi \) is an irreducible principal series that is unitary, then \( \lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) \cdot I(s) = 2I_{st} \). □

Unitary irreducible principal series of \( \text{GL}_2(F) \) are of the form \( \pi = \rho(\chi_1, \chi_2) \) (cf. [10]), where either \( \chi_1 \) and \( \chi_2 \) are unitary or \( \chi_1 = \chi | \cdot |^{1}, \chi_2 = \chi | \cdot |^{-\lambda} \) with \( \chi \) unitary and \( \lambda \in (0, \frac{1}{2}) \). We shall use the asymptotic formula of matrix coefficients to prove Lemma 3.8, and for simplicity, only present the argument when \( F \) is \( p\)-adic. As a compensation, we present in Section 4.3.2 a complete proof of a similar lemma for unitary irreducible principal series of \( \tilde{\text{SL}}_2(\mathbb{R}) \). Here are a few basic facts on \( \rho(\chi_1, \chi_2) \) when
$F$ is $p$-adic:

1. $\rho(\chi_1, \chi_2)$ is the right-translation action of $GL_2(F)$ on the space $B(\chi_1, \chi_2)$ of locally constant functions $f$ on $GL_2(F)$ satisfying

$$f\left(\begin{pmatrix} a_1 & n \\ a_2 \end{pmatrix} g\right) = \chi_1(a_1) \chi_2(a_2) \left| \frac{a_1}{a_2} \right|^{1/2} f(g).$$

2. On $B(|\cdot|^{1/2}, |\cdot|^{-1/2})$, the functional $f \to \int_K f(k) \, dk = \frac{m(K)}{1 + |\sigma|} \int_F f(wn) \, dn$ is $G$-invariant; between $B(\chi_1, \chi_2)$ and $B(\chi_1^{-1}, \chi_2^{-1})$, the pairing

$$\langle f_1, f_2 \rangle = \int_K f_1(k) f_2(k) \, dk = \frac{m(K)}{1 + |\sigma|} \int_N f_1(wn) f_2(wn) \, dn$$

is $G$-invariant. Note that for $f_1 \in B(\chi_1, \chi_2)$ and $f_2 \in B(\chi_1', \chi_2')$, the pairing $\langle f_1, f_2 \rangle_K = \int_K f_1(k) f_2(k) \, dk$ is usually only $K$-invariant but not $G$-invariant.

3. Denote by $r_{w_0}$ and $r_{w_1}$ the intertwining operators associated to $w_0$ and $w_1$: $r_{w_0}$ is the identity map on $B(\chi_1, \chi_2)$ and $r_{w_1}$ is formally defined by

$$r_{w_1} : B(\chi_1, \chi_2) \to B(\chi_2, \chi_1), f(g) \to r_{w_1} f(g) = \int_F f(wng) \, dn.$$

$r_{w_1}$ can be defined via meromorphic continuation for all $\chi_1$ and $\chi_2$ except when $\chi_1 \chi_2^{-1} = 1$; it satisfies $r_{w_1} r_{w_1}^{-1} = \frac{1}{\gamma(0, \chi_1, \chi_2, \psi) \gamma(0, \chi_1^{-1}, \chi_2, \psi)}$ and $\langle r_{w_1} f_1, f_2 \rangle = \langle f_1, r_{w_1}^{-1} f_2 \rangle$.

The following two lemmas on the asymptotic behavior of matrix coefficients are easy to prove and hence we skip the proof.

**Lemma 3.9.** Suppose that $F$ is $p$-adic and $\chi_1 \chi_2^{-1} \neq 1$. Given $f_1, f_2 \in B(\chi_1, \chi_2)$ and $f_2 \in B(\chi_1^{-1}, \chi_2^{-1})$, there exists $\epsilon \in (0, 1]$ such that when $|a| < \epsilon$, one has

$$\left\langle \pi \left(\begin{array}{c} a \\ 1 \end{array} \right) f_1, f_2 \right\rangle = \frac{|a| \frac{m(K)}{1 + |\sigma|} (\chi_2(a) r_{w_1} f_1(wn) r_{w_0} f_2(wn) + \chi_1(a) r_{w_0} f_1(wn) r_{w_1} f_2(wn^{-1}))}{\gamma(0, \chi_1, \chi_2, \psi) \gamma(0, \chi_1^{-1}, \chi_2, \psi)}. \hfill \square$$
Lemma 3.10. Suppose that $F$ is $p$-adic and $\chi^2 = 1$. Given $f_1, f_2 \in B(\chi_1, \chi_2)$, there exists $\epsilon \in (0, 1)$ and a constant $A(f_1, f_2)$ depending on $f_1$ and $f_2$ such that when $|a| < \epsilon$, one has

$$\left\langle \pi \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right), f_1, f_2 \right\rangle = \frac{|a|^\frac{1}{2} m(K)}{1 + |\sigma|} \chi(a)(A(f_1, f_2) + (1 - |\sigma|)v_F(a)f_1(w_0)f_2(w_1)).$$

Proof of Lemma 3.8. As explained, we present the proof when $F$ is $p$-adic. For simplicity, we assume that $\pi$ is a continuous series, that is, $\pi = \rho(\chi_1, \chi_2)$ with $\chi_1$ and $\chi_2$ unitary; the proof for a complementary series is similar. Since $\chi_1$ and $\chi_2$ are unitary, the Hermitian paring on $\pi = \rho(\chi_1, \chi_2)$ can be taken as $(f_1, f_2) = \langle f_1, \bar{f}_2 \rangle$; for convenience, write $k f$ for $\pi(k) f$ when $k \in K$. Our proof distinguishes two cases.

(I) $\chi_1 \chi_2^{-1} \neq 1$. Choose $\sigma$ such that $\chi_1 \chi_2^{-1}(\sigma) \neq 1$; for given $f_i (1 \leq i \leq 4) \in B(\chi_1, \chi_2)$, apply Lemma 3.9 to find $n_0 \geq 0$ such that for all $k_1, k_2 \in K$ and $v(a) \geq n_0$, one has

$$\left( \pi \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right) \right)^{k_1 f_i, k_2 f_j} = \frac{|a|^\frac{1}{2} m(K)}{1 + |\sigma|}\left( \begin{matrix} \chi_2(a) r_{w_1}^{k_1 f_i(w_0)} r_{w_0}^{k_2 f_j(w_1)} + \chi_1(a) r_{w_0}^{k_1 f_i(w_0)} r_{w_1}^{k_2 f_j(w_1)} \end{matrix} \right)$$

Put $G_n = ZK (\sigma^n) K = \{ g \in G | \Delta(g) = |\sigma|^n \}$. Then $G = \bigcup_{n \geq 0} G_n$ and

$$I(s; f_1, f_2, f_3, f_4) = \sum_{n \geq 0} |\sigma|^n \int_{G_n \backslash F^\times} (\pi(g) f_1, f_2) (\pi(g) f_3, f_4) \, dg.$$ 

When $n \geq n_0$, we apply the asymptotic formula to get

$$\int_{G_n \backslash F^\times} (\pi(g) f_1, f_2) (\pi(g) f_3, f_4) \, dg$$

$$= \left( 1 + |\sigma| \right)^{-n} \int_{K \times K} \left( \pi \left( \begin{pmatrix} \sigma^n \\ 1 \end{pmatrix} \right)^{k_1 f_i, k_2 f_j} \right) (\pi \left( \begin{pmatrix} \sigma^{-1} \\ 1 \end{pmatrix} \right)^{k_1 f_i, k_2 f_j}) \, dk_1 \, dk_2$$

$$= (1 - |\sigma|)(r_{w_1} f_1, r_{w_1} f_3)(r_{w_0} f_2, r_{w_0} f_4) + (r_{w_0} f_1, r_{w_0} f_3)(r_{w_1} f_2, r_{w_1} f_4)$$

$$+ \chi_1 \chi_2(-1) \chi_1^{-1} \chi_2(\sigma)^n (r_{w_1} f_1, r_{w_0} f_3)(r_{w_0} f_2, r_{w_1} f_4)$$

$$+ \chi_1 \chi_2^{-1}(\sigma)^n \chi_1 \chi_2(-1)(r_{w_0} f_1, r_{w_1} f_3)(r_{w_1} f_2, r_{w_0} f_4).$$
It follows that $I(s; f_1, f_2, f_3, f_4)$ is a rational function of $|\sigma|^s$ and its main term near $s = 0$ is

\[
\frac{1 - |\sigma|}{1 - |\sigma|^s} (r_{w_1} f_1, r_{w_1} f_3) (r_{w_0} f_2, r_{w_0} f_4) + (r_{w_0} f_1, r_{w_0} f_3) (r_{w_1} f_2, r_{w_1} f_4) = \frac{2}{\gamma(0, \chi_1^{-1}, \psi) \gamma(0, \chi_2^{-1}, \psi)} \frac{1 - |\sigma|}{1 - |\sigma|^s} (f_1, f_3)(f_2, f_4).
\]

Therefore, $\lim_{s \to 0} \gamma(s, \pi, \alpha, \psi) I(s; f_1, f_2, f_3, f_4) = 2 (f_1, f_3)(f_2, f_4)$.

(II) $\chi_1 = \chi_2 = \chi$ and $\chi^2 = 1$. Given $f_i \in B(\chi, \chi)(1 \leq i \leq 4)$, we apply Lemma 3.10 to find $n_0 \geq 1$ such that for all $k_1, k_2 \in K$ and $v(a) \geq n_0$, one has

\[
\left( \pi \left( \begin{array}{cc} a \\
1 \end{array} \right) \right)^{k_1 f_1, k_2 f_j} = \frac{|a| \times m(K)}{1 + |\sigma|} \chi(a) (A(k_1 f_1, k_2 f_j)
\]

\[
+ (1 - |\sigma|) v_F(a) f_i (e)(k_2 f_j(w_1)).
\]

Then when $n \geq n_0$, we apply the asymptotic formula to get

\[
\int_{G_{n/F}} \frac{\pi(g f_1, f_2) \pi(g f_3, f_4)}{m(K)} \frac{(1 + |\sigma|)|\sigma|^{-n}}{m(K)} \int_{K \times K} \pi \left( \begin{array}{cc} \sigma^n \\
1 \end{array} \right)^{k_1 f_1, k_2 f_2} dk_1 dk_2
\]

\[
= (1 - |\sigma|) \int_{K \times K} A(k_1 f_1, k_2 f_2) A(k_3 f_3, k_4 f_4) dk_1 dk_2
\]

\[
+ n(1 - |\sigma|)^2 \int_{K \times K} (A(k_1 f_1, k_2 f_2) k_1 f_3 (w_0) k_2 f_4 (w_1)) \frac{1}{A(k_1 f_1, k_2 f_2)} dk_1 dk_2
\]

\[
+ n^2 (1 - |\sigma|)^3 (f_1, f_3)(f_2, f_4).
\]
It follows that $I(s; f_1, f_2, f_3, f_4) = \sum_{n \geq 0} \int_{G_{n}/F} (\pi(g) f_1, f_2) \pi(g) (f_3, f_4) \, dg$ is a rational function of $|\sigma|^s$ and its main term near $s = 0$ is

$$\langle f_1, f_3 \rangle \langle f_2, f_4 \rangle (1 - |\sigma|)^3 \sum_{n \geq 0} n^2 |\sigma|^n = \langle f_1, f_3 \rangle \langle f_2, f_4 \rangle (1 - |\sigma|)^3 \frac{|\sigma|^s (1 + |\sigma|^2)}{(1 - |\sigma|^s)^3}.$$

Because $\gamma(s, \pi, \text{ad}, \psi) = \frac{(1 - |\sigma|^s)^3}{(1 - |\sigma|^s)^3 - |\sigma|^s}$ in this case, we have $\lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) = 1$ too.

**Theorem 4.1.** $I(s)$ has meromorphic continuation to $C$ and its leading coefficient at $s = 0$ is $\tilde{S}L_2(F)$-invariant. \qed

According to [17], $\tilde{\pi}$ has three possibilities:

(i) an even Weil representation $\omega_{\psi'}^+$ attached to a nontrivial character $\psi'$ of $F$;
(ii) a unitary representation $\tilde{\pi}(\mu)$ (as in [17]) induced from the Borel subgroup of $G$. In this case, $\mu$ is either a unitary character or of the form $\chi_{E/F} | \cdot |^s$ with $\chi_{E/F}$ quadratic and $s \in (0, \frac{1}{2})$.
(iii) a square-integrable representation.

We shall verify Theorem 4.1 for these three cases in Sections 4.2–4.4 separately.

### 4.1 The structure of $\tilde{S}L_2(F)$

To describe the structure of $\tilde{S}L_2(F)$, we identify $\tilde{S}L_2(F)$ with $SL_2(F) \times \mathbb{Z}/2\mathbb{Z}$ as sets.
4.1.1 The group law

For

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(F), \]

set

\[ j(g) = \begin{cases} c & \text{if } c \neq 0, \\
 \quad a & \text{if } c = 0. \end{cases} \]

The group law of \( \tilde{\text{SL}}_2(F) \) can be described as \( [g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 \cdot g_2, \epsilon(g_1, g_2)\epsilon_1\epsilon_2] \), where \( \epsilon(g_1, g_2) = \langle j(g_1)j(g_1g_2)j(g_2), j(g_1)j(g_1g_2) \rangle_F \) and \( \langle \cdot | \cdot \rangle_F \) is the Hilbert symbol over \( F \).

4.1.2 The topology

When \( F \) is \( p \)-adic, we use the identification \( \tilde{\text{SL}}_2(F) = \text{SL}_2(F) \times \mathbb{Z}/2\mathbb{Z} \) to give \( \tilde{\text{SL}}_2(F) \) the product topology; when \( F = \mathbb{R} \), we identify \( \tilde{\text{SL}}_2(F) \) homeomorphically with \( B(\mathbb{R}) \times (\text{SO}_2(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}) \), where \( B(\mathbb{R}) = \left\{ \begin{pmatrix} y & x \\ y \end{pmatrix} \right| y \in \mathbb{R}^\times, x \in \mathbb{R} \right\} \) takes the usual topology and \( \text{SO}_2(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z} \) is topologically identified with \( \mathbb{R}/4\pi\mathbb{Z} \) via the map

\[ \tilde{\gamma} : \mathbb{R}/4\pi\mathbb{Z} \rightarrow \text{SO}_2(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}, \]

\[ [\theta] \rightarrow \left[ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, 1_{(-\pi, \pi]} + 4\pi\mathbb{Z}(\theta) - 1_{(\pi, 3\pi]} + 4\pi\mathbb{Z}(\theta) \right], \]

4.1.3 The subgroups

For a subgroup \( H \) of \( \text{SL}_2(F) \), let \( \tilde{H} \) denote its preimage in \( \tilde{\text{SL}}_2(F) \). Let \( K = \text{SL}_2(O_F) \) when \( F \) is \( p \)-adic and \( \text{SO}_2(\mathbb{R}) \) when \( F = \mathbb{R} \), then \( \tilde{K} \) is a maximal compact subgroup of \( \tilde{\text{SL}}_2(F) \). Let \( B \) be the Borel subgroup consisting of upper triangular matrices of \( \text{SL}_2(F) \), \( N \) the nilpotent radical of \( B \), and \( A \) the diagonal subgroup of \( B \).

4.1.4 The Weil constant

For a nontrivial additive character \( \psi' \) of \( F \), the Weil constant \( \gamma(\psi') \) is a complex number of norm 1 such that the following identity holds for all Bruhat–Schwartz functions
\[ f(x) \text{ on } F, \]
\[ \int_F f(x)\psi'(x^2) \, dx = \gamma(\psi') \int_F \hat{f}(x)\psi'(-x^2) \, dx. \]

Here \( \hat{f}(x) = \int_F f(y)\psi'(2xy) \, dy \) is the Fourier transform of \( f \) with respect to the character of second degree \( x \to \psi'(x^2) \), and the measures \( dx \) and \( dy \) are Haar measures of \( F \) such that \( \hat{f}(x) = f(-x) \). According to [19], \( \gamma(\psi) \) is an eighth root of unity.

### 4.1.5 Abbreviations

For \( g \in \text{SL}_2(F) \), we take it as \( [g, 1] \) when we want to consider it as an element of \( \widetilde{\text{SL}}_2(F) \).

For \( a \in F^\times \), we write \( a \) for \( \left[ \begin{array}{c} a \\ 1 \end{array} \right] \) and \( n \in F \), we write \( n \) for \( \left[ \begin{array}{c} 1 \\ n \end{array} \right] \). If \( (\tilde{\pi}, V_{\tilde{\pi}}) \) is a representation of \( \tilde{\text{SL}}_2(F) \), we write \( \pi v \) or \( g \circ v \) for \( \tilde{\pi}(\tilde{\pi})v \) when \( g \in \text{SL}_2(F) \) and \( v \in V_{\tilde{\pi}} \).

### 4.2 \( \tilde{\pi} \) is an even Weil representation

The Weil representation \( \omega_{\psi'} \) is an action of \( G \) on \( S(F) \), the space of Bruhat–Schwartz functions on \( F \), according to the following rules:

(i) \[ \omega_{\psi'} [I_2, \epsilon] \phi(x) = \epsilon \phi(x); \]

(ii) \[ \omega_{\psi'} \left[ \begin{array}{c} a \\ a^{-1} \end{array} \right], 1 \phi(x) = \chi_{\psi'}(a)|a|^{\frac{1}{2}} \phi(ax); \]

(iii) \[ \omega_{\psi'} \left[ \begin{array}{c} 1 \\ n \end{array} \right], 1 \phi(x) = \psi'(nx^2) \phi(x); \]

(iv) \[ \omega_{\psi'} [w, 1] \phi(x) = \gamma(\psi') \int_F \phi(y)\psi'(2xy) dy, \text{ where } w = (-1)^1 \text{ and } dx \text{ is the self-dual measure of } F \text{ with respect to the character of second order } x \to \psi'(x^2). \]

\( \omega_{\psi'} \) is unitary with respect to the Hermitian pairing \( \langle \phi_1, \phi_2 \rangle = \int_F \phi_1(x)\overline{\phi_2(x)} \, dx \); \( \omega_{\psi'} = \omega_{\psi'}^+ \oplus \omega_{\psi'}^- \), where \( \omega_{\psi'}^+ \) and \( \omega_{\psi'}^- \) consist of even and odd functions in \( S(F) \), respectively, and are both irreducible. Two Weil representations \( \omega_{\psi'} \) and \( \omega_{\psi''} \) are equivalent if and only if there exists \( b \) such that \( \psi''(x) = \psi'(b^2x) \); the set of equivalence classes of even Weil representations over \( F \) are \( \{ \omega_{\psi_{\phi,a}} \mid [a] \in F^\times/F^\times_2 \} \).

In this subsection, we prove the following proposition, which is Theorem 4.1 in the case of \( \tilde{\pi} = \omega_{\psi'}^+ \). We give a detailed proof when \( F \) is \( p \)-adic and guidelines when \( F \) is \( \mathbb{R} \).
Proposition 4.2. When $\tilde{\pi} = \omega^\dagger$, $I(s)$ has meromorphic continuation to $\mathbb{C}$. It has no pole at $s = 0$ and $I(0) = d(\tilde{\pi})^{-1}I_{st}$ with $d(\tilde{\pi}) = \frac{\varphi(2)}{2|\varphi(1)|}$. □

4.2.1 $F$ is $p$-adic

The following asymptotic formula of matrix coefficients is easy to check.

Lemma 4.3. Given $\phi_1, \phi_2 \in \omega \psi'$, there exists $n_0 \geq 0$ such that when $n \geq n_0$, one has

$$(\omega \psi'(\omega^n)\phi_1, \phi_2) = \chi_{\psi'}(\omega^n)^{-1}|\sigma|^n/2\phi_1(0)\omega \psi'(\omega^n)\phi_2(0).$$

□

Proof. Skipped. ■

Now we verify Proposition 4.2 in the $p$-adic case step by step.

Lemma 4.4. When $\tilde{\pi} = \omega^\dagger$, $I(s)$ has meromorphic continuation to $\mathbb{C}$ with no pole at $0$. □

Proof. Given $\phi_i \in \omega^\dagger (1 \leq i \leq 4)$, we apply Lemma 4.3 to find $n_0$ such that

$$(\omega \psi'(\omega^n)\phi_i, \phi_j) = \chi_{\psi'}(\omega^n)^{-1}|\sigma|^n/2\phi_i(0)\omega \psi'(\omega^n)\phi_j(0), (i, j) = (1, 2), (3, 4)$$

for all $n \geq n_0$ and $\tilde{k}_i \in \tilde{K}$. It follows that

$$I(s; \phi_1, \phi_2, \phi_3, \phi_4) = \sum_{n \geq 0} \int_K (\omega^n_{\sigma, \sigma^{-n}}) \frac{m(K)}{m(K)^2} \left(\omega^\dagger_{\psi}(g, 1)\phi_1, \phi_2\right) \left(\omega^\dagger_{\psi}(g, 1)\phi_3, \phi_4\right) \Delta(g)^s \, dg$$

$$= \sum_{n \geq 0} \frac{m \left(\frac{m^n}{m^{-n}}\right)}{m(K)^2} \int_{K \times K} (\omega^\dagger_{\psi}(\omega^n)\phi_1, \phi_2) \times (\omega^\dagger_{\psi}(\omega^n)\phi_3, \phi_4) |\sigma|^n \, dk_1 \, dk_2$$

$$= \sum_{0 \leq n < n_0} (\cdot) + \sum_{n \geq n_0} (\cdot)$$

$$= \sum_{0 \leq n < n_0} (\cdot) + \sum_{n \geq n_0} |\sigma|^{ns-1} \left(1 + |\sigma|^{n-1} \right)$$
\[
\times \int_K k\phi_1(0)^k\phi_3(0) \, dk_1 \int_K u^k\phi_2(0)^u\phi_4(0) \, dk_2 \]
\[
= \sum_{0 \leq n < n_0} (-1)^{n_0} \frac{1 + |\omega|}{1 - |\omega|^{s-1}} \frac{1}{m(K)} \lambda(\phi_1, \phi_3) \lambda(\phi_2, \phi_4).
\]

Here \( \lambda(\cdot, \cdot) \) denotes the functional \( \lambda(\phi_1, \phi_2) = \int_K k\phi_1(0)^k\phi_2(0) \, dk \). This expression tells that \( (1 - |\omega|^{s-1})I(s) \) has holomorphic continuation to \( \mathbb{C} \), hence \( I(s) \) has meromorphic continuation to \( \mathbb{C} \) and 0 is not its pole. 

**Lemma 4.5.** When \( \tilde{\pi} = \omega_{\psi}' \), \( I(0) \) is \( G \times G \)-invariant. 

**Proof.** \( I(0) \) is \( \tilde{K} \times \tilde{K} \)-invariant because \( I(s) \) is so when \( \text{Re}(s) \gg 0 \). Hence, it suffices to show that \( I(0; \phi_1, \phi_2, \phi_3, \phi_4) = I(0; \omega_{\psi}(\omega)\phi_1, \phi_2, \omega_{\psi}(\omega)\phi_3, \phi_4) = I(0; \phi_1, \omega_{\psi}(\omega)\phi_2, \phi_3, \omega_{\psi}(\omega)\phi_4) \). By symmetry, one only needs to verify the first equality. Making a change of variable, one easily sees that

\[
I(s; \omega_{\psi}(\omega)\phi_1, \phi_2, \omega_{\psi}(\omega)\phi_3, \phi_4) = \int_{SL_2(F)} (\omega_{\psi}(g)\phi_1, \phi_2)\omega_{\psi}(g)\phi_3, \phi_4) \Delta \left( g \begin{pmatrix} \omega & -1 \\ \omega & \omega \end{pmatrix} \right)^s \, dg.
\]

The key is to determine the values of \( \Delta \left( g \begin{pmatrix} \omega & -1 \\ \omega & \omega \end{pmatrix} \right) \) on \( SL_2(F) \). For

\[
g = k_2^{-1} \begin{pmatrix} \omega^n & -n \\ \omega^{-n} & \omega \end{pmatrix} k_1,
\]

we have

\[
\Delta \left( g \begin{pmatrix} \omega & -1 \\ \omega & \omega \end{pmatrix} \right) = \Delta \left( \begin{pmatrix} \omega^n & -n \\ \omega^{-n} & \omega \end{pmatrix} k_1 \begin{pmatrix} \omega & -1 \\ \omega & \omega \end{pmatrix} \right);
\]

writing \( k_1 = \begin{pmatrix} u_1 & u_2 \\ u_1 & u_2 \end{pmatrix} \), we further have

\[
\Delta \left( g \begin{pmatrix} \omega & -1 \\ \omega & \omega \end{pmatrix} \right) = \begin{cases} 
|\omega|^{n+1} & \text{if } v_F(u_{21}) = 0, \\
|\omega|^n & \text{if } v_F(u_{21}) = 1, \\
|\omega|^{n-1} & \text{if } v_F(u_{21}) \geq 2.
\end{cases}
\]
This observation suggests us to decompose $K$ as the disjoint union of $K_1, K_2,$ and $K_3$ according to whether $v_F(u_1)$ is 0, 1, or $\geq 2$. Then $\Delta(g(\sigma^{-1}\sigma))$ is constant on $K(\sigma^n, \sigma^{-n}) K_j$ for each $j = 1, 2, 3$.

Now let $n_0$ still be the number such that the asymptotic formulas of matrix coefficients work as in the proof of previous lemma. We decompose $I(s; \omega_1^+, (\sigma) v_1, v_2, \omega_3^+, (\sigma) v_3, v_4)$ into two parts $I_0(s)$ and $I_1(s)$, where

$$I_0(s) = \sum_{0 \leq n < n_0} \int_{K(\sigma^n, \sigma^{-n}) K} (\omega_1^+(g) \phi_1, \phi_2)(\omega_3^+(g) \phi_3, \phi_4) \Delta \left( g \left( \sigma^{-1}\sigma \right) \right)^s dg,$$

$$I_1(s) = \sum_{n \geq n_0} \int_{K(\sigma^n, \sigma^{-n}) K} (\omega_1^+(g) \phi_1, \phi_2)(\omega_3^+(g) \phi_3, \phi_4) \Delta \left( g \left( \sigma^{-1}\sigma \right) \right)^s dg$$

$$= \sum_{n \geq n_0} \frac{m(K(\sigma^n, \sigma^{-n}) K)}{m(K)^2} \int_{K \times K} (\omega_1^+(\sigma^n k) \phi_1, k_2 \phi_2)(\omega_3^+(\sigma^n k) \phi_3, k_4 \phi_4)$$

$$\cdot \Delta \left( \begin{pmatrix} \sigma^n \\ \sigma^{-n} \end{pmatrix} k_1 \begin{pmatrix} \sigma^{-1} \\ \sigma^1 \end{pmatrix} \right) dk_1 dk_2.$$

$I_1(s)$ can be decomposed further as $I_{1,1}(s) + I_{1,2}(s) + I_{1,3}(s)$, where for each $j = 1, 2, 3$, one has

$$I_{1,j}(s) = \sum_{n \geq n_0} \frac{m(K(\sigma^n, \sigma^{-n}) K)}{m(K)^2} \int_{K \times K_j} (\omega_1^+(\sigma^n k) \phi_1, k_2 \phi_2)(\omega_3^+(\sigma^n k) \phi_3, k_4 \phi_4)$$

$$\cdot \Delta \left( \begin{pmatrix} \sigma^n \\ \sigma^{-n} \end{pmatrix} k_1 \begin{pmatrix} \sigma^{-1} \\ \sigma^1 \end{pmatrix} \right) dk_1 dk_2,$$

Particularly,

$$I_{1,1}(s) = \sum_{n \geq 0} \frac{m(K(\sigma^n, \sigma^{-n}) K)}{m(K)^2} \int_{K \times K_1} (\omega_1^+(\sigma^n k) \phi_1, k_2 \phi_2)(\omega_3^+(\sigma^n k) \phi_3, k_4 \phi_4) |\sigma|^{(n+1)s} dk_1 dk_2.$$

The functions $I_{1,1}(s)$, $I_{1,2}(s)$, and $I_{1,3}(s)$ behave just like $I(s; \phi_1, \phi_2, \phi_3, \phi_4)$, with meromorphic continuation to $\mathbb{C}$ and being holomorphic at $s = 0$. 
The key observation is that

\[ I(s; \phi_1, \phi_2, \phi_3, \phi_4) = \sum_{0 \leq n < n_0} \int_{K} (\omega^n_{\psi} (g) \phi_1, \phi_2)(\omega^n_{\psi} (g) \phi_3, \phi_4) \Delta(g)^s \, dg \]

\[ + I_{1,1}(s)|\sigma|^{-s} + I_{1,2}(s) + I_{1,3}(s)|\sigma|^s. \]

Evaluating this equation at \( s = 0 \), we obtain that

\[ I(0; \phi_1, \phi_2, \phi_3, \phi_4) = I_0(0; \phi_1, \phi_2, \phi_3, \phi_4). \]

To determine the ratio of \( I(0) \) to \( I_{st} \), we need the following easy to verify lemma.

Lemma 4.6. In \( K = \text{SL}_2(O_F) \), we put \( N_K = N \cap K \), \( A_K = A \cap K \), and \( N_{K,1} = \{(1, n) \mid n \in \sigma O_F \}. \) Then \( K = N_K w A_K N_K \cup w N_{K,1} w A_K N_K. \)

Proof. Skipped.

Lemma 4.7. When \( \tilde{\pi} = \omega^+_{\psi}(1 \leq i \leq 4) \), we have \( I(0) = d(\tilde{\pi})^{-1} I_{st} \), where \( d(\tilde{\pi}) = \frac{\xi_F(2)}{2|2|F_{\psi}(-1)}. \)

Proof. Note that \( \frac{\xi_F(2)}{2|2|F_{\psi}(-1)} = - \frac{1}{2|2|F_{\psi}(1+|\sigma|)}. \) Let \( \phi(x) \) be the characteristic function of \( O_F \), then \( (\phi, \phi) = |2|F/2 \) and it suffices to show that \( I(0; \phi, \phi, \phi, \phi) = -2|2|F_{\psi}(1+|\sigma|). \)

We shall compute \( I(0; \phi, \phi, \phi, \phi) \) according to whether \( \text{char}(\kappa_F) \neq 2 \). Writing \( \psi' = \psi_a \), we can further assume that \( v_F(a) = 0 \) or \( 1 \) because \( \omega_{\psi_a} \) only depends the class \( aF^x \) in \( F^x/F^2 \).

1. \( \text{char}(\kappa_F) \neq 2 \) and \( v_F(a) = 0 \). In this situation, \( \phi(x) \) is a spherical function of \( \omega^+_{\psi_a} \) and we have

\[ \left| \left( \omega^+_{\psi_a} \left( [k_2, 1] \left( \begin{array}{c} \sigma^n \\ \sigma^{-n} \end{array} \right) [k_1, 1] \right) \phi, \phi \right) \right|^2 = |\sigma|^n \quad \text{for all } k_1, k_2 \in K \text{ and } n \geq 0. \]
It follows that
\[ I(s; \phi, \psi, \phi, \phi) = \sum_{n \geq 0} \int_{K(\varpi^{-n})} |(\omega_{\psi}^+(g \phi, \phi)|^2 \Delta(g)^s \, dg \]
\[ = m(K) \cdot 1 + \sum_{n \geq 1} (1 + |\varpi|)|\varpi|^{-2n} m(K)|\varpi|^{-n} |\varpi|^s \]
\[ = m(K) \left( 1 + \frac{(1 + |\varpi|)|\varpi|^{-1+s}}{1 - |\varpi|^{-1+s}} \right). \]

Hence \( I(0; \phi, \psi, \phi, \phi) = -2|\varpi|(1 + |\varpi|) \).

(2) \( \text{char}(\kappa_F) \neq 2 \) and \( v_F(a) = 1 \). In this situation, one may assume that \( a \) is just \( \varpi \), the uniformizer of \( F \). As we did before,
\[ I(s; \phi, \psi, \phi, \phi) = \sum_{n \geq 0} m(K(\varpi^{-n})) \frac{K(\varpi^{-n})}{m(K)^2} \int_{K \times K} |(\omega_{\psi}^+(\varpi^{-n} \phi, \phi)|^2 \, dk_1 \, dk_2. \]

Putting \( J(s; n, k_1, k_2) = |(\omega_{\psi}^+(\varpi^{-n} \phi, \phi)|^2 \), we shall first compute \( I(s; n) = \int_{K \times K} J(s; -) \, dk_1 \, dk_2 \).

Our first observation is that \( J(s; n, k_1, k_2) \) is right invariant by \( A_K N_K \) in each \( k \)-variable. With the view of Lemma 4.6, we put \( K_1 = w N_K w A_K N_K \), \( K_2 = N_{K_1} w A_K N_K \), \( I_{ij}(s; n) = \int_{K \times K_j} J(s; n, k_1, k_2) \, dk_1 \, dk_2 \) and then obtain a decomposition \( I(s; n) = \sum_{ij} I_{ij}(s; n) \).

Our second observation is that
\[ \omega_{\psi_{\varpi}}^+(|w|, 1) = \gamma(\psi_{\varpi}) \phi(\varpi x)|\varpi|^{1/2}, \]
\[ \omega_{\psi_{\varpi}}^+(\left[ \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right], 1) \omega_{\psi_{\varpi}}^+(|w|, 1) = \gamma(\psi_{\varpi}) \phi(\varpi x) \psi(\varpi n x^2)|\varpi|^{1/2}, \]
\[ \omega_{\psi_{\varpi}}^+(|w|, 1) \omega_{\psi_{\varpi}}^+(\left[ \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right], 1) \omega_{\psi_{\varpi}}^+(|w|, 1) = \gamma(\psi_{\varpi})^2 \phi(x) \quad \text{if } n \in \varpi O_F. \]

It follows that when \( n \geq 1 \), we have
\[ I_{11}(s) = (1 - |\varpi|)^2 \int_{O_F \times O_F} |(\omega_{\psi_{\varpi}}^+(n_1 w) \phi, \omega_{\psi_{\varpi}}^+(\varpi^{-n_2} w) \phi)|^2 \, dn_1 \, dn_2 \]
\[ = |\varpi|^n (1 - |\varpi|)^2 \int_{O_F \times O_F} \left| \int_{O_F} \phi(\varpi x) \psi(\varpi n x^2)|\varpi| \, dx \right|^2 \, dn_1 \, dn_2. \]
\[ I_{12}(s) = (1 - |\sigma|)^2 \int_{\mathcal{O}_F \times \mathcal{O}_F} |(\omega_{\psi_\sigma} (n_1 w) \phi, \omega_{\psi_\sigma} (\overline{n_2 w} n_2 w) \phi)|^2 \, dn_1 \, dn_2 = |\sigma|^n (1 - |\sigma|)^2 |\sigma|(2 - |\sigma|). \]

\[ I_{21}(s) = (1 - |\sigma|)^2 \int_{\mathcal{O}_F \times \mathcal{O}_F} |(\omega_{\psi_\sigma} (w n_1 w) \phi, \omega_{\psi_\sigma} (\overline{n_2 w} n_2 w) \phi)|^2 \, dn_1 \, dn_2 = |\sigma|^n (1 - |\sigma|)^2 |\sigma|^2, \]

\[ I_{22}(s) = (1 - |\sigma|)^2 \int_{\mathcal{O}_F \times \mathcal{O}_F} |(\omega_{\psi_\sigma} (\overline{n_1 w} n_1 w) \phi, \omega_{\psi_\sigma} (\overline{n_2 w} n_2 w) \phi)|^2 \, dn_1 \, dn_2 = |\sigma|^n (1 - |\sigma|)^2 |\sigma|^2. \]

Hence, in the expression of \( I(s; \phi, \phi, \phi, \phi) \), the summation over \( n \geq 1 \) yields

\[
\sum_{n \geq 1} \frac{m(K (\sigma^n) K) |\sigma|^n I(s; n)}{m(K)^2} = \sum_{n \geq 1} \frac{m(K (\sigma^n) K) |\sigma|^n I_{ij}(s; n) |\sigma|^n}{m(K)^2} \sum_{i,j} I_{ij}(s; n) |\sigma|^n
\]

\[
= \frac{4 |\sigma| (1 - |\sigma|) |\sigma|^{s-1}}{1 - |\sigma|^{s-1}}.
\]

On the other hand, the first term in the summation is

\[
\int_K |(k \phi, \phi)|^2 = \int_K |(k \phi, \phi)|^2 + \int_K |(k \phi, \phi)|^2
\]

\[
= \int_{\mathcal{O}_F} (1 - |\sigma|) |(\omega_{\psi_\sigma} (w n w) \phi, \phi)|^2 \, dn + (1 - |\sigma|)
\]

\[
\times \int_{\mathcal{O}_F} |(\omega_{\psi_\sigma} (\overline{w n} \phi, \phi)|^2 \, dn
\]

\[
= 2 |\sigma| (1 - |\sigma|).
\]

Therefore, \( I(0; \phi, \phi, \phi, \phi) \) is the sum of the above two terms evaluated at 0, which is \(-4 |\sigma| + 2 |\sigma|(1 - |\sigma|) = -2 |\sigma|(1 + |\sigma|). \)
(3) \( \text{char}(\kappa_F) = 2. \)

We need to distinguish two cases \( v_F(a) = 0 \) and \( v_F(a) = 1 \) and use the same decomposition of \( K \) into \( K_1 \) and \( K_2 \) as in the case of \( \text{char}(\kappa_F) \neq 2, v_F(a) = 0; \) a very similar computation yields that \( I(0; \phi, \phi, \phi, \phi) = -2|\sigma|^2 F|\omega| (1 + |\sigma|). \)

\[ \blacksquare \]

Combining Lemmas 4.4 and 4.7, we prove Proposition 4.2 for \( \tilde{\pi} = \omega_{\psi}^+ \) when \( F \) is \( p \)-adic.

### 4.2.2 \( F \) is real

One can prove an analog of Lemmas 4.4 and 4.7 for \( \tilde{\pi} = \omega_{\psi, \psi}^+ \) when \( F \) is real, and hence deduce Proposition 4.2 in this case. For brevity, we skip the proof here.

### 4.3 \( \tilde{\pi} \) is a principal series that is unitary

Here are the basic facts about induced representations of \( \tilde{\text{SL}}_2(F) \). Let \( \tilde{A} = \{(a, a^{-1}), e|a \in F^\times, e = \pm 1\} \) and \( \chi_{\psi}(a) = (a-1)\gamma(\psi(a))\gamma(\psi)^{-1}. \) Then \( \chi_{\psi}(a^2) = 1 \) and \( \chi_{\psi}(a_1 a_2) = \chi_{\psi}(a_1)\chi_{\psi}(a_2)(a_1|a_2)F. \)

For a quasi-character \( \mu \) of \( F^\times \), the principal series \( \tilde{\pi}(\mu) \) (cf. [17]) is the right-translation action of \( \tilde{\text{SL}}_2(F) \) on the space \( \tilde{B}(\mu) \) of \( \tilde{K} \)-finite functions \( f: \tilde{\text{SL}}_2(F) \rightarrow \mathbb{C} \) such that

\[
f\left(\left[\begin{array}{cc} a & n \\ a^{-1} & 1 \end{array}\right], e\right) = \chi_{\psi}(a)\mu(a)|a|^{\frac{1}{2}}e f(\sigma).
\]

The standard pairing between \( \tilde{\pi}(\mu) \) and \( \tilde{\pi}(\chi_{-1} \mu^{-1}) \) is \( \langle f_1, f_2 \rangle = \int_K f_1(k) f_2(k) \, dk. \)

Intertwining operators can be defined just as for induced representations of \( \text{GL}_2(F) \). For the element \( w = (-1, 1) \), we formally write

\[
M_\mu^w f(\sigma) = \int_F f\left(\left[\begin{array}{cc} 1 & x \\ 1 & 1 \end{array}\right], 1\right) f(\sigma) \, dx, \quad M_{\mu^{-1}}^w f(\sigma) = \int_F f\left(\left[\begin{array}{cc} 1 & 1 \\ 1 & x \end{array}\right], 1\right) f(\sigma) \, dx.
\]

Put \( |\mu| = |s_0| \), then the above two integrals are absolutely convergent when \( \text{Re}(s_0) > 0 \) and define operators \( \tilde{B}(\mu) \rightarrow \tilde{B}(\mu^{-1}) \) accordingly; \( M_\mu^w \) and \( M_{\mu^{-1}}^w \) can be meromorphically continued to all \( \mu \) with only simple poles at those \( \mu \) satisfying \( \mu^2 = 1. \) It is easy to see that \( M_{\mu^{-1}}^w = \chi_{\psi}(-1)\mu(-1)(-1|1)F M_\mu^w. \)
In this subsection, we prove the following proposition, which is Theorem 4.1 in the case of principal series. The proof relies heavily on the asymptotic formulas of matrix coefficients and we shall treat the $p$-adic and real case separately.

**Proposition 4.8.** Suppose that $\tilde{\pi} = \tilde{\pi}(\mu)$ is irreducible unitary and write $\pi = \pi(\mu, \mu^{-1})$. Then $I(s)$ has meromorphic continuation to $\mathbb{C}$ and $\lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi)I(s) = 2|2|_F I_{st}$. □

4.3.1 $F$ is p-adic

**Lemma 4.9 ([16]).** When $F$ is $p$-adic, $\tilde{\pi}(\mu)$ is irreducible if and only if $\mu^2 \neq ||^\pm 1$. □

Concerning the unitarizability of $\tilde{\pi}(\mu)$, we have the following lemma, whose proof is similar to the case of $\text{GL}_2(F)$ (cf. [4, §1, Theorem 12]) and hence skipped.

**Lemma 4.10.** When $F$ is $p$-adic, $\tilde{\pi}(\mu)$ is unitary if and only if (I) $\mu$ is unitary or (II) $\mu = \chi||^{s_0}$ with $\chi$ a quadratic character and $s_0$ a real number in $(-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. □

A unitary $\tilde{\pi}(\mu)$ is called a continuous series if it is of type (I) and a complementary series if it is of type (II). The $\tilde{\text{SL}}_2(F)$-invariant Hermitian pairing on a unitary $\tilde{\pi}(\mu)$ can be chosen as $(f_1, f_2) = \langle f_1, f_2 \rangle$ for a continuous series or $(f_1, f_2) = \langle f_1, M_w^\mu f_2 \rangle$ for a complementary series. We comment that for $f_1 \in \tilde{\mathcal{B}}(\mu)$ and $f_2 \in \tilde{\mathcal{B}}(\chi^{-1}\mu)$, one has

$\langle f_1, f_2 \rangle = \frac{m(K)}{1 + |\sigma|} \int_K f_1 f_2 \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} dx.$

There exists a similar Hermitian pairing on $\tilde{\pi}(\mu_1) \otimes \tilde{\pi}(\mu_2)$, namely $(f_1, f_2)_K = \int_K f_1(k) f_2(k) dk$ for all $\mu_1$ and $\mu_2$. But such a pairing is usually not $\tilde{\text{SL}}_2(F)$-invariant.

Concerning intertwining operators, we provide the following lemma, whose proof is similar to the case of $\text{GL}_2$ and hence skipped.

**Lemma 4.11.** Suppose that $\mu^2 \neq 1$. Then

1. $(M_w^\mu f_1, f_2) = \langle f_1, M_w^{\mu^{-1}} f_2 \rangle$ for $f_1 \in \tilde{\mathcal{B}}(\mu)$, $f_2 \in \tilde{\mathcal{B}}(\chi^{-1}\mu)$;
2. $M_w^{\mu^{-1}} M_w^\mu = \frac{|2|_F}{\gamma(0, \mu^2, \psi) \gamma(0, \mu^{-2}, \psi)}$. □

Now we investigate the behavior of matrix coefficients of $\tilde{\pi}(\mu)$; let $e$ denote $[I_2, 1] \in \tilde{\text{SL}}_2(F)$ in the following.
Lemma 4.12. Suppose that $\tilde{\pi} = \tilde{\pi}(\mu)$ with $\mu$ unitary, $f_1, f_2 \in \tilde{B}(\mu)$, and $a \in F^\times$.

(1) $\mu^2 \neq 1$. Then when $|a|$ is sufficiently small,

$$\langle \tilde{\pi}(a)f_1, f_2 \rangle = \frac{m(K)|\chi_\psi(a)}{1 + |\sigma|}(\mu^{-1}(a)M_\mu^w f_1(e)f_2(w) + \mu(a)f_1(e)M_\mu^w f_2(w^{-1})).$$

(2) $\mu^2 = 1$. Let $m$ be a positive integer such that $f_1$ and $f_2$ are right invariant by $(\sigma^mO_F)_{1}$ and $(\sigma^mO_F)_{-1}$. Then when $n$ is sufficiently large,

$$\langle \tilde{\pi}(\sigma^n)f_1, f_2 \rangle = \frac{m(K)|\sigma|^n}{1 + |\sigma|} \chi_\psi(\sigma^n)\mu(\sigma^n)(A + B(n)),$$

with

$$A = \int_{v(x) \geq -m} f_1 \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) \overline{f_2(w)} dx$$

$$+ \int_{v(x) < m} \chi_\psi(-x^{-1})\mu(-x^{-1})|x^{-1}|f_1(e)f_2 \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) dx,$$

$$B(n) = c|2|^{1/2}(n - m)m(O_F^\times) f_1(e)f_2(w),$$

where $c$ is a complex number of norm 1.

Proof.

$$\langle \tilde{\pi}(a)f_1, f_2 \rangle$$

$$= \int_k \tilde{\pi}(a)f_1(k)\overline{f_2(k)} dk$$

$$= \frac{m(K)}{1 + |\sigma|} \int_F f_1 \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \left[ \left( \begin{array}{c} a \\ a^{-1} \end{array} \right), 1 \right] \right) \overline{f_2 \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right)} dx$$

$$= \frac{m(K)}{1 + |\sigma|} \int_F f_1 \left( \left[ \left( \begin{array}{c} a^{-1} \\ a \end{array} \right), 1 \right] \left[ w \left( \begin{array}{c} 1 \\ a^{-2}x \\ 1 \end{array} \right), 1 \right] \right) \overline{f_2 \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right)} dx$$
\[ \frac{m(K)}{1 + |\sigma|} \chi_{\psi}(a^{-1}) \mu(a^{-1}) |a^{-1}| \int_{F} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{-2}x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) dx \]

\[ x \to a^{2}x \]

\[ \frac{m(K)}{1 + |\sigma|} \chi_{\psi}(a^{-1}) \mu(a^{-1}) |a| \int_{F} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) dx. \]

For chosen \( f_{1} \) and \( f_{2} \), there exists \( \delta > 0 \) such that when \( |x| \leq \delta \), one has

\[ f_{i} \left( \sigma \left[ \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) = f_{i} \left( \sigma \left[ \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) = f_{i}(\sigma). \]

It follows that when \( |x| \geq \delta^{-1} \),

\[ f_{i} \left( w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right) = f_{i} \left( \left[ \left( \begin{array}{c} 1 \\ -x^{-1} \\ 1 \end{array} \right), 1 \right] \left[ \left( \begin{array}{c} -x^{-1} \\ -x \\ 1 \end{array} \right), 1 \right] \left[ \left( \begin{array}{c} 1 \\ x^{-1} \\ 1 \end{array} \right), 1 \right] \right) \]

\[ = \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| f_{i}(e). \]

So, we break the integral

\[ \int_{F} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) \]

into two parts according to whether \( |a^{2}x| \leq \delta \). When \( |a| \) is small enough such that \( |a|^{-2} \delta > \delta^{-1} \), we have

\[ \int_{F} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) dx \]

\[ = \int_{|a^{2}x| \leq \delta} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) dx \]

\[ + \int_{|a^{2}x| > \delta} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) dx \]

\[ = \int_{|x| \leq |a|^{-2} \delta} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) dx f_{2}(w) \]

For chosen \( f_{1} \) and \( f_{2} \), there exists \( \delta > 0 \) such that when \( |x| \leq \delta \), one has

\[ f_{i} \left( \sigma \left[ \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) = f_{i} \left( \sigma \left[ \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) = f_{i}(\sigma). \]

It follows that when \( |x| \geq \delta^{-1} \),

\[ f_{i} \left( w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right) = f_{i} \left( \left[ \left( \begin{array}{c} 1 \\ -x^{-1} \\ 1 \end{array} \right), 1 \right] \left[ \left( \begin{array}{c} -x^{-1} \\ -x \\ 1 \end{array} \right), 1 \right] \left[ \left( \begin{array}{c} 1 \\ x^{-1} \\ 1 \end{array} \right), 1 \right] \right) \]

\[ = \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| f_{i}(e). \]

So, we break the integral

\[ \int_{F} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) \]

into two parts according to whether \( |a^{2}x| \leq \delta \). When \( |a| \) is small enough such that \( |a|^{-2} \delta > \delta^{-1} \), we have

\[ \int_{F} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) dx \]

\[ = \int_{|a^{2}x| \leq \delta} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) dx \]

\[ + \int_{|a^{2}x| > \delta} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) f_{2} \left( \left[ w \left( \begin{array}{c} 1 \\ a^{2}x \\ 1 \end{array} \right), 1 \right] \right) dx \]

\[ = \int_{|x| \leq |a|^{-2} \delta} f_{1} \left( \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right) dx f_{2}(w) \]
+ \int_{|x|>|a|^{-2\delta}} \chi_{\psi}(-x^{-1})\mu(-x^{-1})|x^{-1}| f_1(e) f_2\left(\left[w\left(\begin{array}{c} 1 \\ a^2x \\ 1 \end{array}\right), 1\right]\right) dx

= \int_{|x|\leq|a|^{-2\delta}} f_1\left(\left[w\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right), 1\right]\right) dx f_2(w)

+ \mu(a)^2 \int_{|x|>|a|^{-2\delta}} \chi_{\psi}(-x^{-1})\mu(-x^{-1})|x^{-1}| f_1(e) f_2\left(\left[w\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right), 1\right]\right) dx

= I + II.

(1) When \mu^2 \neq 1, the following two improper integrals are well defined:

$$\int_F f_1\left(\left[w\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right), 1\right]\right),$$

$$\int_F \chi_{\psi}(-x^{-1})\mu(-x^{-1})|x^{-1}| f_2\left(\left[w\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right), 1\right]\right) dx.$$

The first one is $M^w_{\mu} f_1(e)$ and the second one is $M^w_{\mu} w^{-1} f_2(e)$. We further have

$$I + II = \int_F f_1\left(\left[w\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right), 1\right]\right) dx f_2(w)$$

$$+ \mu(a)^2 \int_F \chi_{\psi}(-x^{-1})\mu(-x^{-1})|x^{-1}| f_1(e) f_2\left(\left[w\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right), 1\right]\right) dx$$

$$= M^w_{\mu} f_1(e) f_2(w) + \mu(a)^2 f_1(e) M^w_{\mu} w^{-1} f_2(e).$$

Therefore,

$$\tilde{\pi}(\{a\} f_1, f_2) = \frac{m(K)|a|}{1 + |a\sigma|} \chi_{\psi}(a)\mu^{-1}(a)(I + II)$$

$$= \frac{m(K)|a|}{1 + |a\sigma|} (\chi_{\psi}(a)\mu^{-1}(a)M^w_{\mu} f_1(e) f_2(w) + \chi_{\psi}(a)\mu(a) f_1(e) M^w_{\mu} w^{-1} f_2(e)).$$
(2) When $\mu^2 = 1$, we put $a = \varpi^n$, $\delta = |\varpi^m|$ and break integral $I$ into two parts.

$$I = \int_{|x| \leq |\varpi|^{-2n+m}} f_1 \left[ \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right] \, dx f_2(w)$$

$$= \int_{v(x) \geq -m} f_1 \left[ \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right] \, dx f_2(w)$$

$$+ \int_{-2n+m \leq v(x) < -m} f_1 \left[ \left[ w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right), 1 \right] \right] \, dx f_2(w)$$

The second summand in the above formula is exactly $B(n)$. Therefore,

$$\langle \tilde{\pi} (\varpi^n) f_1, f_2 \rangle = \frac{m(K) |\varpi|^n}{1 + |\varpi|} \frac{\chi \psi (\varpi^{-n} \mu (\varpi^{-n})) (I + II)}{A + B(n)}. \quad \blacksquare$$

**Lemma 4.13.** Suppose that $\mu = \chi \cdot |s|^s$ with $\chi$ quadratic and $s \in (0, \frac{1}{2})$ and that $f_1, f_2 \in \tilde{B}(\mu)$. Then when $|a|$ is sufficiently small, one has

$$\langle \tilde{\pi} (a) f_1, f_2 \rangle = \frac{m(K) |a|}{1 + |\varpi|} \frac{\chi \psi (a) \mu^{-1}(a) M^w \mu f_1(e) M^w \mu f_2(w) + \chi \psi (a) \mu(a) f_1(e) M^w \mu^{-1} w^{-1} M^w \mu f_2(e)}{A + B(n)}. \quad \blacksquare$$

**Proof.** The proof is similar to the one for the previous lemma and hence skipped. \quad \blacksquare

**Proof of Proposition 4.8 when $F$ is $p$-adic.** We distinguish between continuous series and complementary series.

(A) $\mu$ is unitary. Given $f_i \in \tilde{B}(\mu)(1 \leq i \leq 4)$, we write

$$I(s; f_1, f_2, f_3, f_4) = \sum_{n \geq 0} |\varpi|^n \int_{K(\varpi^n \varpi^{-n})} (\tilde{\pi} (g) f_1, f_2) (\tilde{\pi} (g) f_3, f_4) \, dg$$

Let $n_0$ be big enough such that when $n \geq n_0$, Lemma 4.12 applies to $(\tilde{\pi} (\varpi^n) k_1 f_i, k_2 f_j)$ for $(i, j) = (1, 2), (3, 4)$ and all $k_1, k_2 \in K$, then we decompose
We need to estimate the matrix coefficients according to whether \( \mu \) is a polynomial function of \( s \) and \( n \geq 0 \), where

\[
I_1 = \sum_{n \geq n_0} |\omega|^n \int_{K} \left( \hat{\pi}(g) f_1, f_2 \right) (\hat{\pi}(g) f_3, f_4) \, dg,
\]

\[
I_2 = \sum_{n \geq n_0} |\omega|^n \int_{K} \left( \hat{\pi}(g) f_1, f_2 \right) (\hat{\pi}(g) f_3, f_4) \, dg.
\]

\( I_1 \) is a polynomial function of \( |\omega|^s \) and \( I_2 \) is equal to

\[
\sum_{n \geq n_0} m(K)^{-1} (1 + |\omega|) |\omega|^{n(s-2)} \int_{K \times K} \left( \hat{\pi}(\omega^n) f_1, f_2 \right) (\hat{\pi}(\omega^n) f_3, f_4) \, dk_1 \, dk_2.
\]

We need to estimate the matrix coefficients according to whether \( \mu^2 \) is \( 1 \).

(A1) \( \mu \) is unitary and \( \mu^2 \neq 1 \). When \( n \geq n_0 \), we have

\[
\left( \hat{\pi}(\omega^n) f_1, f_2 \right) = \frac{m(K) |\omega|^n}{1 + |\omega|} (\chi_\psi(\omega^n) \mu^{-1}(\omega^n) M^w \mu_1 f_1(e) k_2 f_j(w)
+ \chi_\psi(\omega^n) \mu(\omega^n) k_1 f_1(e) M^w \hat{\pi}(w^{-1}) f_j(e)).
\]

Hence \( I_2 = \frac{m(K)}{(1 + |\omega|)} (I_{21} + I_{22} + I_{23} + I_{24}) \), where

\[
I_{21} = \sum_{n \geq n_0} |\omega|^n \int_{K \times K} M^w \mu_1 f_1(e) k_2 f_2(w) M^w \mu_1 f_3(e) k_2 f_4(w) \, dk_1 \, dk_2
= \frac{|\omega|}{1 - |\omega|^s} (M^w \mu_1 f_1, M^w \mu_1 f_3)(f_2, f_4)
= \frac{|\omega|}{1 - |\omega|^s} \left| 2 \right|_F \gamma(0, \mu^2, \psi) \gamma(0, \mu^{-2}, \psi) (f_1, f_3)(f_2, f_4),
\]

\[
I_{22} = \sum_{n \geq n_0} |\omega|^n \int_{K \times K} k_1 f_1(e) M^w \mu_1 f_1(w) k_2 f_2(e) k_1 f_3(e) M^w \mu_1 f_4(w) \, dk_1 \, dk_2
= \frac{|\omega|}{1 - |\omega|^s} (f_1, f_3)(M^w \mu_1 f_2, M^w \mu_1 f_4)
= \frac{|\omega|}{1 - |\omega|^s} \left| 2 \right|_F \gamma(0, \mu^2, \psi) \gamma(0, \mu^{-2}, \psi) (f_1, f_3)(f_2, f_4),
\]
\[ I_{23} = \sum_{n \geq n_0} |\sigma|^n \mu(\sigma)^{-2n} \int_{K \times K} M_{\mu}^{w} k_1 f_1(e) k_2 f_2(w) k_1 f_3(e) M_{\mu}^{w} k_2 f_4(e) \, dk_1 \, dk_2 \]
\[ = \frac{\mu(\sigma)^{-2n} |\sigma|^{n_0}}{1 - \mu(\sigma)^{-2} |\sigma|^2} (M_{\mu}^{w} f_1, f_3 k(f_2, M_{\mu}^{w} f_4)), \]
\[ I_{24} = \sum_{n \geq n_0} |\sigma|^n \mu(\sigma)^{2n} \int_{K \times K} k_1 f_1(e) M_{\mu}^{w} k_2 f_2(e) M_{\mu}^{w} k_3 f_3(e) k_2 f_4(w) \, dk_1 \, dk_2 \]
\[ = \frac{\mu(\sigma)^{2n} |\sigma|^{n_0}}{1 - \mu(\sigma)^{2} |\sigma|^2} (f_1, M_{\mu}^{w} f_3 k(M_{\mu}^{w} f_2, f_4)). \]

Note that \( I_{23} \) and \( I_{24} \) are zero when \( \mu^2 \) is ramified. The above expressions clearly show that \( (1 - |\sigma|^3)(1 - \mu(\sigma)^2 |\sigma|^5)(1 - \mu(\sigma)^{-2} |\sigma|^4)I(s; f_1, f_2, f_3, f_4) \) is a polynomial of \( |\sigma|^s \) for all \( f_i \), hence \( I(s) \) has meromorphic continuation to \( \mathbb{C} \). The main term of \( I(s; f_1, f_2, f_3, f_4) \) near \( s = 0 \) is \( \frac{m(K)}{(1 + |\sigma|)} (I_{21} + I_{22}) \), that is,

\[ m(K) |\sigma|^{n_0} \frac{2|2|}{(1 + |\sigma|) 1 - |\sigma|^2} \gamma(0, \mu^2, \psi) \gamma(0, \mu^{-2}, \psi^*) (f_1, f_3)(f_2, f_4). \]

Therefore \( \lim_{s \to 0} \gamma(s, \text{ad}, \pi, \psi) I(s; f_1, f_2, f_3, f_4) = 2|2|(f_1, f_3)(f_2, f_4). \)

(A2) \( \mu \) is quadratic. We apply the second part of Lemma 4.12 and find that \( I_2 = \frac{m(K)}{(1 + |\sigma|)} (I_{21} + I_{22} + I_{23}) \), where \( I_{22} \) and \( I_{23} \) are rational functions of \( |\sigma|^s \) with denominator \( (1 - |\sigma|^3), (1 - |\sigma|^5)^2 \) separately and

\[ I_{21} = \frac{|2|}{(1 + |\sigma|)} \sum_{n \geq n_0} n^2 |\sigma|^n m(\mathcal{O}_F^s)^2 \int_{K \times K} k_1 f_1(e) k_2 f_2(w) k_1 f_3(e) k_2 f_4(w) \, dk_1 \, dk_2 \]
\[ = \frac{|2|}{(1 + |\sigma|)} \sum_{n \geq n_0} n^2 |\sigma|^n m(\mathcal{O}_F^s)^2 (f_1, f_3)(f_2, f_4) \]
\[ = \frac{|2|}{(1 + |\sigma|)} m(\mathcal{O}_F^s)^2 (f_1, f_3)(f_2, f_4) \left( \frac{|\sigma|^s (1 + |\sigma|^3)}{(1 - |\sigma|^5)^3} - \sum_{1 \leq n_0} n^2 |\sigma|^n \right). \]

Hence \( I(s) \) has meromorphic continuation to \( \mathbb{C} \) and its main term near \( s = 0 \) is

\[ \frac{|2|m(K)m(\mathcal{O}_F^s)^2}{(1 + |\sigma|)} \frac{|\sigma|^s (1 + |\sigma|^3)}{(1 - |\sigma|^5)^3} (f_1, f_3)(f_2, f_4). \]
It follows that \( \lim_{s \to 0} \gamma(s, \text{ad}, \pi, \psi) I(s; f_1, f_2, f_3, f_4) = 2|2|(f_1, f_3) (f_2, f_4) \).

\[ \mu = \chi \cdot |s| \] with \( \chi \) quadratic and \( s \in (0, \frac{1}{2}) \). The argument is similar to the one for (A1) and only different in that one uses Lemma 4.13 to estimate the matrix coefficients. For brevity, we skip the proof.

### 4.3.2 \( F = \mathbb{R} \)

For a quasi-character \( \mu \) of \( \mathbb{R}^\times \), we write as \( \mu(x) = |x|^{s_0} \text{sgn}(x)^0 \).

**Lemma 4.14.** [16, 17] \( \tilde{\pi}(\mu) \) is irreducible if and only \( s_0 \) is not a half integer.

**Remark 4.15.** When \( s_0 \) is a half integer, \( \tilde{\pi}(\mu) \) has a unique infinite-dimensional subrepresentation. This subrepresentation and the corresponding subquotient of \( \tilde{\pi}(\mu) \) form the packet \( \{ \tilde{\pi}^+_{|s_0+1|}, \tilde{\pi}^-_{|s_0+1|} \} \) of weight-\( |s_0+1| \) holomorphic and anti-holomorphic discrete series of \( \tilde{SL}_2(\mathbb{R}) \).

The following two properties of \( \tilde{\pi}(\mu) \) are as just as those for induced representations of \( GL_2(\mathbb{R}) \) and we skip their proof for brevity.

**Lemma 4.16.** \( \tilde{\pi}(\mu) \) is unitary if and only if (I) \( \mu \) is unitary or (II) \( s_0 \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \).

**Proposition 4.17.** Suppose that \( \mu^2 \neq 1 \). Then

1. \( \langle M^\mu_{\mu} f_1, f_2 \rangle = \langle f_1, M^{-1}_{\mu} f_2 \rangle \) for \( f_1 \in \tilde{B}(\mu), f_2 \in \tilde{B}(\chi^{-1} \mu) \);
2. \( M^{-1}_{\mu} M^\mu_{\mu} = \gamma(0, \mu^4, \psi) \gamma(0, \mu^{-4}, \psi)^{-1} \).

**Remark 4.18.** When \( s_0 \) is a half integer, \( M^{-1}_{\mu} M^\mu_{\mu} = 0 \).

The \( \tilde{SL}_2(\mathbb{R}) \)-invariant Hermitian pairing on an unitary \( \tilde{\pi}(\mu) \) can be taken as

1. \( \langle f_1, f_2 \rangle = \langle f_1, \tilde{f}_2 \rangle \), if \( \tilde{\pi}(\mu) \) is of type (I);
2. \( \langle f_1, f_2 \rangle = \langle f_1, M^{-1}_{\mu} f_2 \rangle \), if \( \tilde{\pi}(\mu) \) is of type (II).

When \( F \) is real, the key to prove Proposition 4.8 is the asymptotic formula of matrix coefficients, but we need information not only about the main term, but also the remainder term. We first point out a few facts that will be used.
(1) A basis of $\tilde{B}(\mu)$. Put $\chi(\psi(-1)\mu(-1)) = e^{iv\pi}$ with $v \in \{\pm \frac{1}{2}\}$. Then a basis of $\tilde{B}(\mu)$ is given by $\{\varphi_{n}|n \in v + 2\mathbb{Z}\}$, with the restriction of $\varphi_{n}$ to $\tilde{K} \cong \mathbb{R}/4\pi\mathbb{Z}$ given by $\varphi_{n}(\tilde{\gamma}(\theta)) = e^{in\theta}$. The map $\tilde{\gamma}$ is described in Section 4.1.2.

(2) The Haar measure of $SL_2(\mathbb{R})$ is given by

$$dg = |y^2 - y^{-2}|dx \cdot dy \cdot d\theta_1 \cdot d\theta_2 \quad \text{if} \quad g$$

$$= \begin{pmatrix}
\cos \theta_1 & \sin \theta_1 \\
-\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
y \\
y^{-1}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_2 & \sin \theta_2 \\
-\sin \theta_2 & \cos \theta_2
\end{pmatrix},$$

$$dg = |y|^{-2}dx \cdot dy \cdot d\theta \quad \text{if} \quad g = \begin{pmatrix}
1 & x \\
1 & y
\end{pmatrix}
\begin{pmatrix}
y \\
y^{-1}
\end{pmatrix}
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix},$$

$$dg = |y|^{-2}dx_1 \cdot dx_2 \quad \text{if} \quad g = \begin{pmatrix}
1 & x_1 \\
1 & y
\end{pmatrix}
\begin{pmatrix}
y \\
y^{-1}
\end{pmatrix}w
\begin{pmatrix}
1 & x_2 \\
1 & 1
\end{pmatrix}.$$

Hence for $f_1 \in \tilde{B}(\mu)$ and $f_2 \in \tilde{B}(\chi(\psi(\mu)^{-1}))$, one has

$$\langle f_1, f_2 \rangle = 2 \int_{\mathbb{R}} f_1 f_2 \left( w \begin{pmatrix}
1 & x \\
1 & 1
\end{pmatrix} \right) \, dx.$$

(3) For $\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in SL_2(\mathbb{R}),$

one has

$$\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, 1 = \begin{pmatrix}
1 & \frac{ac + bd}{(c^2 + d^2)^{\frac{1}{2}}} \\
\frac{d}{(c^2 + d^2)^{\frac{1}{2}}} & 1
\end{pmatrix} \cdot \begin{pmatrix}
\frac{d}{(c^2 + d^2)^{\frac{1}{2}}} & \frac{-c}{(c^2 + d^2)^{\frac{1}{2}}} \\
\frac{c}{(c^2 + d^2)^{\frac{1}{2}}} & \frac{-d}{(c^2 + d^2)^{\frac{1}{2}}}
\end{pmatrix}, 1$$

$$w \begin{pmatrix}
1 & x \\
1 & 1
\end{pmatrix}, 1 = \begin{pmatrix}
1 & -x^{-1} \\
1 & 1
\end{pmatrix} \begin{pmatrix}
-x^{-1} & 1 \\
-x & 1
\end{pmatrix} \begin{pmatrix}
1 & -x^{-1} \\
1 & 1
\end{pmatrix}. $
Suppose that $f \in \tilde{\mathcal{B}}(\mu)$; then when $f = \varphi_n$, one has
\[ f \left( \begin{bmatrix} 1 & x \\ w & 1 \end{bmatrix} \right) = (1 + x^2)^{-\frac{n+1}{2}} e^{i \text{arccos} \frac{x}{1 + x^2}} = e^{i \pi f} \left( \begin{bmatrix} 1 & -x \\ w & 1 \end{bmatrix} \right). \]

Similarly,
\[ f \left( \begin{bmatrix} x & 1 \\ y & 1 \end{bmatrix} \right) = (1 + x^2)^{-\frac{n+1}{2}} e^{-i \text{arctan} x} = f \left( \begin{bmatrix} 1 & 1-x \\ -y & 1 \end{bmatrix} \right). \]

Particularly,
\[ f \left( \begin{bmatrix} w & 1 \\ 1 & 1 \end{bmatrix} \right) \text{ and } f \left( \begin{bmatrix} 1 & 1-x \\ -y & 1 \end{bmatrix} \right) \]

can be expanded as the Taylor series at $x = 0$ which converge in $[-1, 1]$.

(4) Let $1$ be the trivial representation of $\mathbb{R}^\times$. Then $\gamma(s, 1, \psi) = 1/2 s + \cdots$ near $s = 0$. In the following, $e$ denotes $[I_2, 1]$ and $y$ denotes $\left( \begin{array}{cc} y & 1 \\ y^{-1} & 1 \end{array} \right)$.

Lemma 4.19. Suppose that $\tilde{\pi} = \tilde{\pi}(\mu)$ with $\mu$ unitary, $f_1, f_2 \in \tilde{\mathcal{B}}(\mu)$ and $0 < y < 1$.

1. If $\mu^2 \neq 1$, then
\[ (\tilde{\pi}(y) f_1, f_2) = 2 y \mu^{-1} (y) (M_{\mu}^w f_1(e) \overline{f_2(w)} + L_1(y)) + 2 y \mu(y) (f_1(e) M_{\mu}^w w^{-1} f_2(e) + L_2(y)), \]
where $L_1(y)$ and $L_2(y)$ are determined by $f_1$ and $f_2$, analytic in $(-1, 1)$, and of size $y^2 O(1)$ near $0$.

2. If $\mu^2 = 1$, then $(\tilde{\pi}(y) f_1, f_2) = -4 y (1 + e^{i \pi y}) (\ln(y) f_1(e) \overline{f_2(w)} + L(y))$, where $L(y)$ is determined by $f_1$ and $f_2$, analytic in $(-1, 1)$, and of size $O(1)$ near $0$. \hfill \Box

Proof.
\[ \langle \tilde{\pi}(y) f_1, f_2 \rangle = \int_{\mathbb{R}} \tilde{\pi} \left( \begin{bmatrix} y & 1 \\ y^{-1} & 1 \end{bmatrix} \right) f_1(k) \overline{f_2(k)} \, dk \]
\[ = 2 \int_{\mathbb{R}} f_1 \left( \begin{bmatrix} w & 1 \\ 1 & 1 \end{bmatrix} \right) \left( \begin{bmatrix} y & 1 \\ y^{-1} & 1 \end{bmatrix} \right) f_2 \left( \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \right) \, dx \]
\[= 2 \int_{\mathbb{R}} f_1 \left( \left[ \begin{array}{c} y^{-1} \\ y \end{array} \right], 1 \right) \left[ \begin{array}{c} w \left( 1 \ y^{-2} x \\ 1 \end{array} \right), 1 \right] f_2 \left( \left[ \begin{array}{c} w \left( 1 \ x \\ 1 \end{array} \right), 1 \right] \right) \, dx\]

\[= 2 \mu^{-1}(y)y^{-1} \int_{\mathbb{R}} f_1 \left( \left[ \begin{array}{c} w \left( 1 \ y^{-2} x \\ 1 \end{array} \right), 1 \right] \right) f_2 \left( \left[ \begin{array}{c} w \left( 1 \ x \\ 1 \end{array} \right), 1 \right] \right) \, dx\]

\[= 2 \mu^{-1}(y)y \int_{\mathbb{R}} f_1 \left( \left[ \begin{array}{c} w \left( 1 \ x \\ 1 \end{array} \right), 1 \right] \right) f_2 \left( \left[ \begin{array}{c} w \left( 1 \ y^2 x \\ 1 \end{array} \right), 1 \right] \right) \, dx\]

(1) \(\mu^2 \neq 1\). Then

\[\int_{\mathbb{R}} f_1 \left( \left[ \begin{array}{c} w \left( 1 \ x \\ 1 \end{array} \right), 1 \right] \right) \frac{f_2(w)}{f_2\left( \left[ \begin{array}{c} w \left( 1 \ y^2 x \\ 1 \end{array} \right), 1 \right] \right)} \, dx\]

\[= \int_{\mathbb{R}} f_1 \left( \left[ \begin{array}{c} w \left( 1 \ x \\ 1 \end{array} \right), 1 \right] \right) \frac{f_2(w)}{f_2\left( \left[ \begin{array}{c} w \left( 1 \ y^2 x \\ 1 \end{array} \right), 1 \right] \right)} \, dx\]

\[+ \int_{\mathbb{R}} \chi_{\psi}(-x^{-1})\mu(-x^{-1})|x^{-1}| f_1(e) f_2\left( \left[ \begin{array}{c} w \left( 1 \ y^2 x \\ 1 \end{array} \right), 1 \right] \right) \, dx\]

\[+ \int_{\mathbb{R}} \left( f_1 \left( \left[ \begin{array}{c} w \left( 1 \ x \\ 1 \end{array} \right), 1 \right] \right) - \chi_{\psi}(-x^{-1})\mu(-x^{-1})|x^{-1}| f_1(e) \right)\]

\[\times \left( f_2\left( \left[ \begin{array}{c} w \left( 1 \ y^2 x \\ 1 \end{array} \right), 1 \right] - f_2(w) \right) \right) \, dx\]

\[= M^w_{\mu} f_1(e) f_2(w) + \mu^2(y) f_1(e) M^w_{\mu} f_2(e) + \int_{\mathbb{R}} F(x; y) \, dx,\]

with

\[F(x; y) = \frac{f_1 \left( \left[ \begin{array}{c} w \left( 1 \ x \\ 1 \end{array} \right), 1 \right] \right) - \chi_{\psi}(-x^{-1})\mu(-x^{-1})|x^{-1}| f_1(e)\}

\times \left( f_2\left( \left[ \begin{array}{c} w \left( 1 \ y^2 x \\ 1 \end{array} \right), 1 \right] - f_2(w) \right) \right).\]
We shall compute \( \int_{\mathbb{R}} F(x; y) \, dx \) by dividing the domain \( \mathbb{R} \) into three pieces. For convenience, let \( T_1(x) \) and \( T_2(x) \) be such that

\[
\begin{align*}
  f_1 \left( \left[ \begin{array}{c} 1 \\ x \\ 1 \end{array} \right], 1 \right) - f_1(e) &= xT_1(x), \\
  f_2 \left( \left[ \begin{array}{c} 1 \\ x \\ 1 \end{array} \right], 1 \right) - f_2(w) &= xT_2(x).
\end{align*}
\]

Then \( T_1(x) \) and \( T_2(x) \) are bounded analytic functions on \((-1, 1)\).

(i) We have

\[
\int_{|x| < 1} F(x; y) \, dy = y^2 \int_{|x| < 1} \left( f_1 \left( \left[ \begin{array}{c} 1 \\ x \\ 1 \end{array} \right], 1 \right) \right) \\
  - \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| f_1(e) \\
  \times xT_1(y^2 x) \, dx \\
  = y^2 S_2(y).
\]

Here

\[
S_2(y) = \int_{|x| < 1} \left( f_1 \left( \left[ \begin{array}{c} 1 \\ x \\ 1 \end{array} \right], 1 \right) \right) \\
  - \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| f_1(e) \\
  \times xT_1(y^2 x) \, dx
\]

is analytic in \((-1, 1)\) and of size \( O(1) \) near 0.

(ii) We note that

\[
\begin{align*}
  f_1 \left( \left[ \begin{array}{c} 1 \\ x \\ 1 \end{array} \right], 1 \right) - \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| f_1(e) \\
  = \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| f_1 \left( \left[ \begin{array}{c} 1 \\ x^{-1} \\ 1 \end{array} \right], 1 \right) - f_1(e) \\
  = \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| x^{-1} T_1(x^{-1}).
\end{align*}
\]

It follows that

\[
\int_{|x| > y^{-2}} F(x; y) \, dx = \int_{|x| > y^{-2}} \chi_{\psi}(-x^{-1}) \mu(-x^{-1}) |x^{-1}| x^{-1} T_1(x^{-1}) \\
  \times \left( f_2 \left( \left[ \begin{array}{c} 1 \\ y^2 x \\ 1 \end{array} \right], 1 \right) - f_2(w) \right) \, dx
\]
\[ \begin{align*}
= \mu^2(y) y^2 \int_{|x| < 1} \chi_\psi(-x) \mu(-x) \text{sgn}(x) T_1(y^2 x) \\
\times \left( f_2 \left( \left[ w \left( \begin{array}{c}
1 \\
1 \\
\end{array} \right), 1 \right) \right) - f_2(w) \right) \, dx \\
= \mu^2(y) y^2 S_1(y),
\end{align*} \]

where

\[ S_1(y) = \int_{|x| < 1} \chi_\psi(-x) \mu(-x) \text{sgn}(x) T_2(y^2 x) \\
\times \left( f_2 \left( \left[ w \left( \begin{array}{c}
1 \\
1 \\
\end{array} \right), 1 \right) \right) - f_2(w) \right) \, dx \]

is analytic on \((-1, 1)\) and of size \(O(1)\) near 0.

(iii)

\[ \int_{1 < |x| < y^2} F(x; y) \, dx \]

\[ = \int_{1 < |x| < y^2} \chi_\psi(-x^{-1}) \mu(-x^{-1}) |x|^{-1} T_1(x^{-1}) y^2 x T_2(y^2 x) \, dx \]

\[ = y^2 \int_{1 < |x| < y^2} \chi_\psi(-x^{-1}) \mu(-x^{-1}) |x|^{-1} T_1(x^{-1}) \overline{T_2(y^2 x)} \, dx. \]

Because \(\mu^2 \neq 1\), we have \(\mu \neq 1, \text{sgn}\). Hence the integral

\[ \int_{1 < |x| < y^2} \chi_\psi(-x^{-1}) \mu(-x^{-1}) |x|^{-1} T_1(x^{-1}) \overline{T_2(y^2 x)} \, dx \]

is of the form \(S_3(y) + S_4(y) \mu^2(y)\), where \(S_3(y)\) and \(S_4(y)\) are analytic in \((-1, 1)\) and of size \(O(1)\) near 0.

If we put \(L_1(y) = y^2 (S_2(y) + S_3(y))\) and \(L_2(y) = y^2 (S_1(y) + S_4(y))\), then \(L_1(y)\) and \(L_2(y)\) are analytic in \((-1, 1)\) and of size \(y^2 O(1)\) near 0. We have
\[
\int_{\mathbb{R}} F(x; y) \, dx = L_1(y) + \mu^2(y)L_2(y) \quad \text{and}
\]
\[
\left( \tilde{\pi} \left( \begin{bmatrix} \frac{y}{y^{-1}} & 1 \end{bmatrix} \right) f_1, f_2 \right) = 2y\mu^{-1}(y)(M_{\mu}^w f_1(e)f_2(w) + L_1(y)) + 2y\mu(y)(f_1(e)M_{\mu}^w w^{-1}f_2(e) + L_2(y)),
\]

(2) \( \mu^2 = 1 \). Then \( \mu = 1 \) or \( \text{sgn} \) and hence \( \mu(y) = 1 \) for \( y \in (0, 1) \). We need to compute \[
\int_{\mathbb{R}} f_1 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) f_2 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ y^2x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) \, dx
\]
as in the previous case.

(i)
\[
\int_{|x| < 1} f_1 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) f_2 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ y^2x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) \, dx
\]
is a bounded analytic function of \( y \) on \( (-1, 1) \).

(ii)
\[
\int_{|x| > y^{-2}} f_1 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) f_2 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ y^2x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) \, dx
\]
\[
= \int_{|x| < 1} f_1 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) f_2 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) \, dx
\]
is also a bounded function of \( y \) on \( (-1, 1) \).

(iii)
\[
\int_{1 < |x| < y^{-2}} f_1 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) f_2 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ y^2x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) \, dx
\]
\[
= \int_{1 < |x| < y^{-2}} \chi(y^{-1})\mu(x^{-1})|x|^{-1}
\]
\[
\times f_1 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ x^{-1} \\ 1 \end{array} \right), 1 \end{bmatrix} \right) f_2 \left( \begin{bmatrix} w \left( \begin{array}{c} 1 \ y^2x \\ 1 \end{array} \right), 1 \end{bmatrix} \right) \, dx
\]
\[
\begin{align*}
&= \int_{1<|x|<\gamma^{-2}} \chi_\psi(-x^{-1}) \mu(-x^{-1}) |x|^{-1} (f_1(e) + x^{-1} T_1(x^{-1})) \\
&\quad \times \left( \frac{f_2(w) + y^2 x T_2(y^2 x)}{f_2(w)} \right) \, dx \\
&= \int_{1<|x|<\gamma^{-2}} \chi_\psi(-x^{-1}) \mu(-x^{-1}) |x|^{-1} f_1(e) f_2(w) \, dx \\
&\quad + \int_{1<|x|<\gamma^{-2}} \chi_\psi(-x^{-1}) \mu(-x^{-1}) |\text{sgn}(x)| \\
&\quad \times \left( x^{-2} T_1(x^{-1}) f_2(w) + y^2 f_1(e) T_2(y^2 x) + \frac{y^2}{x} T_2(y^2 x) \right) \, dx \\
&= (1 + \chi_\psi(-1) \mu(-1)) \ln(y^{-2}) f_1(e) f_2(w) \\
&\quad + \text{an analytic function in } y \text{ of size } O(1) \text{ near } 0.
\end{align*}
\]

Since \( \chi_\psi(-1) \mu(-1) = e^{i\nu} \), the above calculations shows that
\( (\tilde{\sigma}(y) f_1, f_2) = -4y(1 + e^{i\nu})(\ln(y) f_1(e) f_2(w) + L(y)), \) where \( L(y) \) is an analytic function on \((-1, 1)\) and of size \( O(1) \) near \( 0 \).

**Lemma 4.20.** Suppose that \( \tilde{\sigma} = \tilde{\sigma}(\mu) \) with \( \mu = \|s_0 \text{sgn}(x)^n \), \( s_0 \in (0, 1/2) \) and \( f_1, f_2 \in \tilde{B}(\mu), \ 0 < \gamma < 1 \). Then \( (\tilde{\sigma}(y) f_1, f_2) = 2y\mu^{-1}(y)(M_\mu^{\nu} f_1(e) M_\mu^{\nu} f_2(w) + L_1(y)) + 2y\mu(y)(f_1(e) M_\mu^{\nu} w^{-1} M_\mu^{\nu} f_2(e) + L_2(y)), \) where \( L_1(y) \) and \( L_2(y) \) are determined by \( f_1 \) and \( f_2 \), analytic in \((-1, 1)\), and of size \( y^2 O(1) \) near \( 0 \).

**Proof.** The argument is similar to that of the previous lemma and we skip it for brevity.

**Proof of Proposition 4.8 when \( F = \mathbb{R} \).** \( \{\varphi_n|n \in \nu + 2\mathbb{Z}\} \) is a basis of \( \tilde{B}(\mu) \). When \( f_i = \varphi_n \),

\[
I(s; f_1, f_2, f_3, f_4) = \int_{\text{SL}_2(\mathbb{R})} (\tilde{\sigma}(g) f_1, f_2)(\tilde{\sigma}(g) f_3, f_4)|\Delta(g)|^s \, dg \\
= 2\pi^2 \delta_n \delta_n \delta_n \delta_n \int_0^1 (\tilde{\sigma}(y) f_1, f_2)(\tilde{\sigma}(y) f_3, f_4)(y + y^{-1})^{-s} |y^2 - y^{-2}| \, dy \\
= I_1(s) + I_2(s).
\]
where

\[ I_1(s) = 2\pi^2 \delta_{n_1, n_2} \delta_{n_2, n_3} \int_{0}^{1/2} (\tilde{\pi}(y) f_1, f_2)(\tilde{\pi}(y) f_3, f_4)(y + y^{-1})^{-s} |y^2 - y^{-2}|d^s y \]

\[ I_2(s) = 2\pi^2 \delta_{n_1, n_2} \delta_{n_2, n_3} \int_{1/2}^1 (\tilde{\pi}(y) f_1, f_2)(\tilde{\pi}(y) f_3, f_4)(y + y^{-1})^{-s} |y^2 - y^{-2}|d^s y. \]

\( I_2(s) \) is holomorphic in \( s \); to analyze \( I_1(s) \), we need the estimate of matrix coefficients near \( y = 0 \) given in Lemmas 4.19 and 4.20 and we need two situations (A) and (B).

(A) \( \mu \) is unitary. We apply Lemma 4.19 for the estimate of matrix coefficients.

(A1) \( \mu^2 \neq 1 \). Then \( I_1(s) = 8\pi^2 \delta_{n_1, n_2} \delta_{n_2, n_3} (J_1 + J_2 + J_3) \), where

\[
J_1(s) = (M^w u f_1(e) \bar{f}_2(w) M^w u f_3(e) \bar{f}_4(w)) + f_1(e) M^w u f_2(e) f_3(e) M^w u f_4(e) - \mu^{-2}(y) y^2 - y^{-2}|d^s y \]

\[
J_2(s) = (M^w u f_1(e) \bar{f}_3(e) f_2(w) M^w u f_4(e)) + f_1(e) M^w u f_2(e) f_4(w) - \mu^2(y) y^2 - y^{-2}|d^s y \]

\[
J_3(s) = \int_{0}^{1/2} (L_{11}(y) + L_{12}(y) \mu^2(y) + L_{13}(y) \mu^{-2}(y)) y^2 - y^{-2}|d^s y. \]

Write \( \mu(y)^2 = |y|^{r_0} \), then \( r_0 \neq 0 \). We observe that for an analytic function \( \lambda(y) \) on \((-1, 1)\), the functions

\[
\int_{0}^{1/2} \frac{\lambda(y) y^2 - y^{-2}|d^s y}{(y + y^{-1})^s}, \quad \int_{0}^{1/2} \frac{\lambda(y) \mu^2(y) y^2 - y^{-2}|d^s y}{(y + y^{-1})^s}, \quad \text{and} \]

\[
\int_{0}^{1/2} \frac{\lambda(y) \mu^{-2}(y) y^2 - y^{-2}|d^s y}{(y + y^{-1})^s}
\]

have meromorphic continuation to \( \mathbb{C} \) with at most simple poles at \(-n(n \leq 0), -n - r_0(n \leq 0), -n + r_0(n \leq 0)\), respectively; also,

\[
\text{Res}_{s=0} \int_{0}^{1/2} \frac{\lambda(y) y^2 - y^{-2}|d^s y}{(y + y^{-1})^s} = \lambda(0).
\]
Applying this observation to \(J_1(s), J_2(s),\) and \(J_3(s),\) one sees easily that \(I_1(s)\) and \(I(s; f_1, f_2, f_3, f_4)\) have meromorphic continuation to \(\mathbb{C}\) with at most simple poles at \(-n, \pm n - r_0, \pm n + r_0 (n \leq 0).\) Hence the functional-valued function \(I(s)\) has meromorphic continuation to \(\mathbb{C}.

On the other hand, when \(n_1 = n_3\) and \(n_2 = n_4,\) one has

\[
M_w^u f_1(e) f_2(w) M_w^u f_3(e) f_4(w) + f_1(e) M_w^u w^{-1} f_2(e)
\]

\[
= \frac{1}{4\pi^2} \left((M_w^u f_1, M_w^u f_3)(f_2, f_4) + (f_1, f_3)(M_w^u w^{-1} f_2, M_w^u w^{-1} f_4)\right)
\]

\[
= \frac{1}{4\pi^2} \left((M_w^{-1} M_w^u f_1, f_3)(f_2, f_4) + (f_1, f_3)(M_w^{-1} M_w^u w^{-1} f_2, w^{-1} f_4)\right)
\]

\[
= \frac{1}{\pi^2} \gamma(0, \mu^2, \psi) \gamma(0, \mu^{-2}, \psi).
\]

It follows that

\[
\lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) I(s; f_1, f_2, f_3, f_4) = \lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) I_1(s; f_1, f_2, f_3, f_4)
\]

\[
= \lim_{s \to 0} 8\pi^2 \delta_{n_1, n_3} \delta_{n_2, n_4} \gamma(s, \pi, \text{ad}, \psi) J_1
\]

\[
= \lim_{s \to 0} 8(f_1, f_3)(f_2, f_4) \gamma(s, 1, \psi) \left(\frac{1}{s} + \cdots\right)
\]

\[
= 4(f_1, f_3)(f_2, f_4)
\]

\[
= 2 \vert\vert f_1, f_3 \vert\vert(f_2, f_4).
\]

(A2) \(\mu^2 = 1.\) Then \(I_1(s; f_1, f_2, f_3, f_4) = 64\pi^2 \delta_{n_1, n_3} \delta_{n_2, n_4} (\tilde{J}_1(s) + \tilde{J}_2(s)),\) where

\[
\tilde{J}_1(s) = f_1(e) f_3(e) f_2(w) f_4(w) \int_0^{1/2} \frac{y^2 (\ln y)^2 |y^2 - y^{-2}| d^x y}{(y + y^{-1})^s},
\]

\[
\tilde{J}_2(s) = f_1(e) f_3(e) f_2(w) f_4(w) \int_0^{1/2} \frac{y^2 (-\ln y + \tilde{L}(y)) |y^2 - y^{-2}| d^x y}{(y + y^{-1})^s}.
\]

\(\tilde{J}_1(s)\) is meromorphic in \(s\) and expanded as \(\frac{2}{s^3} + \cdots\) near \(s = 0;\) \(\tilde{J}_2(s)\) is meromorphic in \(s\) and expanded as \(\frac{1}{s^2} + \cdots\) near \(s = 0.\) Hence \(I_1(s)\) and \(I(s; f_1, f_2, f_3, f_4)\) are meromorphic in \(s\) with a pole at \(s = 0\) of order 3. It follows that \(I(s)\) has meromorphic continuation to \(\mathbb{C}.\)
On the other hand, \( \gamma(s, \pi, \text{ad}, \psi) = \frac{1}{8}s^3 + \cdots \) near \( s = 0 \) and we have
\[
\lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) I(s; f_1, f_2, f_3, f_4) = \lim_{s \to 0} \gamma(s, \pi, \text{ad}, \psi) I_1(s; f_1, f_2, f_3, f_4)
\]
\[
= \lim_{s \to 0} 64\pi^2 \delta_{n_1,n_2}\delta_{n_3,n_4} \gamma(s, \pi, \text{ad}, \psi) J_1
\]
\[
= \lim_{s \to 0} 16(f_1, f_3)(f_2, f_4) \gamma(s, 1, \psi)^3 \left( \frac{2}{s^3} + \cdots \right)
\]
\[
= 4(f_1, f_3)(f_2, f_4)
\]
\[
= 2|2|_R(f_1, f_3)(f_2, f_4).
\]

(B) \( \mu(x) = |x|^{s_0} \text{sgn}(x)^0 \) with \( s_0 \in (0, 1/2) \). The argument is similar to the one for (A1) and only different in that one uses Lemma 4.13 to estimate the matrix coefficients. For brevity, we skip the proof.

### 4.4 \( \tilde{\pi} \) is square-integrable

Now we deal with the case that \( \tilde{\pi} \) is square-integrable.

When \( F \) is \( p \)-adic, \( \tilde{\pi} \) is either supercuspidal or a special representation. If \( \pi \) is supercuspidal, then the matrix coefficients of \( \tilde{\pi} \) are compactly supported and \( I_{\tilde{\pi}}(s) \) is holomorphic in \( s \). If \( \tilde{\pi} \) is a special representation, then \( \tilde{\pi} \) is a subrepresentation of \( \tilde{\pi}(\chi \cdot | \cdot |^{1/2}) \) where \( \chi \) is a quadratic character; the method for principal series easily shows that \( I_{\tilde{\pi}}(s) \) has meromorphic continuation to \( \mathbb{C} \). In either case, \( I_{\tilde{\pi}}(0) \) is \( G \times G \)-invariant because \( \tilde{\pi} \) is square-integrable.

When \( F = \mathbb{R} \), \( \tilde{\pi} \) is part of a principal series \( \tilde{\pi}(\mu) \) (cf. Remark 4.15). The method for principal series easily shows that \( I_{\tilde{\pi}}(s) \) has meromorphic continuation to \( \mathbb{C} \); \( I_{\tilde{\pi}}(0) \) is \( G \times G \)-invariant because \( \tilde{\pi} \) is square-integrable.

### 4.5 Concluding remark

Propositions 4.2, 4.8, and the argument in Section 4.4 for square-integrable case give the complete proof of Theorem 4.1.

### 5 The Pair \((\widetilde{SL}_2, SO_3)\)

When \( F \) is \( \mathbb{R} \) or a \( p \)-adic field, there are two nonisomorphic \( SO_3(F) \). Let \( B \) be a quaternion over \( F \) and put \( V = \{ x \in B \mid \text{Tr}(x) = 0 \} \), \( q(x) = -\nu(x) \), where \( \nu \) is the reduced norm of \( B \).
Then \( \text{SO}(V, q) \cong PB^\times = B^\times / F^\times \) and \( B \) acts on \( V \) by \( b \circ v = bv b^{-1} \). Any \( \text{SO}_3(F) \) is isomorphic to an \( \text{SO}(V, q) \) obtained in this way; \( B \) is either the matrix algebra \( M_{2 \times 2}(F) \) or the division algebra, and accordingly we obtain the noncompact or compact \( \text{SO}_3 \).

Let \( \psi' \) be a nontrivial character of \( F \) and \( \omega_{\psi', q} \) be the Weil representation of \( \widetilde{\text{SL}}_2(F) \times PB^\times \) on \( S(V) \) (as given in [17, Section I.3]). For a genuine irreducible admissible representation \( \tilde{\pi} \) of \( \widetilde{\text{SL}}_2(F) \) and an irreducible admissible representation \( \pi \) of \( PB^\times \), we say that they are in local theta correspondence with respect to \( \psi' \) if \( \dim \mathbb{C} \text{Hom}_{\widetilde{\text{SL}}_2(F) \times PB^\times} (\omega_{\psi', q}, \tilde{\pi} \otimes \pi) \neq 0 \). Local Howe duality assures that the dimension is 1 when it is nonzero. In this case, we also know that unitary representations are in correspondence with unitary representations.

**Theorem 5.1.** Let \( \tilde{\pi} \) be a genuine irreducible unitary representation of \( \widetilde{\text{SL}}_2(F) \) and \( \pi \) be an irreducible unitary representations of \( PB^\times \). If \( \tilde{\pi} \) and \( \pi \) are in local theta correspondence with respect to a nontrivial character \( \psi' \) of \( F \), then \( d_{\tilde{\pi}}(s) = |2|^{-1} d_{\pi}(s) \).

We first treat the case that \( B \) is the division algebra and then the case that \( B = M_{2 \times 2}(F) \).

### 5.1 Compact \( \text{SO}_3 \)

Now we let \( B \) be the division algebra over \( F \), whence \( V \) is anisotropic and \( \text{SO}(V, q) \cong PB^\times \) is compact. The advantage of \( PB^\times \) being compact is that

\[
\omega_{\psi', q} = \bigoplus_{\pi_i \in \text{Irr}(PB^\times)} \tilde{\pi}_i \otimes \pi_i,
\]

as an \( \widetilde{\text{SL}}_2(F) \times PB^\times \)-module, where \( \text{Irr}(PB^\times) \) is the isomorphism classes of irreducible admissible representations of \( PB^\times \) and \( \tilde{\pi}_i \) are in local theta correspondence with \( \pi_i \).

Hence for a particular \( \pi \in \text{Irr}(PB^\times) \), one has

\[
\tilde{\pi} \cong \text{Hom}_{PB^\times}(\pi, \omega_{\psi', q}) \cong (\pi^\vee \otimes \omega_{\psi', q})^{PB^\times} \cong (\pi \otimes \omega_{\psi', q})^{PB^\times} = S(V, V_\pi)^{PB^\times},
\]

where \( V_\pi \) is the space on which \( \pi \) acts and \( S(V, V_\pi) \) are \( V_\pi \)-valued Bruhat–Schwartz functions on \( V \). It leads us to the following realization of \( \tilde{\pi} \).

\[
(i) \quad V_\tilde{\pi} = S(V, V_\pi)^{PB^\times} = \{ f: V \to V_\pi | f(b \circ v) = \pi(b) f(v) \text{ for all } b \in B^\times, v \in V \};
\]
(ii) \( \tilde{\pi} \) is the action of \( \widetilde{SL}_2 \) on \( V_\pi \) induced from the action of \( \widetilde{SL}_2 \) on \( S(V) \), namely

\[
\tilde{\pi} \left( \begin{pmatrix} a & \epsilon \\ a^{-1} & 1 \end{pmatrix} \right) f(X) = |a|^{3/2} \frac{\gamma(\psi')}{\gamma(a)} \epsilon f(aX),
\]

\[
\tilde{\pi} \left( \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) f(X) = \psi'(nq(X)) f(X),
\]

\[
\tilde{\pi}(w, 1) f(X) = \gamma(\psi', q) \int_V f(Y) \psi'(q(X, Y)) dY.
\]

Here \( w = (1, -1) \), \( q(X, Y) = q(X + Y) - q(X) - q(Y) \), and \( dY \) is the Haar measure of \( V \) that is self-dual with respect to \( \psi'(q(X, Y)) \). If we choose an orthogonal basis \( e_1, e_2, e_3 \) of \( V \) and write \( q(x_1 e_1 + x_2 e_2 + x_3 e_3) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \), then \( \gamma(\psi', q) = \gamma(\psi'_{a_1}) \gamma(\psi'_{a_2}) \gamma(\psi'_{a_3}) \) (cf. Section 4.1.4);

(iii) if \( \pi \) is unitary, then \( \tilde{\pi} \) is unitary with respect to the pairing

\[
(f_1, f_2) = \int_V (f_1(X), f_2(X))_{\psi} dX.
\]

When \( B \) is the division algebra over \( F \), \( \tilde{\pi} \) is a square-integrable representation. Then Theorem 5.1 is equivalent to say that for all \( f_i \in V_\tilde{\pi} \ (1 \leq i \leq 4) \),

\[
I_\pi(0; f_1, f_2, f_3, f_4) = |2|_F d(\pi)^{-1}(f_1, f_3)(f_2, f_4) dg,
\]

where \( I_\pi(s) \) denotes the functional-valued function associated to \( \tilde{\pi} \). We shall prove this equality by direct computation. To simplify the writing, we assume that \( \psi' = \psi \) and write \( \omega_{\psi} \) for \( \omega_{\psi', q} \); but the argument below works for arbitrary \( \psi' \).

The first step is to decompose the measures of \( SL_2(F) \), \( B^\times \) and \( V \) so that integration on them becomes easy. Recall that \( |·| \) denotes the absolute value function on \( F \) (cf. Section 2). The following three lemmas are easy to prove and we skip their proofs.

**Lemma 5.2.** The measure of \( SL_2(F) \) at

\[
g = \begin{pmatrix} 1 & n_1 \\ n_1 & 1 \end{pmatrix} \begin{pmatrix} a & \epsilon \\ a^{-1} & 1 \end{pmatrix} w \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix}
\]

is \( dg = |a|^{-2} |d^\times a| \, dn_1 \, dn_2 \). \( \square \)
Lemma 5.3. Let $B$ be a division algebra over $F$ and write $B = F \cdot 1 \oplus F v_1 \oplus F v_2 \oplus F v_3$ with $v_1^2 = \delta$, $v_2^2 = \tau$, and $v_3 = v_1 v_2 = -v_2 v_1$.

(1) The multiplicative Haar measure of $B^\times$ at $b = x_0 + x_1 v_1 + x_2 v_2 + x_3 v_3$ is

$$d^\times b = \frac{|4 \delta \tau| \, dx_0 \, dx_1 \, dx_2 \, dx_3}{|v(b)|^2}.$$ 

(2) Write $K_\delta = F \oplus F e_1$, $T_\delta = K_\delta^\times$, $B_\delta^\circ = T_\delta \oplus T_3 v_2$. Then $B^\times / T_\delta = ([1]) \cup \{[v_2]\} \cup (B_\delta^\circ / T_\delta)$ and $B_\delta^\circ / T_3 = ([1 + tv_2]| t \in T_3\}$, where $[\cdot]$ denotes a left coset of $T_3$. The induced measure on $B^\times / T_\delta$, when restricted to $B_\delta^\circ / T_3$, takes the form

$$d[1 + tv_2] = \frac{|\tau| \, dt}{|1 - \tau v(t)|^2}, \quad t \in T_3.$$ 

(3) $T_3 \setminus B^\times / T_\delta = ([1]) \cup \{[v_2]\} \cup (T_3 \setminus B_\delta^\circ / T_3)$ and $T_3 \setminus B_\delta^\circ / T_3 = \{[1 + tv_2]| t \in T_3 / T_3, 1\}$, where $[\cdot]$ denotes a double coset of $T_3$. With respect to the parameterization $\alpha : T_3 \setminus B_\delta^\circ / T_3 \to \nu(T_3), [1 + tv_2] \to s = \nu(t)$, the induced measure on $T_3 \setminus B_\delta^\circ / T_3$ takes the form

$$d[1 + tv_2] = \frac{|\tau| \, d\nu(t)}{|1 - \tau \nu(t)|^2} = \frac{|\tau| \, ds}{|1 - \tau s|^2}.$$ 

Lemma 5.4. Denote by $\Sigma_t$ the $t$-sphere in $V$, namely $\Sigma_t := \{v \in V : q(v) = t\}$. Then there is an $\text{SO}(V)$-invariant measure $d\mu_t$ on $\Sigma_t$ such that for $\varphi \in S(V)$,

$$\int_V \varphi(X) \, dX = \int_{q(V)} \int_{\Sigma_t} \varphi(X) \, d\mu_t(X) \, dt.$$ 

We have $d\mu_{t^2} = |r| \, d\mu_t$, $m(\Sigma_{t^2}) = |r| m(\Sigma_t)$, and $m(\Sigma_\delta) = |2|^{3/2} |\delta|^{1/2} m(B^\times / T_\delta)$ for $\delta \in q(V - \{0\})$. 

The second step is to decompose the pairing on $V_\delta$ with respect to the above measure decomposition of $V$. Write $q(V - \{0\}) / F^\times = \{[\delta_i]\}$, with $\{\delta_i\}$ being a choice of representatives and $[\delta_i]$ denoting $\delta_i F^\times$. For each $\delta \in \{\delta_i\}$, we write $\Sigma_\delta = \bigcup_{t \in [\delta]} \Sigma_t$, fix a vector $X_\delta \in V$ with $q(X_\delta) = \delta$, and put $K_\delta = F \oplus F \cdot X_\delta \subseteq B$, $T_\delta = K_\delta^\times$. Then $V_\delta$ contains functions that are nonvanishing on $\Sigma_\delta$ if and only there are nonzero $T_\delta$-invariant vectors in $V_\tau$. (For example, $X_\delta$ is fixed by $T_\delta$. Hence $f$ belonging to $V_\delta$ implies that $f(X_\delta)$ is fixed by $T_\delta$.) The following lemma on $T_\delta$-invariant vectors in $\pi$ are well known.
Lemma 5.5. Let $\pi$ be a finite-dimensional irreducible admissible representation of $PB^\times$. Then

1. There exists $\delta \in \{\delta_i\}$ such that $\pi$ has nonzero $T_\delta$-invariant subspace $[16]$.
2. For each $\delta \in \{\delta_i\}$, the $T_\delta$-invariant subspace of $\pi$ is at most one dimensional $[12]$. $\square$

Denote by $(F^\times F^\times^2)_\pi$ the union of $F^\times^2$-classes $[\delta_i]$ for those $\delta_i$ such that $\pi$ has nonzero $T_\delta$-invariant subspace. Then for $f_1, f_2 \in V_\pi$, one has

$$(f_1, f_2) = \int_{V} (f_1(X), f_2(X)) \ dX$$

$$= \sum_{[\delta] \in (F^\times F^\times^2)_\pi} \int_{[\delta]} \int_{\Sigma_\delta} (f_1(X), f_2(X)) \ d\mu_\delta(X) \ dt$$

$$= \sum_{[\delta] \in (F^\times F^\times^2)_\pi} |\delta| \int_{F^\times^2} \int_{\Sigma_\delta} (f_1(t^{1/2} X), f_2(t^{1/2} X)) |t|^{1/2} \ d\mu_\delta(X) \ dt.$$ 

We point out that (i) $\Sigma_\delta = \{ b \circ X_\delta | [b] \in B^\times / T_\delta \}$ because $B^\times$ acts transitively on each $\Sigma_\delta$ and the stabilizer of $X_\delta$ is $T_\delta$; (ii) For each $\delta \in \{\delta_i\}$ such that $[\delta] \in (F^\times F^\times^2)_\pi$, we fix a vector $e_\delta$ of norm 1 on the $T_\delta$-invariant line of $V_\pi$, then $f|_{F X_\delta}$ is valued in $\mathbb{C} \cdot e_\delta$.

These two facts enable us to decompose $(f_1, f_2)$ further.

$$(f_1, f_2)$$

$$= \sum_{[\delta] \in (F^\times F^\times^2)_\pi} \frac{|2\delta|m(\Sigma_\delta)}{2 \cdot m(G/T_\delta)} \int_{F^\times^2} \int_{G/T_\delta} (f_1(t^{1/2} b \circ X_\delta), f_2(t^{1/2} b \circ X_\delta)) |t|^{1/2} \ d[b] \ dt$$

$$= \sum_{[\delta] \in (F^\times F^\times^2)_\pi} \frac{2 \cdot \delta|m(\Sigma_\delta)}{2 \cdot m(G/T_\delta)} \int_{F^\times} \int_{G/T_\delta} (b \circ f_1(t^{1/2} \circ X_\delta), b \circ f_2(t^{1/2} \circ X_\delta)) |t|^{1/2} \ d[b] \ dt$$

$$= \sum_{[\delta] \in (F^\times F^\times^2)_\pi} \frac{|2\delta|m(\Sigma_\delta)}{2} \int_{F^\times} (f_1(t^{1/2} X_\delta), f_2(t^{1/2} X_\delta)) |t|^{1/2} \ dt$$

$$= \sum_{[\delta] \in (F^\times F^\times^2)_\pi} \frac{|2\delta|m(\Sigma_\delta)}{2} \int_{F^\times} (f_1(t^{1/2} X_\delta), e_\delta)(f_2(t^{1/2} X_\delta), e_\delta) |t|^{1/2} \ dt.$$ 

The third ingredient we need is the following lemma.
Lemma 5.6. Suppose that \([\delta] \in (F^\times/F^{\times 2})_\pi\) and \(f_1, f_2 \in V_{\pi}\) are supported in \(\Sigma_{[\delta]}\). Then

\[
\int_{F} \int_{P_{F}^\times} (w_n \circ f_1(aX_\delta), w_n \circ f_2(aX_\delta)) \, da \, dn = |2| m(\Sigma_{\delta})^{-1} d(\pi)^{-1}(f_1, f_2).
\]

Proof. We first expand the functions involved in the integral,

\[
w_n \circ f_i(aX_\delta) = \int_X n \circ f_i(Y_i) \psi(q(Y_i, aX_\delta)) \, dY_i
\]

\[
= \int_{[\delta]} \int_{\Sigma_i} f_i(Y_i) \psi(nt_i) \psi(q(Y_i, aX_\delta)) \, d\mu_\delta(Y_i) \, dt_i
\]

\[
t_i \mapsto \delta \int_{P_{F}^\times} \int_{\Sigma_i} f_i(t_i^{1/2} X_\delta) \psi(nt_i \delta) \psi(q(Y_i, X_\delta)at_i^{-1/2}) \, d\mu_\delta(Y_i) |t_i|^{1/2} \, dt_i.
\]

Because \(B^\times/T_\delta \cong \Sigma_{\delta}, [g] \rightarrow g \circ X_\delta\), we may substitute \(Y_i\) by \(b_i \circ X_\delta\) and get

\[
w_n \circ f_i(aX_\delta)
\]

\[
= \frac{|\delta| m(\Sigma_{\delta})}{m(B^\times/T_\delta)} \int_{P_{F}^\times} \int_{B^\times/T_\delta} f_i(t_i^{1/2} b_i \circ X_\delta) \psi(nt_i \delta) \psi(q(b_i \circ X_\delta, X_\delta)at_i^{-1/2}) \, d|b_i||t_i|^{1/2} \, dt_i
\]

\[
= \frac{|\delta| m(\Sigma_{\delta})}{m(B^\times/T_\delta)} \int_{P_{F}^\times} \int_{B^\times/T_\delta} b_i \circ f_i(t_i^{1/2} X_\delta) \psi(nt_i \delta) \psi(q(b_i \circ X_\delta, X_\delta)at_i^{-1/2}) \, d|b_i||t_i|^{1/2} \, dt_i
\]

\[
= \frac{|\delta| m(\Sigma_{\delta})}{m(B^\times/T_\delta)} \int_{P_{F}^\times} \int_{B^\times/T_\delta} b_i \circ e_i(f_i(t_i^{1/2} X_\delta), e_i) \psi(nt_i \delta) \psi(q(b_i \circ X_\delta, X_\delta)at_i^{-1/2}) \, d|b_i||t_i|^{1/2} \, dt_i.
\]

It follows that \((w_n \circ f_1(aX_\delta), w_n \circ f_2(aX_\delta)) = (w_n \circ f_1(aX_\delta), e_i)(w_n \circ f_2(aX_\delta), e_i)\). It is not hard to compute that

\[
(w_n \circ f_i(aX_\delta), e_i)
\]

\[
= \frac{|\delta| m(\Sigma_{\delta})}{m(B^\times/T_\delta)} \int_{P_{F}^\times \times B^\times/T_\delta} (b_i \circ e_i, e_i)(f_i(t_i^{1/2} X_\delta), e_i) \psi(nt_i \delta) \psi(q(b_i \circ X_\delta, X_\delta)at_i^{-1/2}) \, d|b_i||t_i|^{1/2} \, dt_i
\]

\[
= \frac{|\delta| m(\Sigma_{\delta}) m(T_\delta/F^\times)}{m(B^\times/T_\delta)} \int_{P_{F}^\times \times (T_\delta/F^\times)} (b_i \circ e_i, e_i)(f_i(t_i^{1/2} X_\delta), e_i) \psi(nt_i \delta)
\]

\[
\times \psi(q(b_i \circ X_\delta, X_\delta)at_i^{-1/2}) \, d|b_i||t_i|^{1/2} \, dt_i.
\]

We put \(v_1 = X_\delta\), choose \(v_2 \in V\) orthogonal to \(v_1\), and write \(v_3 = v_1 v_2\), \(\tau = v_3^2\). Then we can apply Lemma 5.3 to express the measure on \(T_\delta \setminus B^\circ_{\delta}/T_\delta\) with respect to the
parameterization $\alpha : T_3 \backslash B_3^\circ / T_3 \to q(T_3), \langle 1 + tv_2 \rangle \to s = v(t)$. To facilitate our calculation, we introduce another parameterization $\beta : T_3 \backslash B^\times / T_3 \to F, \langle [b] \rangle \to \lambda = q(b \circ X_\delta, X_\delta)$. The two parameterizations and related measures can be compared with $T_3 \backslash B_3^\circ / T_3$ as below:

$$
\lambda = \beta \alpha^{-1}(s) = \frac{2\delta(1 + \tau s)}{1 - \tau s}, \quad s = \alpha \beta^{-1}(\lambda) = \frac{\lambda - 2\delta}{\tau(\lambda + 2\delta)}, \quad d\lambda = \frac{|4\delta \tau| ds}{|1 - \tau s|^2} = |4\delta| [d[b]].
$$

Hence

$$(w \eta \circ f_1(a X_\delta), e_\delta) = \left| \frac{4^{-1} m(\Sigma_\delta) m(T_3 / F^\times)}{m(B^\times / T_3)} \right| \int_{F^\times} \int_{\text{Im}(\beta)} (\beta^{-1} \lambda_i \circ e_\delta, e_\delta) (f_1(t_i^{1/2} X_\delta), e_\delta)

\times \psi(nt_i \delta) \psi(\lambda_i a t_i^{1/2}) \, d\lambda_i |t_i|^{1/2} \, dt_i$$

$$
\lambda_i = \lambda_i^{1/2} \left| \frac{4^{-1} m(\Sigma_\delta) m(T_3 / F^\times)}{m(B^\times / T_3)} \right| \int_{F^\times} \int_{F^\times} (\beta^{-1} \lambda_i t_i^{-1/2} \circ e_\delta, e_\delta) 1_{\text{Im}(\beta)}(\lambda_i t_i^{-1/2})

\times (f_1(t_i^{1/2} X_\delta), e_\delta) \psi(nt_i \delta) \psi(\lambda_i a) \, d\lambda_i \, dt_i.
$$

The inner $\lambda_i$-integral can be regarded as a Fourier transform from variable $\lambda_i$ to variable $a$. Therefore,

$$
\int_F (w \eta \circ f_1(a X_\delta), w \eta \circ f_2(a X_\delta)) \, da

= \left| \frac{4^{-2} |\delta|^{-1} m(\Sigma_\delta)^2 m(T_3 / F^\times)^2}{m(B^\times / T_3)^2} \right| \int_{F^\times \times F^\times} \int_F (\beta^{-1} (\lambda t_1^{-1/2} \circ e_\delta, e_\delta) \overline{(\beta^{-1} (\lambda t_2^{-1/2} \circ e_\delta, e_\delta) 1_{\text{Im}(\beta)}(\lambda t_1^{-1/2}) \lambda t_2^{-1/2}) (f_1(t_1^{1/2} X_\delta), e_\delta)(f_2(t_2^{1/2} X_\delta), e_\delta) \psi(nt_1 \delta) \overline{\psi(nt_2 \delta)}} \, d\lambda \, dt_1 \, dt_2.
$$

Now we move the $\lambda$-integral to the outside. The $t_i$-integrals can be considered as Fourier transforms from variable $t_i$ to variable $n$. It follows that

$$
\int_F \int_{F^\times} (w \eta \circ f_1(a X_\delta), w \eta \circ f_2(a X_\delta)) \, da \, dn

= \left| \frac{4^{-2} |\delta|^{-1} m(\Sigma_\delta)^2 m(T_3 / F^\times)^2}{m(B^\times / T_3)^2} \right| \int_F \int_{F^\times} (\beta^{-1} (\lambda t_1^{-1/2} \circ e_\delta, e_\delta) \overline{(\beta^{-1} (\lambda t_2^{-1/2} \circ e_\delta, e_\delta) 1_{\text{Im}(\beta)}(\lambda t_1^{-1/2}) \lambda t_2^{-1/2}) (f_1(t_1^{1/2} X_\delta), e_\delta)(f_2(t_2^{1/2} X_\delta), e_\delta) \psi(nt_1 \delta) \overline{\psi(nt_2 \delta)}} \, dt_1 \, dt_2.
$$

$$
1_{\text{Im}(\beta)}(\lambda t_i^{-1/2})(f_1(t_i^{1/2} X_\delta), e_\delta)(f_2(t_i^{1/2} X_\delta), e_\delta) \, dt_1 \, dt_2.
$$
\[
\lambda \rightarrow \lambda^{1/2} \frac{|4|^{-2} |\delta|^{-1} m(\Sigma_{\delta})^2 m(T_{\delta}/F^\times)^2}{m(B^\times/T_{\delta})^2} \int_{E} \int_{F^\times} (\beta^{-1}(\lambda) \circ e_3, e_3)(\beta^{-1}(\lambda) \circ e_3, e_3) 1_{\text{Im}(\beta)}(\lambda) \, d\lambda \, dt \, d\lambda \\
1_{\text{Im}(\beta)}(\lambda)(f_1(t^{1/2}X_{\delta}), e_3)(f_2(t^{1/2}X_{\delta}), e_3)|t|^{1/2} \, dt \, d\lambda \\
= \frac{|4|^{-2} |\delta|^{-2} m(\Sigma_{\delta}) m(T_{\delta}/F^\times)^2}{m(B^\times/T_{\delta})^2} \int_{\text{Im}(\beta)} (\beta^{-1}(\lambda) \circ e_3, e_3)(\beta^{-1}(\lambda) \circ e_3, e_3) \, d\lambda(f_1, f_2).
\]

We then move the \(\lambda\)-integral back to the environment of \(T_{\delta}\backslash B^\times/T_{\delta}\),

\[
\int_{\text{Im}(\beta)} (\beta^{-1}(\lambda) \circ e_3, e_3)(\beta^{-1}(\lambda) \circ e_3, e_3) \, d\lambda = |4\delta| \int_{T_{\delta}\backslash B^\times/T_{\delta}} (b \circ e_3, e_3)(b \circ e_3, e_3) \, d[[b]] \\
= |4\delta| m(T_{\delta}/F^\times)^{-2} \int_{F^\times} (b \circ e_3, e_3)(b \circ e_3, e_3) \, d\pi \\
= |4\delta| m(T_{\delta}/F^\times)^{-2} d(\pi)^{-1}.
\]

Therefore,

\[
\int_{F} \int_{F^\times} (w n \circ f_1(aX_{\delta}), w n \circ f_2(aX_{\delta})) \, da \, dn = \frac{|4\delta|^{-1} m(\Sigma_{\delta})}{m(B^\times/T_{\delta})^2} \, d(\pi)^{-1}(f_1, f_2).
\]

By Lemma 5.4, the constant \(\frac{|4\delta|^{-1} m(\Sigma_{\delta})}{m(B^\times/T_{\delta})^2} = 2|\delta| m(\Sigma_{\delta})^{-1}\). Hence we obtained the listed equality. \(\blacksquare\)

**Proof of Theorem 5.1 when \(B\) is the division algebra.** For simplicity, we assume that \(\psi' = \psi\). Choose one \(\delta \in \{\delta_i\}\) such that \(\delta \in (F^\times/F^\times)^\times\). Then it suffices to verify

\[
I_{\tilde{\pi}}(0; f_1, f_2, f_3, f_4) = |2| F d(\pi)^{-1}(f_1, f_2) \overline{(f_3, f_4)} \, dg
\]

for functions \(f_i(1 \leq i \leq 4) \in V_{\tilde{\pi}}\) supported on \(\Sigma_{\delta}\).

By Lemma 5.2, one has

\[
I_{\tilde{\pi}}(0; f_1, f_2, f_3, f_4) = \int_{\text{SL}_2(F)} (\tilde{\pi}(g) f_1, f_2)(\tilde{\pi}(g) f_3, f_4) \, dg \\
= \int_{F^\times F^\times F^\times} (n_1 aw n_2 \circ f_1, f_2)(n_1 aw n_2 \circ f_3, f_4)|a|^{-2} \, dn_1 \, d^\times a \, dn_2.
\]
Since the $f_i$’s are supported on $\Sigma_{[\delta]}$, we have
\[
(n_1 \circ a \circ w \circ n_2 \circ f_i, f_j) = \frac{|2\delta|m(\Sigma_{\delta})}{2} \int_{F^\times} (\psi(t^2u\delta n_1)awn_2 \circ f_i(uX_\delta), f_j(uX_\delta))|u|^2 \, du
\]
\[
= \int_{F^\times} \psi(t^2\delta n_1)(awn_2 \circ f_i(t^2X_\delta), f_j(t^2X_\delta))|t^2| \, dt.
\]
Here $t^2$ denotes one of the square roots of $t$ for $t \in F^{\times 2}$. The integrand in the above $t$-integral does not depend on the choice of the square root of $t$. Also, the measure $dt$ used above is the Haar measure of $F$ that is self-dual with respect to $\psi(2st)$. We may regard the above expression as a Fourier transform from variable $t_i$ to variable $t$ to variable $n_i$. Hence,
\[
\int_F (n_1 awn_2 \circ f_i, f_2)(n_1 awn_2 \circ f_3, f_4) \, dn_1
\]
\[
= |\delta|m(\Sigma_{\delta})^2 \int_{F^{\times 2} \times F^\times} (aw \circ f_1(t^{1/2}X_\delta), f_2(t^{1/2}X_\delta))(aw \circ f_3(t^{1/2}X_\delta), f_4(t^{1/2}X_\delta))|t| \, dt \, da
\]
Therefore,
\[
\int_F (n_1 awn_2 \circ f_i, f_2)(n_1 awn_2 \circ f_3, f_4) \, dn_1 |a|^{-2} \, da
\]
\[
= |\delta|m(\Sigma_{\delta})^2 \int_{F^{\times 2} \times F^\times} (w_1 \circ f_1(at^{1/2}X_\delta), f_2(t^{1/2}X_\delta))(w_2 \circ f_3(at^{1/2}X_\delta), f_4(t^{1/2}X_\delta))|t| \, dt \, da
\]
\[
= |\delta|m(\Sigma_{\delta})^2 \int_{F^{\times 2} \times F^\times} (w_2 \circ f_1(at^{1/2}X_\delta), w_1 \circ f_3(at^{1/2}X_\delta))(f_2(t^{1/2}X_\delta), f_4(t^{1/2}X_\delta))|t|^{1/2} \, dt \, da
\]
\[
a \rightarrow at^{-1/2} \rightarrow |\delta|m(\Sigma_{\delta})^2 \int_{F^{\times 2} \times F^\times} (w_2 \circ f_1(aX_\delta), w_1 \circ f_3(aX_\delta))(f_2(t^{1/2}X_\delta), f_4(t^{1/2}X_\delta))|t|^{1/2} \, dt \, da
\]
\[
= (m(\Sigma_{\delta}) \int_{F^\times} (w_2 \circ f_1(aX_\delta), w_1 \circ f_3(aX_\delta)) \, da)(f_2, f_4).
\]

It follows that $I_{\pi}(0; f_1, f_2, f_3, f_4) = (m(\Sigma_{\delta}) \int_{F^\times} (wn_2 \circ f_1(aX_\delta), wn_2 \circ f_3(aX_\delta)) \, da \, dn) \cdot (f_2, f_4)$. Now we apply Lemma 5.6 and obtain that $I_{\pi}(0; f_1, f_2, f_3, f_4) = |2|d(\pi)^{-1}(f_1, f_3)(f_2, f_4)$.

5.2 Noncompact $SO_3$

If $B = M_{2 \times 2}(F)$, then $V$ is isotropic and $SO(V, q) \cong PGL_2(F)$ is noncompact.
Proof of Theorem 5.1 when $B = M_{2 \times 2}(F)$. There are three possibilities of $\pi$.

1. $\pi$ is one dimensional. Then $\pi = \chi_a(\det)$ where $\chi_a = \langle a | \cdot \rangle_F$ is a quadratic character associated to $a \in F^\times$. It follows that $\tilde{\pi} = \omega_{\psi_a}$. By Lemma 3.7 and Proposition 4.2, we have

$$d_{\tilde{\pi}}(s) = \frac{\zeta_F(2)}{2|2|_F \zeta_F(-1)} = |2|_F^{-1} d_{\pi}(s).$$

2. $\pi = \pi(\mu, \mu^{-1})$ is an unitary irreducible principal series. If $\psi' = \psi_a$, then $\tilde{\pi} = \tilde{\pi}(\chi_a \mu)$. Hence by Proposition 4.8, we have

$$d_{\tilde{\pi}}(s) = \frac{\gamma(s, \pi \otimes \chi_a, \text{ad}\psi)}{2|2|_F} = |2|_F^{-1} d_{\pi}(s).$$

3. $\pi$ is square-integrable. Let $D$ be the division algebra over $F$ and $\pi'$ be the Jacquet–Langlands lift of $\pi$ from $\text{PGL}_2$ to $PD^\times$. By [17], there exists $a \in F^\times$ such that $\tilde{\pi}$ and $\pi' \otimes \chi_a$ are in local theta correspondence with respect to $\omega_{\psi_a}$. Theorem 5.1 in the compact case tells that $d(\tilde{\pi}) = |2|_F^{-1} d(\pi' \otimes \chi_a) = |2|_F^{-1} d(\pi')$. Since $d(\pi') = d(\pi)$, we get $d(\tilde{\pi}) = |2|_F^{-1} d(\pi)$.

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References


