NORMALIZED LOCAL THETA CORRESPONDENCE
AND THE DUALITY OF INNER PRODUCT FORMULAS

BY

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ABSTRACT

We conjecture that local theta correspondence can be normalized by the
leading coefficient of a weighted local period integral, and that there exists
a duality of local and global inner product formulas. The conjecture is
verified for the pair (\(\widetilde{SL}_2, PGL_2\)) and (\(SL_2, SO(2, 2)\)). As an application,
global inner product formulas are obtained for liftings in the directions
\(PGL_2 \to \widetilde{SL}_2, GSO(2, 2) \to GL_2\).

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Introduction

Local Howe duality [6] asserts that the oscillator representation provides a local theta correspondence between irreducible admissible representations of a dual reductive pair. In this paper, we shall use period integrals of matrix coefficients to quantitatively study this correspondence, or more specifically, to normalize the homomorphism from the oscillator representation to unitary representations that are in correspondence. Such an approach reveals a quantitative duality between inner product formulas and leads to a new concept, the relative degree of local theta lifting.

Our theory, though conjectural in general, is verified in this paper for the dual reductive pairs $(\widetilde{SL}_2, PGL_2)$ and $(SL_2, SO(2, 2))$ over real and $p$-adic fields, and, as an application, we have deduced global inner product formulas in the directions $PGL_2 \to \widetilde{SL}_2$, $GSO(2, 2) \to GL_2$ from the inner product formulas in the inverse directions.

Now let us explain the setting so that the main results can be stated precisely. Let $W$ be a symplectic space over a $p$-adic field or $\mathbb{R}$, $\widetilde{Sp}(W)$ the 2-fold metaplectic cover of the symplectic group $Sp(W)$, and $\omega_\psi$ the oscillator representation of $\widetilde{Sp}(W)$ associated to a non-trivial character $\psi$ of $F$; let $(G, H)$ be a reductive dual pair in $Sp(W)$ and $\widetilde{G}, \widetilde{H}$ their preimages in $\widetilde{Sp}(W)$; let $\sigma$ and $\pi$ be genuine irreducible admissible unitary representations of $\widetilde{G}$ and $\widetilde{H}$ that are in local theta correspondence with respect to $\omega_\psi$ (cf. Section 1.1). Then $\text{Hom}_{\widetilde{G} \times \widetilde{H}}(\omega_\psi, \sigma \otimes \pi)$ has dimension 1 and we call a non-zero element $\theta$ in this space a local theta correspondence between $\sigma$ and $\pi$; a chosen $\theta$ leads to explicit local theta liftings in two directions: for $\phi \in \omega_\psi, \varphi \in \sigma, f \in \pi$, one defines $\theta(\phi, \varphi) := (\theta(\phi), \varphi)_{\sigma}$ and $\theta(\phi, f) := (\theta(\phi), f)_{\pi}$.

We plan to normalize $\theta$ (up to a complex unit) by normalizing the following invariant pairings associated to $\theta$:

\[ J_\theta(\phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2) = \left( \theta(\phi_1), \varphi_1 \otimes f_1 \right) \left( \theta(\phi_2), \varphi_2 \otimes f_2 \right), \]
\[ J_{\theta, \sigma}(\phi_1, \phi_2, \varphi_1, \varphi_2) := \left( \theta(\phi_1, \varphi_1), \theta(\phi_2, \varphi_2) \right)_{\pi}, \]
\[ J_{\theta, \pi}(\phi_1, \phi_2, f_1, f_2) := \left( \theta(\phi_1, f_1), \theta(\phi_2, f_2) \right)_{\sigma}. \]

Accordingly, we introduce three functional-valued functions by integrating matrix coefficients against $\Delta(\cdot)^s$, where $\Delta(\cdot)$ is a height function (cf. Section 1.1,
respectively. Then according to $J$ (a) The height function $\Delta^J_X$ $(\otimes X^J_0$ and give rational $J$ $GL \times \tau X\tau \otimes \otimes X_|$ be their leading coefficients at denote the dual, the $\otimes$ is holomorphic on a right half-

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When Re$(s) >> 0$, the height functions $\Delta G(\cdot), \Delta H(\cdot)$ make the above weighted integrals absolutely convergent and hence the functionals $J_{\sigma,\pi,\omega_\psi}(s), J_{\sigma,\omega_\psi}(s), J_{\pi,\omega_\psi}(s)$ are defined by assigning their values at vectors $\phi_i, \varphi_i, f_i$ according to these integrals.

For an admissible representation $\tau$, let $\tau^\vee, \tau$ and $\tau^*$ denote the dual, the conjugate, and the conjugate dual of $\tau$ respectively; let $X_{\sigma,\pi,\omega_\psi}, X_{\sigma,\omega_\psi}, X_{\pi,\omega_\psi}$ be the space of linear functionals on $\omega_\psi \otimes \overline{\omega_\psi} \otimes \sigma \otimes \sigma \otimes \tau, \omega_\psi \otimes \overline{\omega_\psi} \otimes \sigma \otimes \tau, \omega_\psi \otimes \overline{\omega_\psi} \otimes \sigma \otimes \tau$ respectively. Then $J_{\sigma,\pi,\omega_\psi}(s), J_{\sigma,\omega_\psi}(s), J_{\pi,\omega_\psi}(s)$ are $X_{\sigma,\pi,\omega_\psi}$-valued, $X_{\sigma,\omega_\psi}$-valued, and $X_{\pi,\omega_\psi}$-valued functions on a right half-plane. Concerning them, we make the following conjecture.

**Conjecture A:** $J_{\sigma,\pi,\omega_\psi}(s), J_{\sigma,\omega_\psi}(s), J_{\pi,\omega_\psi}(s)$ is holomorphic on a right half-plane and have meromorphic continuation to the whole $s$-plane. Let $J_{\sigma,\pi,\omega_\psi}, J_{\sigma,\omega_\psi}, J_{\pi,\omega_\psi}$ be their leading coefficients at $s = 0$. Then $J_{\sigma,\pi,\omega_\psi}, J_{\sigma,\omega_\psi}, J_{\pi,\omega_\psi}$ are invariant by $\tilde{G} \times \tilde{G} \times \tilde{H} \times \tilde{H}, \tilde{G} \times \tilde{G} \times \tilde{H}, \tilde{G} \times \tilde{H} \times \tilde{H}$ respectively.

**Remark 1:** (a) The height function $\Delta G(\cdot)$ is a generalization of the determinant function $\det(\cdot)$ on $GL_n(F)$ and the integrals involved in $J_{\sigma,\pi,\omega_\psi}(s), J_{\sigma,\omega_\psi}(s), J_{\pi,\omega_\psi}(s)$ can be considered as a generalization of the Godement–Jacquet integrals. Particularly, when $F$ is $p$-adic, these integrals are essentially multi-variable geometric series which converge when Re$(s) >> 0$ and give rational functions of $|\varpi|^s$, where $\varpi$ is a uniformizer of $F$. When $F = \mathbb{R}$, the asymptotic behavior of matrix coefficients easily tells the convergence of these integrals when Re$(s) >> 0$; very often, a close examination of the asymptotics of matrix coefficients shows the meromorphic continuation of these integrals in the parameter of $s$.

(b) We fix the following terms for one-complex-variable functional-valued functions. Let $F(s)$ be an $X_{\sigma,\pi,\omega_\psi}$-valued function on the complex plane;
$F(s)$ is called holomorphic if $F(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)$ is holomorphic for all $\phi_i \in \omega_\psi, \varphi_i \in \sigma, f_i \in \pi$; $F(s)$ is called meromorphic if there exists a numeric meromorphic function $f(s)$ such that $f(s)F(s)$ is holomorphic; suppose that $F(s)$ is meromorphic, then its order of zero at $s = s_0$ is defined to be $\min\{\operatorname{ord}_{s=s_0} F(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2) : \phi_i, \varphi_i, f_i\}$, (which is always finite by the definition of meromorphicity); if $F(s)$ is meromorphic and $n_0$ is its order of zero at $s = s_0$, the leading coefficient of $F(s)$ at $s = s_0$ is defined to be 

$$
\left. \left( (s - s_0)^{-n_0} F(s) \right) \right|_{s=s_0}.
$$

Similar terms can be defined for $X_{\sigma, \omega_\psi}$-valued and $X_{\pi, \omega_\psi}$-valued functions.

Let $X_{\sigma, \pi, \omega_\psi}$ be the space of $\tilde{G} \times \tilde{G} \times \tilde{H} \times \tilde{H}$-invariant functionals in $X_{\sigma, \pi, \omega_\psi}$. Because $X_{\sigma, \pi, \omega_\psi}$ has dimension 1, Conjecture A tells that one can use $J_{\sigma, \pi, \omega_\psi}$ to normalize $J_\theta$; when an extra multiplicity assumption (cf. Section 2.1) is verified, one can also use $J_{\sigma, \omega_\psi}, J_{\pi, \omega_\psi}$ to normalize $J_{\theta, \sigma}, J_{\theta, \pi}$. These three normalizations must possess certain compatibility.

**Conjecture B** (Local Duality): Given a local theta correspondence $\theta : \omega_\psi \to \sigma \otimes \pi$, there exist meromorphic functions $L_{\sigma, \pi}(s), L_{\sigma}(s), L_{\pi}(s)$ such that

$$
\lim_{s \to 0} \frac{J_{\sigma, \pi, \omega_\psi}(s)}{L_{\sigma, \pi}(s)} = J_\theta, \quad \lim_{s \to 0} \frac{J_{\sigma, \omega_\psi}(s)}{L_{\sigma}(s)} = J_{\theta, \sigma}, \quad \lim_{s \to 0} \frac{J_{\pi, \omega_\psi}(s)}{L_{\pi}(s)} = J_{\theta, \pi}.
$$

Let $d_\sigma(s), d_\pi(s)$ be the formal degree factors of $\sigma, \pi$ (cf. Definition 2). Then

$$
d_{\sigma, \pi} := \lim_{s \to 0} \frac{d_\sigma(s)L_{\sigma}(s)}{d_\pi(s)L_{\pi}(s)}
$$

is a finite number and we call $d_{\sigma, \pi}$ the relative degree of the local theta lifting from $\sigma$ to $\pi$.

**Remark 2:** (a) Formal degree factor is a conjectural generalization of formal degree for non-square-integrable unitary representations (cf. Section 1.2), that is, $d_\sigma(s)$ is such that

$$
\lim_{s \to 0} d_\sigma(s) \int_G (\sigma(g)v_1, v_2)(\sigma(g)v_3, v_4)\Delta(g)^* dg = (v_1, v_3)(v_2, v_4);
$$

it is expected to be a scalar multiple of the adjoint root number and we prove in [12] their existence for irreducible unitary representations of $GL_n(F)$ and $SL_2(F)$.
(b) When \( \sigma \) and \( \pi \) are both square-integrable, it is expected that
\[
J_{\sigma,\pi,\omega}(s), \quad J_{\sigma,\omega}(s), \quad J_{\pi,\omega}(s)
\]
are absolutely convergent at \( s = 0 \), whence \( d_{\sigma,\pi} = 1 \) as a consequence of changing the order of integration; when \( \sigma \) and \( \pi \) are non-square-integrable, the situation is more subtle.

(c) Conjecture B leads to the duality of global inner product formulas (cf. Proposition 3).

Our main theorems in this paper are as below.

**Theorem A:** Let \( F \) be a \( p \)-adic field or \( \mathbb{R} \), \(( G, H ) = ( \text{SL}_2(F), \text{PGL}_2(F) ) \) or \(( \text{SL}_2(F), \text{SO}(2,2)(F) ) \), and the height functions be as in Section 1.2. Then Conjectures A and B are true.

1. When \(( G, H ) = ( \text{SL}_2(F), \text{PGL}_2(F) )\), \( d_{\sigma,\pi} = 1 \).
2. When \(( G, H ) = ( \text{SL}_2(F), \text{SO}(2,2)(F) )\), \( d_{\sigma,\pi} \) is 1 when \( \sigma \) is square-integrable, \( \frac{1}{2} \) when \( \sigma \) is trivial, and 2 when \( \sigma \) is a direct summand of a principal series of \( \text{SL}_2(F) \).

We prove Theorem A by applying the asymptotic formulas of matrix coefficients to estimate the leading terms and remainder terms of \( J_{\sigma,\pi,\omega}(s), \quad J_{\sigma,\omega}(s), \quad J_{\pi,\omega}(s) \) at \( s = 0 \). On the other hand, a general discussion of the mechanism behind the local duality is presented in Section 2.

Let \( \text{GSO}(2,2) \) denote the connected component of \( \text{GO}(V) \), where \( V \) is a split 4-dimensional quadratic space over \( F \). Then \( \text{GSO}(2,2) \cong ( \text{GL}_2 \times \text{GL}_2 )/\Delta G_m \), where \( G_m \) is the diagonally embedded multiplicative group; under this identification,

\[
\text{SO}(2,2) \cong \{(g_1, g_2) | g_i \in \text{GL}_2, \det g_1 = \det g_2\}/\Delta G_m.
\]

**Theorem B:** (1) Let \( \pi \) be an irreducible cuspidal representation of \( \text{PGL}_2(\mathbb{A}_Q) \) and \( \omega_\psi \) an oscillator representation of \( \widetilde{\text{SL}}_2(\mathbb{A}_Q) \times \text{PGL}_2(\mathbb{A}_Q) \). Let \( \phi_i \in \omega_\psi \) and \( \varphi_i \in \pi \ (i = 1, 2) \) be decomposable vectors. Then

\[
\left( \Theta(\phi_1, f_1), \Theta(\phi_2, f_2) \right) = \frac{L(\frac{1}{2}, \pi)}{\zeta_Q(2)} \prod_v \frac{\zeta_{Q_v}(2)}{L(\frac{1}{2}, \pi_v)} \int_{\text{PGL}_2(Q_v)} (\omega_{\psi_v}(h_v)\phi_{1,v}\phi_{2,v}) \cdot (\pi_v(h_v)f_{1,v}, f_{2,v}) dh_v.
\]

(2) Let \( F \) be a totally real number field and \( \omega_\psi \) an oscillator representation of \( \text{SL}_2(\mathbb{A}_F) \times \text{O}(2,2)(\mathbb{A}_F) \). Let \( \Pi = \tau \boxtimes \tau' \) be an irreducible cuspidal representation of \( \text{GSO}(2,2)(\mathbb{A}_F) \), where \( \tau \) is an irreducible unitary cuspidal representation.
of $PGL_2(\mathbb{A}_F)$; write $H = SO(2, 2)$. Then for decomposable vectors $\phi_i \in \omega_\psi$, $f_i \in \Pi \ (i = 1, 2)$, we have

$$\left( \Theta(\phi_1, f_1), \Theta(\phi_2, f_2) \right) = \frac{L(0, \tau, \text{ad})}{2^{\beta-1} \zeta_F(2)} \prod_v \lim_{s \to 0} \frac{\int_{H(F_v)} (\omega_{\psi_v}(h_v)\phi_{1,v}, \phi_{2,v})(\Pi(h_v)f_{1,v}, f_{2,v}) \Delta(h_v)^s dh_v}{L(s, \tau_v, \text{ad})}.$$

Here $\beta$ is the number of places $v$ of $F$ such that $\tau_v$ is square-integrable, $\Delta(\cdot)$ is the local height function on $SO(2, 2)(F_v)$ defined in Section 1.2.

We remark that the global inner product formulas in the inverse directions $\widetilde{SL}_2 \to PGL_2$ and $GL_2 \to GSO(2, 2)$ are proved in [13] and [3] separately and the formulas in Theorem B are deduced from them by applying the global duality principle (cf. Proposition 3). Note that the pairings involved here are all Petersson inner product pairings (cf. Sections 4.5, 5.2).

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1. Preliminary

This section recalls the local results we need; throughout the section, $F$ will be $\mathbb{R}$ or a $p$-adic field.

For a real reductive group $G$, an admissible $G$-representation refers to an admissible $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ is the Lie algebra of $G$ and $K$ is a maximal compact subgroup of $G$. For a $p$-adic group $G$, an admissible $G$-representation refers to a smooth $G$-representation $V$ such that $\dim V^K$ is finite for each compact open subgroup $K$ of $G$.

Irreducibility of admissible representations is defined in the algebraic way; an admissible representation is call unitary (or unitarizable) if the underlying vector space can be given a positive definite Hermitian pairing that is $(\mathfrak{g}, K)$-invariant (when $F = \mathbb{R}$) or $G$-invariant (when $F$ is $p$-adic).

When $G$ is a real reductive group, a $G$-invariant homomorphism between two $G$-admissible representations is understood as a homomorphism that is $(\mathfrak{g}, K)$-invariant.
1.1. LOCAL THETA CORRESPONDENCE. Let $(\mathbb{W}, \langle \rangle)$ be a symplectic space over $F$ and $(G, H)$ a reductive dual pair (cf. [6]) contained in $\text{Sp}(\mathbb{W})$. We put

$$\tilde{\text{Sp}}(\mathbb{W}) = \text{the unique 2-fold cover of } \text{Sp}(\mathbb{W}),$$

$$\tilde{G} = \text{the preimage of } G \text{ in } \tilde{\text{Sp}}(\mathbb{W}),$$

$$\tilde{H} = \text{the preimage of } H \text{ in } \tilde{\text{Sp}}(\mathbb{W}).$$

We choose a non-trivial additive character $\psi$ of $F$ and denote by $\omega_{\psi}$ the oscillator representation of $\tilde{\text{Sp}}(\mathbb{W})$ associated to the character $\psi$. One may consider $\omega_{\psi}$ as a representation of $\tilde{G} \times \tilde{H}$ via the map $\tilde{G} \times \tilde{H} \rightarrow \tilde{G} \cdot \tilde{H} \hookrightarrow \text{Sp}(\mathbb{W})$.

We are only concerned with “genuine” admissible representations of $\tilde{G}$ and $\tilde{H}$. One writes the kernel of $\tilde{\text{Sp}}(\mathbb{W}) \rightarrow \text{Sp}(\mathbb{W})$ as $\{ \pm 1 \}$; then a “genuine” admissible representation of $\tilde{G}$ (respectively $\tilde{H}$) is an admissible representation of $\tilde{G}$ (respectively $\tilde{H}$) on which $-1$ acts by $-1$. We denote by $\text{Irr}(\tilde{G})$ (respectively $\text{Irr}(\tilde{H})$) the set of genuine irreducible admissible representations of $\tilde{G}$ (respectively $\tilde{H}$).

For $\sigma \in \text{Irr}(\tilde{G})$, the maximal $\sigma$-isotypic quotient $\omega_{\psi}[\sigma]$ of $\omega_{\psi}$ is a $\tilde{G} \times \tilde{H}$-module, namely

$$\omega_{\psi}[\sigma] = \omega_{\psi} / \bigcap_{\alpha} \text{Ker}(\alpha)$$

with $\alpha$ running over all non-zero $\tilde{G}$-equivariant maps $\alpha : \omega_{\psi} \rightarrow \sigma$. Because the action of $\tilde{G}$ and $\tilde{H}$ on the underlying space of $\omega_{\psi}$ is commutative, $\omega_{\psi}[\sigma]$ is of the form

$$\omega_{\psi}[\sigma] = \sigma \otimes \theta_0(\sigma),$$

where $\theta_0(\sigma)$ is a finitely generated admissible $\tilde{H}$-module (cf. [7], [11]). Let $\theta(\sigma)$ denote the maximal semisimple quotient of $\theta_0(\sigma)$; we call $\theta(\sigma)$ the local theta lift of $\sigma$.

**Conjecture (Local Howe Duality):** Suppose that $\sigma \in \text{Irr}(\tilde{G})$. If $\theta_0(\sigma)$ is non-zero. Then $\theta(\sigma)$ is non-zero and irreducible.

**Definition 1:** If $\sigma \in \text{Irr}(\tilde{G})$, $\pi \in \text{Irr}(\tilde{H})$ satisfy $\theta(\sigma) = \pi$ and $\theta(\pi) = \sigma$, we say that $\sigma$ and $\pi$ are in local theta correspondence.
Proposition 1 ([7], [11], [9]): The local Howe duality conjecture is true

(i) when $F = \mathbb{R}$ or a p-adic field with odd residue characteristic,
(ii) when $F$ is a p-adic local field and $\sigma$ is supercuspidal.

Remark 3: When $F$ is a dyadic field, local Howe duality can be checked explicitly when $\{G, H\} = \{SL_2, O_3\}$, $\{SL_2, O_4\}$, $\{SL_2, O_5\}$, as remarked by Wee Teck Gan in [1], [2]. Hence for these four dual reductive pairs one has local Howe duality for $\mathbb{R}$ and all p-adic local fields.

Remark 4: Suppose that $\sigma \in \text{Irr}(G \times H)$, $\pi \in \text{Irr}(G \times H)$. Then the local Howe duality conjecture implies that $\dim_{\mathbb{C}} \text{Hom}_{G \times H}(\omega, \sigma \otimes \pi) \leq 1$ and that $\dim_{\mathbb{C}} \text{Hom}_{G \times H}(\omega, \sigma \otimes \pi) = 1 \iff \theta(\sigma) = \pi \iff \theta(\pi) = \sigma$. If $\sigma$ and $\pi$ are in local theta correspondence, we call a non-zero element $\theta \in \text{Hom}_{G \times H}(\omega, \sigma \otimes \pi)$ a local theta correspondence between $\sigma$ and $\pi$.

1.2. Formal degree factor. Let $G$ be a finite central cover of a connected reductive group over $F$, $Z$ the center of $G$, and $\pi$ an irreducible unitary representation of $H$. We have introduced in [12] a functional-valued function $I_\pi(s)$ which is formally defined by

$$I_\pi(s; v_1, v_2, v_3, v_4) = \int_{G/Z} \langle \pi(h)v_1, v_2 \rangle \langle \pi(h)v_3, v_4 \rangle \Delta(g)^s dg,$$

where $v_i \in \pi(1 \leq i \leq 4)$, $s \in \mathbb{C}$, and $\Delta(g)$ is a height function on $G$ as specified in section 1.1 of [12]; $I_\pi(s)$ is well-defined and holomorphic on a right half-plane of $s$.

Conjecture 1 ([12]):

(i) $I_\pi(s)$ can be meromorphically continued to $\mathbb{C}$.
(ii) Write $m_\pi \overset{\Delta}{=} -\text{ord}_{s=0} I_\pi(s)$ and put $L_\pi = \lim_{s \to 0} s^{m_\pi} I_\pi(s)$. Then $L_\pi$ is $G \times G$-invariant in the sense of $L_\pi(gv_1, g'v_2, gv_3, g'v_4) = L_\pi(v_1, v_2, v_3, v_4)$.
(iii) If $G$ is a connected reductive group, then $m_\pi = \text{ord}_{s=0} \gamma(s, \pi, \text{ad}, \psi)$, where $\gamma(s, \pi, \text{ad}, \psi) = \epsilon(s, \pi, \text{ad}, \psi) \cdot \frac{L(1-s, \pi^\vee, \text{ad})}{L(s, \pi, \text{ad})}$ is the adjoint $\gamma$-factor of $\pi$.

Definition 2: When conjecture 1 is true, we call a meromorphic function $d_\pi(s)$ a formal degree factor of $\pi$ if

$$\lim_{s \to 0} d_\pi(s) I_\pi(s; v_1, v_2, v_3, v_4) = (v_1, v_3)(v_2, v_4) \quad \text{for all } v_i \in V_\pi \text{ } (1 \leq i \leq 4).$$

The leading coefficient of $d_\pi(s)$ at $s = 0$ is called the (generalized) formal
degree of $\pi$ and denoted by $d(\pi)$.

Formal degree and formal degree factor depend on the choice of the measure on $G$ in a simple manner. For convenience, we fix a non-trivial additive character $\psi_0$ of $F$ and let the measure on a connected reductive group be the one determined by $\psi_0$ as in [5] (when $F = \mathbb{R}$) and [4] (when $F$ is $p$-adic); any finite central cover of a connected reductive group is then given an induced measure. For simplicity, we further put $\psi_0(x) = e^{2\pi \sqrt{-1}x}$ when $F = \mathbb{R}$ and require that the conductor of $\psi_0$ is $O_F$ when $F$ is $p$-adic. ($O_F$ denotes the ring of integers of a $p$-adic field.)

We recall the following specific height functions on $GL_n(F)$ and $SL_2(F)$ used in [12].

1. When $G = GL_n(F)$,
   $$\Delta_{GL_n}(g) := \begin{cases} \frac{|\det(g)|}{\text{Tr}(gg^t)^{n/2}}, & F = \mathbb{R}; \\ \frac{|\det(g)|}{|x|^{n-\min\{\text{ord}(g_{ij})\}}}, & F \text{ is } p\text{-adic}. \end{cases}$$

2. When $G = SL_2(F)$, we fix a choice of a maximal compact subgroup $K$ in $H$, say $K = SO(2, \mathbb{R})$ or $SL_2(O_F)$; $\Delta_{SL_2}(\cdot)$ is defined with respect to the $KAK$ decomposition of $H$, where $A$ is the diagonal subgroup; for each $g \in H$, write $g = k_1 \left( \begin{smallmatrix} a & -1 \\ 1 & a \end{smallmatrix} \right) k_2$ with $k_1, k_2 \in K$ and $a \in F^\times$. Then
   $$\Delta_{SL_2} \left( k_1 \left( \begin{smallmatrix} a & -1 \\ 1 & a \end{smallmatrix} \right) k_2 \right) := \begin{cases} \frac{1}{|a| + |a^{-1}|}, & F = \mathbb{R}; \\ \min\{|a|, |a|^{-1}\}, & F \text{ is } p\text{-adic}. \end{cases}$$

These height functions will also be used to study the local theta correspondence concerning the pair $(\widetilde{SL}_2, PGL_2)$. When investigating the pair $(SL_2, SO(2, 2))$, we shall use the height functions on $GL_2(F)$ and $GSO(2, 2)(F) = (GL_2(F) \times GL_2(F))/\Delta G_m(F)$, and then restrict them to $SL_2(F), SO(2, 2)(F)$ (cf. Section 5.1); this is because we eventually want to derive global inner product formulas concerning the liftings between groups of similitudes. Note that for $SL_2$ and $PGL_2 \cong SO(2, 1)$, our height function is the same as the spherical section used in the local doubling integral of Piatetski-Shapiro and Rallis; while for $SO(2; 2)$, our height function is slightly different from the spherical section in the local doubling integral.
Proposition 2 ([12]): Let $F$ be $\mathbb{R}$ or a $p$-adic field.

1. Conjecture 1 is true when $G = GL_2(F), SL_2(F)$.

2. Let $\pi$ be an irreducible unitary representation of $GL_2(F)$. Then its formal degree factor can be taken as

\[
d_\pi(s) = \begin{cases} 
\frac{\pi(-1)}{2} \gamma(s, \pi, \text{ad}, \psi_0), & \text{if } \pi \text{ is square-integrable}, \\
\frac{1}{2} \gamma(s, \pi, \text{ad}, \psi_0), & \text{if } \pi \text{ is a principal series that is unitary}, \\
-\frac{1}{2} \gamma(s, \pi, \text{ad}, \psi_0), & \text{if } \pi \text{ is one-dimensional}.
\end{cases}
\]

3. Let $\tilde{\pi}$ be an irreducible admissible unitary representation of $\tilde{SL}_2(F)$ and $\pi$ its non-zero theta lift to $PGL_2(F)$ through a Weil representation $\omega_\psi$. Then we can take $d_\pi(s) = |2|^{-1} d_\pi(s)$.

2. Local duality

This section analyzes the mechanism behind the local duality in Conjecture B. In this section, $F$ will be a $p$-adic field or $\mathbb{R}$ and we fix the following notations:

1. For meromorphic functions $f_1(s), f_2(s)$, we write $f_1 \sim^k f_2$ if

\[
\lim_{s \to 0} s^k \left( f_1(s) - f_2(s) \right) = 0.
\]

1. Let $\sigma \in \text{Irr}(\tilde{G}), \pi \in \text{Irr}(\tilde{H})$ be unitary and in local theta correspondence with respect to $\omega_\psi$; let $J_{\sigma,\pi,\omega_\psi}(s), J_{\sigma,\omega_\psi}(s), J_{\pi,\omega_\psi}(s)$ be as in the introduction; let $n_{\sigma,\pi}, n_{\sigma}, n_{\pi}$ be their orders of pole at $s = 0$ and $J_{\sigma,\omega_\psi}, J_{\pi,\omega_\psi}$ be their leading coefficients at $s = 0$.

1. For a non-zero element $\theta \in \text{Hom}_G(\tilde{\omega}_\psi, \sigma \otimes \pi)$, we associate functionals $J_\theta, J_{\theta,\sigma}, J_{\theta,\pi}$ as in the introduction.

1. Let $I_{\sigma}(s), I_{\pi}(s)$ be as in Section 1.2 and $m_{\sigma}, m_{\pi}$ be their orders of pole at $s = 0$; let $d_{\sigma}(s), d_{\pi}(s)$ be the formal degree factors of $\sigma, \pi$.

1. Let $I_{\sigma,\text{std}}, I_{\pi,\text{std}}$ be the following standard functionals:

\[
I_{\sigma,\text{std}}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (\varphi_1, \varphi_3)(\varphi_2, \varphi_4),
\]

\[
I_{\pi,\text{std}}(f_1, f_2, f_3, f_4) = (f_1, f_3)(f_2, f_4).
\]

1. Let $R_{\sigma}(s), R_{\pi}(s)$ be the remainder terms of $J_{\sigma,\omega_\psi}(s)$ and $J_{\pi,\omega_\psi}(s)$ at $s = 0$, that is,
\[
R_{\sigma,\omega}(s) = J_{\sigma,\omega}(s) - \frac{J_{\sigma,\omega}}{s^{n_{\sigma}}}, \quad R_{\pi,\omega}(s) = J_{\pi,\omega}(s) - \frac{J_{\pi,\omega}}{s^{n_{\pi}}}. 
\]

(6) Put
\[
R_{\sigma,\pi}(s;\phi_1,\phi_2,\varphi_1,\varphi_2,f_1,f_2) = \int_{H} R_{\sigma,\omega}(s;\omega\psi(h) \circ \phi_1,\phi_2,\varphi_1,\varphi_2)(\pi(h)f_1,f_2)\Delta(h)^*dh, \\
R_{\pi,\sigma}(s;\phi_1,\phi_2,\varphi_1,\varphi_2,f_1,f_2) = \int_{G} R_{\pi,\omega}(s;\omega\psi(g) \circ \phi_1\phi_2,f_1,f_2)(\sigma(g)\varphi_1,\varphi_2)\Delta(g)^*dg.
\]

2.1. Multiplicity assumption. In order that \(J_{\sigma,\pi,\omega}, J_{\sigma,\omega}, J_{\pi,\omega}\) can be used to normalize \(\theta\), it suffices to have the following multiplicity one statements:
\[
\dim_{\mathbb{C}} X_{\sigma,\pi,\omega,\psi} = 1, \quad \dim_{\mathbb{C}} X_{\sigma,\omega,\psi} = 1, \quad \dim_{\mathbb{C}} X_{\pi,\omega,\psi} = 1.
\]

The first equality directly follows from the definition of local theta correspondence but the other two do not; however, the other two equalities will be true if the following assumption is verified.

**MULTIPLICITY ASSUMPTION:** Let \(\sigma \in \text{Irr}(G)\) and \(\pi \in \text{Irr}(H)\) be unitary and in local theta correspondence. Then the composition factors of \(\theta_0(\sigma)\) contains only one \(\theta(\sigma)\) and the composition factors of \(\theta_0(\pi)\) contains only one \(\theta(\pi)\).

**Remark 5:** When \((G,H) = (SL_2,O_3)\) or \((SL_2,O(2,2))\), the multiplicity assumption holds for all irreducible admissible unitary representations of \(\widetilde{G}\) and \(H\) that are in local theta correspondence. Actually, when \(\pi\) is not 1-dimensional, one has \(\theta_0(\sigma) = \theta(\sigma), \theta_0(\pi) = \theta(\pi)\); when \(\pi\) is 1-dimensional, although \(\theta_0(\sigma), \theta_0(\pi)\) are not irreducible, they do contain only one copy of \(\theta(\sigma), \theta(\pi)\) separately. (The verification is straight-forward and we skip it.)

**Lemma 1:** If the multiplicity assumption is true, then
\[
\dim_{\mathbb{C}} X_{\sigma,\omega,\psi} = \dim_{\mathbb{C}} X_{\pi,\omega,\psi} = 1.
\]

**Proof.** Because of symmetry, one only needs to argue for the first dimension. Recalling that \(X_{\sigma,\omega,\psi} = \text{Hom}_{\widetilde{G} \times \widetilde{G} \times \widetilde{H}}(\omega_{\psi} \otimes \overline{\omega_{\psi}} \otimes \sigma \otimes \overline{\sigma}, \mathbb{C})\), we make the
following observation:
\[
\text{Hom}_G(\omega_\psi \otimes \sigma, \mathbb{C}) = \text{Hom}_G(\omega_\psi, \sigma) \otimes \text{Hom}_G(\sigma \otimes \sigma, \mathbb{C}) = \text{Hom}(\theta_0(\sigma), \mathbb{C}) \otimes \text{Hom}_G(\sigma \otimes \sigma, \mathbb{C}),
\]
\[
\text{Hom}_G(\omega_\psi \otimes \sigma, \mathbb{C}) = \text{Hom}_G(\omega_\psi, \sigma) \otimes \text{Hom}_G(\sigma \otimes \sigma, \mathbb{C}) = \text{Hom}(\theta_0(\sigma), \mathbb{C}) \otimes \text{Hom}_G(\sigma \otimes \sigma, \mathbb{C}).
\]
Hence \(\text{dim}_\mathbb{C} \text{Hom}_{\widetilde{G} \times \widetilde{G} \times \widetilde{H}}(\omega_\psi \otimes \omega_\psi \otimes \sigma \otimes \sigma, \mathbb{C}) = \text{dim}_\mathbb{C} \text{Hom}_{\widetilde{H}}(\theta_0(\sigma) \otimes \theta_0(\sigma), \mathbb{C}).\)

We shall show that any \(\widetilde{H}\)-invariant Hermitian pairing on \(\theta_0(\sigma) \times \theta_0(\sigma)\) factors through \(\theta(\sigma) \times \theta(\sigma) = \pi \times \pi\), whence \(\text{dim}_\mathbb{C} \text{Hom}_{\widetilde{H}}(\theta_0(\sigma) \otimes \theta_0(\sigma), \mathbb{C}) = \text{dim}_\mathbb{C} \text{Hom}_{\widetilde{H}}(\pi \otimes \pi, \mathbb{C}) = 1\). If the assertion is false, then there exists an \(\widetilde{H}\)-invariant Hermitian pairing \(\alpha : \theta_0(\sigma) \times \theta_0(\sigma)\) that does not factor through \(\theta(\sigma) \times \theta(\sigma)\). Denote by \(N\) the Kernel of \(\theta_0(\sigma) \rightarrow \theta(\sigma)\). Then \(\alpha(\theta_0(\sigma), N) \neq 0\) and hence there is a non-zero \(\widetilde{H}\)-homomorphism \(\overline{N} \rightarrow \theta_0(\sigma)^\vee\). Because \(\theta(\sigma)^\vee\) is the unique non-zero irreducible submodule of \(\theta_0(\sigma)^\vee\), we have \(\text{Im}(\overline{N}) \supset \theta(\sigma)^\vee = \overline{\theta(\sigma)}\). So \(\theta(\sigma)\) occurs in the composition factors of \(N\) and hence it occurs at least twice in the composition factors of \(\theta_0(\sigma)\). This contradicts the multiplicity assumption.

**Corollary 1:** Let \(\theta : \omega_\psi \rightarrow \sigma \otimes \pi\) be a local theta correspondence between unitary representations \(\sigma \in \text{Irr}(\widetilde{G})\) and \(\pi \in \text{Irr}(\widetilde{H})\). If Conjecture A and the Multiplicity Assumption hold for \((\sigma, \pi)\), then there are non-zero constants \(c(\sigma, \pi), c(\sigma), c(\pi)\) such that
\[
J_{\sigma, \pi, \omega_\psi} = c(\sigma, \pi)^{-1}J_\theta, \quad J_{\sigma, \omega_\psi} = c(\sigma)^{-1}J_{\theta, \sigma}, \quad J_{\pi, \omega_\psi} = c(\pi)^{-1}J_{\theta, \pi}.
\]

**2.2. Asymptotic Estimate.** The duality equation in Conjecture B is suggested by the computation of \(J_{\sigma, \pi, \omega_\psi}(s)\) in two different orders of integration.

**Lemma 2:** Suppose that Conjecture A, Conjecture 1 and the Multiplicity Assumption hold for \((\sigma, \pi)\). Let \(\theta : \omega_\psi \rightarrow \sigma \otimes \pi\) be a local theta correspondence and write \(J_{\sigma, \omega_\psi} = c(\sigma)^{-1}J_{\theta, \sigma}\) and \(J_{\pi, \omega_\psi} = c(\pi)^{-1}J_{\theta, \pi}\). Then
\[
J_{\sigma, \pi, \omega_\psi}(s) \overset{n_\sigma + m_\pi}{\sim} \frac{c(\sigma)^{-1}d(\pi)^{-1}}{s^{n_\sigma + m_\pi}}J_\theta + R_{\sigma, \pi}(s),
\]
\[
J_{\sigma, \pi, \omega_\psi}(s) \overset{n_\sigma + m_\pi}{\sim} \frac{c(\pi)^{-1}d(\sigma)^{-1}}{s^{n_\pi + m_\sigma}}J_\theta + R_{\pi, \sigma}(s).
\]

**Proof.** When \(\text{Re}(s) \gg 0\), the double integral defining \(J_{\sigma, \pi, \omega_\psi}(s)\) is absolutely convergent and we can compute it in two orders.
The $g$-integration followed by the $h$-integration yields

$$J_{\sigma,\tau,\omega}(s; \phi_1, \phi_2, f_1, f_2, \varphi_1, \varphi_2)$$

$$= \int H J_{\sigma,\tau,\omega}(s; \omega \psi(h)\phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(h)f_1, f_2)\Delta(h)^s dh$$

$$= \frac{1}{s^{n_\sigma}} \int H J_{\sigma,\tau,\omega}(\omega \psi(h)\phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(h)f_1, f_2)\Delta(h)^s dh$$

$$+ R_{\sigma,\tau}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)$$

$$= \frac{c(\sigma)^{-1}}{s^{n_\sigma}} I_\pi(s; \theta(\phi_1, \varphi_1), \theta(\phi_2, \varphi_2), f_1, f_2)$$

$$+ R_{\sigma,\tau}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)$$

$$= \frac{c(\sigma)^{-1}d(\pi)^{-1}}{s^{n_\sigma+m_\pi}} J_\theta(\phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)$$

$$+ \frac{c(\sigma)^{-1}}{s^{n_\sigma}} T_\pi(s; \theta(\phi_1, \varphi_1), \theta(\phi_2, \varphi_2), f_1, f_2)$$

$$+ R_{\sigma,\tau}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2),$$

where

$$T_\pi(s) = I_\pi(s) - \frac{d(\pi)^{-1}}{s^{m_\pi}} I_{\pi,\text{std}}$$

is the remainder term of $I_\pi(s)$. Because $T_\pi(s) \sim 0$, we obtain that

$$J_{\sigma,\tau,\omega}(s) \sim \frac{c(\sigma)^{-1}d(\pi)^{-1}}{s^{n_\sigma+m_\pi}} J_\theta + R_{\sigma,\tau}(s).$$

The $h$-integration followed by the $g$-integration gives the second estimate. □

**Corollary 2:** Suppose that Conjecture A, Conjecture 1 and the Mutiplicity Assumption hold for $(\sigma, \pi)$. If

$$n_\sigma + m_\pi = n_\pi + m_\pi = n_{\sigma,\pi}, R_{\tau,\sigma}(s) \sim c_1 J_{\sigma,\tau,\omega}(s)$$

and $R_{\tau,\sigma}(s) \sim c_2 J_{\sigma,\tau,\omega}(s)$, then Conjecture B is true with

$$d_{\sigma,\tau} = \frac{1-c_1}{1-c_2}.$$ 

**Proof.** By Corollary 1, there exist meromorphic functions $L_{\sigma,\tau}(s)$, $L_\sigma(s)$, $L_\pi(s)$ such that

$$\lim_{s \to 0} \frac{J_{\sigma,\tau,\omega}(s)}{L_{\sigma,\tau}(s)} = J_\theta, \quad \lim_{s \to 0} \frac{J_{\sigma,\omega}(s)}{L_\sigma(s)} = J_{\theta,\sigma}, \quad \lim_{s \to 0} \frac{J_{\pi,\omega}(s)}{L_\pi(s)} = J_{\theta,\pi}.$$ 

This is the first part of Conjecture B.
By Lemma 2 and the hypothesis, we get that
\[ J_{\sigma,\pi,\omega,\psi}(s) \sim \frac{L_{\sigma}(s)J_{\theta}}{(1 - c_1)d_{\pi}(s)} \sim \frac{L_{\pi}(s)J_{\theta}}{(1 - c_2)d_{\sigma}(s)}. \]
It follows that that
\[ d_{\sigma,\pi} := \lim_{s \to 0} \frac{d_{\sigma}(s)L_{\sigma}(s)}{d_{\pi}(s)L_{\pi}(s)} = \frac{1 - c_1}{1 - c_2} \]
is a finite number. 

3. Global duality

In this section, we define global theta correspondence and deduce the global duality of inner product formulas from Conjecture B. Throughout this section, \( F \) is a totally real number field and \( \mathbb{A} \) denotes the group of \( F \)-adels.

Let \( (\mathbb{W}, \langle \rangle) \) be a symplectic space over \( F \) and \((G, H)\) be a reductive dual pair contained in \( \text{Sp}(\mathbb{W}) \). Let \( \text{Sp}(\mathbb{W})_{\mathbb{A}} \) be the 2-fold metaplectic cover of \( \text{Sp}(\mathbb{W})_{\mathbb{A}} \) and \( \widetilde{G}_{\mathbb{A}}, \widetilde{H}_{\mathbb{A}} \) the preimages of \( G_{\mathbb{A}}, H_{\mathbb{A}} \) in \( \text{Sp}(\mathbb{W})_{\mathbb{A}} \). We choose an additive character \( \psi \) of \( \mathbb{A}/F \) and denote by \( \omega_{\psi} \) the Weil representation of \( \text{Sp}(\mathbb{W})_{\mathbb{A}} \) associated to \( \psi \). When writing \( \mathbb{W} = X \oplus Y \) with \( X, Y \) two isotropic subspaces in \( \mathbb{W} \), we can realize \( \omega_{\psi} \) as an action of \( \text{Sp}(\mathbb{W})_{\mathbb{A}} \) on \( S(X_{\mathbb{A}}) \). Here \( S(X_{\mathbb{A}}) \) is the space of Bruhat–Schwartz functions on \( X_{\mathbb{A}} \) and the related model of \( \omega_{\psi} \) is called the Schrödinger model.

Let \( \sigma = \otimes_v \sigma_v \) (respectively \( \pi = \otimes_v \pi_v \)) be a genuine irreducible cuspidal representation of \( \widetilde{G}_{\mathbb{A}} \) (respectively \( \widetilde{H}_{\mathbb{A}} \)). For \( \phi \in S(X_{\mathbb{A}}) \), we associate a theta kernel \( \Theta_{\phi,\psi}(g, h) \) with respect to \( \psi \),
\[ \Theta_{\phi,\psi}(g, h) = \sum_{\xi \in X} \omega_{\psi}(g, h)\phi(\xi), \quad g \in \widetilde{G}_{\mathbb{A}}, h \in \widetilde{H}_{\mathbb{A}}. \]
For \( \varphi \in \sigma \), we define the global theta lift of \( \varphi \) through \( \phi \) with respect to \( \psi \) as
\[ \Theta_{\psi}(\phi, \varphi)(h) = \int_{G(F)\backslash G(\mathbb{A})} \Theta_{\phi,\psi}(g, h)\varphi(g)dg. \]
The space \( \Theta(\sigma; \psi) = \{ \Theta_{\psi}(\phi, \varphi) | \phi \in S(X_{\mathbb{A}}), \varphi \in \sigma \} \) is called the global theta lift of \( \sigma \) with respect to \( \psi \). Note that the right-translation action of \( \widetilde{H}_{\mathbb{A}} \) on \( \Theta(\sigma; \psi) \) is automatically an automorphic representation. Similarly, one can define \( \Theta_{\psi}(\phi, f) \) (for \( f \in \pi \)) and \( \Theta(\pi; \psi) \).

As a consequence of local Howe duality, one has the following easy-to-check lemma on global theta lift.
Lemma 3: Suppose that $\sigma$ is a genuine irreducible cuspidal representation of $\widetilde{G}_{\mathbb{A}}$. Assume that $\Theta(\sigma; \psi)$ consists of square-integrable automorphic forms and that the local Howe duality conjecture is true for $\sigma_v$ at all places $v$ of $F$. Then $\Theta(\sigma; \psi) = \bigotimes_v \theta_{\psi_v}(\sigma_v)$.

Definition 3: Let $\sigma$ and $\pi$ be genuine irreducible cuspidal representations of $\widetilde{G}_{\mathbb{A}}$ and $\widetilde{H}_{\mathbb{A}}$, respectively. If $\Theta(\sigma; \psi) = \pi$ and $\Theta(\pi; \psi) = \sigma$, we say that $\sigma$ and $\pi$ are in global theta correspondence with respect to $\psi$.

Remark 6: In general, when $\sigma$ is a genuine irreducible cuspidal representation of $\widetilde{G}_{\mathbb{A}}$, $\Theta(\sigma; \psi)$ may neither be cuspidal nor irreducible. In the current definition of global theta correspondence, we essentially treat the “regular” situation, that is, both $\sigma$ and $\pi$ are cuspidal; the “singular” situation needs to be treated separately when regularization tools are developed.

For genuine irreducible cuspidal $\widetilde{G}(\mathbb{A})$-representation $\sigma$ and $\widetilde{H}(\mathbb{A})$-representation $\pi$, let their Petersson inner product pairing be

\[
(\varphi_1, \varphi_2) = \int_{G(F) \backslash G(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}dg, \quad (f_1, f_2) = \int_{H(F) \backslash H(\mathbb{A})} f_1(h)\overline{f_2(h)}dh,
\]

where $dg, dh$ are the Tamagawa measures on $G(\mathbb{A}), H(\mathbb{A})$. Let the unitary pairing on $S(X_\mathbb{A})$ be $(\phi_1, \phi_2) = \int_{X_\mathbb{A}} \phi_1(x)\overline{\phi_2(x)}dx$, where $dx$ is an additive Haar measure on $X_\mathbb{A}$. Then we have the following duality on global inner product formulas.

Proposition 3 (Global Duality): Let $\sigma = \bigotimes_v \sigma_v$, $\pi = \bigotimes_v \pi_v$ be irreducible unitary cuspidal representations of $\widetilde{G}(\mathbb{A})$, $\widetilde{H}(\mathbb{A})$ that are in global theta correspondence; suppose that at all places $v$ of $F$, $(\sigma_v, \pi_v)$ are in local theta correspondence and Conjecture A, Conjecture 1, Conjecture B hold for them.

If there exists a global inner product formula concerning the lifting $\sigma \rightarrow \pi$,

\[
\left( \Theta(\phi_1, \varphi_1), \Theta(\phi_2, \varphi_2) \right) = C \prod_v \lim_{s \to 0} \int_{G(F_v)} (\omega_{\psi_v}(g_v)\phi_{1,v}(g_v)\phi_{2,v}(g_v)) (\sigma_v(g_v)\varphi_{1,v}(g_v)\varphi_{2,v}(g_v)) \Delta(g_v)^s dg_v
\]

\[
\frac{L_{\sigma_v}(s)}{L_{\pi_v}(s)},
\]

where $\Delta(g_v)$ is the Tamagawa measure on $G(F_v)$.
then there is a dual global inner product formula concerning the lifting \( \pi \to \sigma \),

\[
\left( \Theta(\phi_1, f_1), \Theta(\phi_2, f_2) \right) = C \prod_v \lim_{s \to 0} \int_{H(F_v)} (\omega_{\psi_v}(h_v)\phi_{1,v}(\phi_1)\phi_{2,v}(\phi_2)) \frac{\pi_v(h_v)f_{1,v}f_{2,v}}{L_{\pi_v}(s)} \Delta(h_v)^s \, dg_v,
\]

where

\[
L_{\pi_v}(s) = \frac{d_{\pi_v}(s)L_{\pi_v}(s)}{d_{\pi_v}(s)}.
\]

**Proof.** The global functional \( \omega_\psi \otimes \sigma \otimes \pi \to \mathbb{C}, (\phi, \varphi, f) \to (\Theta(\phi, \varphi), f)_\pi \) determines a global theta correspondence \( \Theta : \omega_\psi \to \sigma \otimes \pi \). We fix local decompositions of unitary representations, \( \omega_\psi = \otimes \omega_{\psi_v}, \sigma = \sigma_v, \pi = \otimes \pi_v \), where the global Petersson inner product pairing is accordingly decomposed as the product of local unitary pairings; let \( \phi_{v,0}, \varphi_{v,0}, f_{v,0} \) be the chosen spherical vectors for the unramified components \( \omega_{\psi_v}, \sigma_v, \pi_v \) of \( \omega_\psi, \sigma, \pi \). At each place \( v \), we choose a non-zero element \( \theta_v \in \text{Hom}_{G_v \times H_v}(\omega_{\psi_v}, \sigma_v \otimes \pi_v) \) such that \( \theta(\phi_{v,0}) = \varphi_{v,0} \otimes f_{v,0} \) at almost all unramified places. Then the global theta correspondence admits a local decomposition \( \Theta = C_0 \otimes_v \theta_v \). It follows that

\[
\Theta(\phi, \varphi) = (\Theta(\phi), \varphi)_\pi = C_0 \otimes_v (\theta_v(\phi_v), \varphi_v)_{\pi_v} = C_0 \otimes_v \theta_v(\phi_v, \varphi_v),
\]

\[
\Theta(\phi, f) = (\Theta(\phi), f)_\sigma = C_0 \otimes_v (\theta_v(\phi_v), f_v)_{\sigma_v} = C_0 \otimes_v \theta_v(\phi_v, f_v).
\]

Therefore, we obtain two theoretical global inner product formulas,

\[
\left( \Theta(\phi_1, \varphi_1), \Theta(\phi_2, \varphi_2) \right)_\pi = |C_0|^2 \prod_v \left( \theta_v(\phi_1,v, \varphi_1,v), \theta_v(\phi_2,v, \varphi_2,v) \right)_{\pi_v},
\]

\[
\left( \Theta(\phi_1, f_1), \Theta(\phi_2, f_2) \right)_\sigma = |C_0|^2 \prod_v \left( \theta_v(\phi_1,v, f_1,v), \theta_v(\phi_2,v, f_2,v) \right)_{\sigma_v}.
\]

Because there is a global inner product formula concerning the lifting \( \sigma \to \pi \)

\[
\left( \Theta(\phi_1, \varphi_1), \Theta(\phi_2, \varphi_2) \right)_\pi = C \prod_v \lim_{s \to 0} \frac{J_{\sigma_v, \omega_{\psi_v}}(s; \phi_1,v, \phi_2,v, \varphi_1,v, \varphi_2,v)}{L_{\sigma_v}(s)};
\]

we can choose \( \theta_v \) at every local place \( v \) such that

\[
\left( \theta_v(\phi_1,v, \varphi_1,v), \theta_v(\phi_2,v, \varphi_2,v) \right)_{\sigma_v} = \lim_{s \to 0} \frac{J_{\sigma_v, \omega_{\psi_v}}(s; \phi_1,v, \phi_2,v, \varphi_1,v, \varphi_2,v)}{L_{\sigma_v}(s)}.
\]

Then \( |C_0|^2 = C \). Applying the local duality, we get that

\[
\left( \theta_v(\phi_1,v, f_1,v), \theta_v(\phi_2,v, f_2,v) \right)_{\pi_v} = \lim_{s \to 0} \frac{d_{\pi_v}(s)L_{\pi_v}(s)}{d_{\pi_v}(s)} \frac{J_{\pi_v, \psi_v}(s; \phi_1,v, \phi_2,v, f_1,v, f_2,v)}{d_{\pi_v}(s)L_{\pi_v}(s)}.
\]
This expression, along with the second theoretical global inner product formula, yields the dual inner product formula
\[ (\Theta(\phi_1, f_1), \Theta(\phi_2, f_2))_\sigma = C \prod_v \lim_{s \to 0} \frac{d_{\sigma_v, \pi_v} d_{\pi_v}(s) J_{\pi_v, \psi_q}(s; \phi_1, v, \phi_2, v, f_1, v, f_2, v)}{d_{\sigma_v}(s) L_{\sigma_v}(s)}. \]

**Remark 7:** Under the assumption of Proposition 3, we see from the proof that there are a pair of inner product formulas:
\[
\begin{align*}
(\Theta(\phi_1, \varphi_1), \Theta(\phi_2, \varphi_2)) & = C \prod_v \lim_{s \to 0} \frac{J_{\sigma_v, \omega_\psi}(s; \phi_1, v, \phi_2, v, \varphi_1, v, \varphi_2, v)}{L_{\sigma_v}(s)}, \\
(\Theta(\phi_1, f_1), \Theta(\phi_2, f_2)) & = C \prod_v \lim_{s \to 0} \frac{J_{\pi_v, \psi_q}(s; \phi_1, v, \phi_2, v, f_1, v, f_2, v)}{L_{\pi_v}(s)}.
\end{align*}
\]

Here \( C \) is a positive constant and \( L_{\sigma_v}(s) \), \( L_{\pi_v}(s) \) are connected by
\[
\lim_{s \to 0} \frac{d_{\sigma_v}(s) L_{\sigma_v}(s)}{d_{\pi_v}(s) L_{\pi_v}(s)} = d_{\sigma_v, \pi_v}.
\]

In practice, \( L_{\sigma_v}(s) \) can be chosen to be certain local \( L \)-factors, as suggested by Theorem 3.1 in [10]. Let \( L_{\sigma}(s) = \prod_v L_{\sigma_v}(s) \) be the global \( L \)-function. Then the expectation is that \( C/L_{\sigma}(0) \) is a universal constant only depending on the reductive dual pair \((G, H)\).

### 4. The pair \((\widetilde{SL}_2, PGL_2)\)

In this section, we prove Conjectures A and B for the pair \((\widetilde{SL}_2, PGL_2)\). Throughout the section, \( F \) denotes a \( p \)-adic field or \( \mathbb{R} \) and \( \psi \) a non-trivial character of \( F \); we put \( G = SL_2(F) \), \( H = PGL_2(F) \) and identify \( H \) with \( SO(V, q) \), where \( V = \{ X \in M_2 \times 2(F)|\text{Tr}(X) = 0 \} \), \( q(X) = -\det(X) \) and \( h \circ X = hX h^{-1} \).

The Weil representation \( \omega_\psi \) of \( \tilde{G} \times H \) on \( S(V) \) is as in [15], that is,
\[
\omega_\psi(h)\phi(X) = \phi(h^{-1} \circ X);
\]
\[
\omega_\psi([\begin{array}{cc} a & -1 \\ 1 & \end{array}], e)\phi(X) = |a|^{3/2} \chi_\psi(a) e \phi(aX),
\]
\[
\omega_\psi([\begin{array}{cc} 1 & \end{array}], 1)\phi(X) = \psi(nq(X))\phi(X),
\]
\[
\omega_\psi([w, 1])\phi(X) = \gamma(\psi, q) \int_V \psi(Y) \psi(q(X, Y)) dY.
\]

Here \( w = \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix} \), \( q(X, Y) = q(X + Y) - q(X) - q(Y); \)
\[
\chi_\psi(a) = \langle a | -1 \rangle \frac{\gamma(\psi, a)}{\gamma(\psi)},
\]
\[
\gamma(\psi, q) = q^{\frac{1}{2}} \frac{\Gamma(q)}{\Gamma(q)}.
\]
where \( \langle \cdot | \cdot \rangle \) is the Hilbert symbol on \( F \); \( \gamma(\psi, q) = \gamma(\psi_{a_1}) \gamma(\psi_{a_2}) \gamma(\psi_{a_3}) \) if \( q = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle a_3 \rangle \); the measure on \( V \) is the Haar measure that is self-dual with respect to the bi-character \( X \to \psi(q(X)) \). Recall that \( \gamma(\psi) \) is the Weil constant satisfying

\[
\int_{\mathcal{A} / \mathcal{G}} f(x) \psi(x^2) = \gamma(\psi) \int_{\mathcal{A} / \mathcal{G}} \hat{f}(x) \psi(-x^2) \, dx,
\]

where \( f \in \mathcal{S}(F) \), \( \hat{f}(x) = \int_{\mathcal{A} / \mathcal{G}} f(y) \psi(2xy) \, dy \) is the Fourier transform of \( f \) with respect to \( \psi(x^2) \), and \( dx, dy \) are the self-dual Haar measures of \( F \) such that \( \hat{\hat{f}}(x) = f(-x) \); \( \gamma(\psi) \) is an eighth root of unity.

Let \( \sigma \in \text{Irr}(\widetilde{G}) \), \( \pi \in \text{Irr}(H) \) be unitary and in local theta correspondence with respect to \( \omega_\psi \); let the height function \( \Delta(g), \Delta(h) \) on \( G, H \) be as chosen in Section 1.2. Conjecture 1 is true for both \( \sigma \) and \( \pi \) by Proposition 2; the multiplicity assumption is true for the pair \( (\sigma, \pi) \) by Remark 5. In this section, we prove the following theorem.

**Theorem 1:** Conjectures A and B are true for \( (\sigma, \pi) \) with \( d_{\sigma, \pi} = 1 \).

We use [14], [15] as the main reference for the local theta correspondence concerning the pair \( (\widetilde{SL}_2(F), \text{PGL}_2(F)) \). For this pair, the meromorphic continuation of the functional-valued functions \( J_{\sigma, \pi, \omega_\psi}(s), J_{\sigma, \omega_\psi}(s), J_{\pi, \omega_\psi}(s) \) can be easily verified (cf. Remark 1(a)). We then need to verify the remaining part of Conjecture A and follow the mechanism in Section 2 to prove Conjecture B; such a proof needs to distinguish three cases according to the type of \( \pi \), and for convenience, we keep the notations in Section 2.

The representation \( \pi \) falls into one and only one of the following three types.

(i) \( \pi \) is square-integrable;
(ii) \( \pi \) is 1-dimensional;
(iii) \( \pi \) is a unitary principal series. (By a unitary principal series, we mean a principal series that is unitary, not just the principal series induced from unitary characters of the Borel group.)

The argument of Theorem 1 for case (i) and (ii) is easy:

(i) If \( \pi \) is square-integrable, then \( \sigma \) is also square-integrable and the integrals defining \( J_{\sigma, \pi, \omega_\psi}(s), J_{\sigma, \omega_\psi}(s), J_{\pi, \omega_\psi}(s) \) are both absolutely convergent at \( s = 0 \). It is easy to see that \( J_{\sigma, \omega_\psi}(s) \neq 0 \) (cf. the proof of Proposition 6.1 in [13]) and hence \( J_{\sigma, \pi, \omega_\psi}(0) \neq 0, J_{\pi, \omega_\psi}(0) \neq 0 \). It follows that \( n_{\sigma, \pi} = n_{\sigma} = n_{\pi} = 0, m_{\sigma} = m_{\pi} = 0 \) and that Conjectures A and B hold for \( (\sigma, \pi) \). The equality \( d_{\sigma, \pi} = 1 \) is derived by changing the order of integration in the integral defining \( J_{\sigma, \pi, \omega_\psi}(0) \).
(ii) When $\pi$ is 1-dimensional, direct computation easily shows that $n_{\sigma,\pi} = n_{\sigma} = n_{\pi} = 1$, $m_{\sigma} = m_{\pi} = 0$ and that $J_{\sigma,\pi,\omega_\psi}, J_{\sigma,\omega_\psi}, J_{\pi,\omega_\psi}$ are $\widetilde{G} \times \widetilde{G} \times H \times H$, $\widetilde{G} \times \widetilde{G} \times H$, $\widetilde{G} \times H \times H$-invariant respectively; hence Conjectures A and B hold in this case. One also computes easily that $R_{\sigma,\pi}(s)$ and $R_{\pi,\sigma}(s)$ are $G \times H \times H$, $G \times H \times H$-invariant respectively; hence Conjectures A and B hold in this case. One also computes easily that $J_{\sigma,\pi,\omega_\psi}(s)$ and $J_{\pi,\omega_\psi}(s)$; applying Corollary 2, we get that $d_{\sigma,\pi} = 1$.

When $\pi$ is of type (iii), that is, $\pi$ is a unitary principal series $\rho(\mu,\mu^{-1})$ (as in [8]), we have either (I) $\mu$ is unitary or (II) $\mu = \mu_0 | \cdot |^{s_0}$ with $\mu_0$ quadratic and $s_0 \in (-1/2,0) \cap (0,1/2)$. Accordingly, $\sigma = \pi(\mu)$ (which is denoted by $\pi(\mu)$ in [15]). We shall use the remaining part of this section to prove the following two more explicit statements.

**Proposition 4:** Suppose that $\pi = \rho(\mu,\mu^{-1})$ is unitary and $\mu^2 \neq 1$. Then Conjectures A and B are true with $m_{\sigma} = m_{\pi} = n_{\sigma,\pi} = 1$, $n_{\sigma} = n_{\pi} = 0$, $d_{\sigma,\pi} = 1$.

**Proposition 5:** Suppose that $\pi = \rho(\mu,\mu^{-1})$ is unitary and $\mu^2 = 1$. Then Conjectures A and B are true with $m_{\sigma} = m_{\pi} = n_{\sigma,\pi} = 3$, $n_{\sigma} = n_{\pi} = 0$, $d_{\sigma,\pi} = 1$.

The main idea of the proof of the above two propositions is to use the asymptotic formulas of matrix coefficients to analyze the integrals defining $J_{\sigma,\pi,\omega_\psi}(s)$, $J_{\sigma,\omega_\psi}(s)$, $J_{\pi,\omega_\psi}(s)$, and more specifically, to separate their leading terms and remainder terms at $s = 0$. For simplicity, we only present the argument when $F$ is $p$-adic; when $F$ is real, the proof is similar and hence skipped.

In the following, when $g \in SL_2(F)$ is considered as an element of $\widetilde{SL}_2(F)$, it is regarded as $[g,1]$. We first present the asymptotic formulas of matrix coefficients and then do the computation.

**4.1. Matrix Coefficients of $PGL_2(F)$-representations.** We describe the asymptotic formulas for the matrix coefficients of principal series of $PGL_2(F)$; their proofs are simple and hence skipped. Denote $K = K''/\{\pm I_2\}$, where

$$K'' = \begin{cases} GL_2(O_F), & F \text{ is } p\text{-adic,} \\
O_2(\mathbb{R}), & F = \mathbb{R}. \end{cases}$$

Put $w = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$. Let $dx$ be the measure on $F$ such that $O_F$ or $[0,1]$ has measure 1. Let $M$ be the intertwining operator on $\pi = \rho(\chi_1,\chi_2)$, which is
formally defined by $Mf(g) = \int_F f(w(\frac{1}{x})g)dx$; $M$ can be defined via meromorphic continuation for all $\chi_1, \chi_2$ with simple poles at representations such that $\chi_1\chi_2^{-1} = 1$.

Let the $PGL_2(F)$-invariant Hermitian pairing on $\pi = \rho(\mu, \mu^{-1})$ be

$$(f_1, f_2) = \left\{ \begin{array}{ll}
f_K f_1(k)f_2(k)dk, & \text{case (I)}, \\
f_K f_1(k)Mf_2(k)dk, & \text{case (II)}. \end{array} \right.$$ 

Put $e = I_2$ and denote by $\delta_g$ the evaluation functional $f \to f(g)$; for $g \in GL_2(F)$, write $g \circ f$ or $gf$ for the action of $g$ on $f$. We introduce functionals

$$\lambda_+(f_1, f_2) = \delta_e f_1 \cdot \overline{\delta_{w^{-1}}(Mf_2)}, \quad \lambda_-(f_1, f_2) = \delta_e (Mf_1) \cdot \overline{\delta_w f_2}.$$ 

(1) $F$ is $p$-adic. Let $\varpi$ be a uniformizer of $F$. Then there exists $\ell > 0$ such that when $n \geq \ell$, the following asymptotic formula holds for all $a \in \varpi^n\mathcal{O}_F^\times$, $k_i \in K (i = 1, 2)$:

$$\left( \pi \left( a 1 \right)^{k_1} f_1, k_2 f_2 \right) = \left\{ \begin{array}{ll}
\frac{|a|^{1/2} m(K)}{1 + |\varpi|} \sum_{\eta = \pm 1} \lambda_+(k_1 f_1, k_2 f_2) \chi_{1 + \frac{1}{2} \eta}(a), & \text{case (I) } \mu^2 \neq 1, \\
\frac{|a|^{1/2} m(K)}{1 + |\varpi|} \sum_{\eta = \pm 1} \lambda_-(k_1 f_1, k_2 M f_2) \chi_{1 + \frac{1}{2} \eta}(a), & \text{case (II)}, \\
\frac{|a|^{1/2} m(K)\mu(a)}{1 + |\varpi|} \left( (1 - |\varpi|) v(a) k_1 f_1(e) k_2 f_2(w) + O(1) \right), & \text{case (I) } \mu^2 = 1. \end{array} \right.$$ 

(2) $F = \mathbb{R}$. There exists $\epsilon > 0$ such that when $y \in (0, \epsilon)$, the following asymptotic formula holds for all $k_i \in K (i = 1, 2)$:

$$\left( \pi \left( y 1 \right)^{k_1} f_1, k_2 f_2 \right) = \left\{ \begin{array}{ll}
2y^{1/2} \sum_{\eta = \pm 1} \left( \lambda_+(k_1 f_1, k_2 f_2) + R_\eta(y, k_1 \varphi_1, k_2 \varphi_2) \right) \chi_{1 + \frac{1}{2} \eta}(y), & \text{(I) } \mu^2 \neq 1, \\
2y^{1/2} \sum_{\eta = \pm 1} \left( \lambda_-(k_1 f_1, k_2 M f_2) + R_\eta(y, k_1 f_1, k_2 M f_2) \right) \chi_{1 + \frac{1}{2} \eta}(y), & \text{case (II)}, \\
y^{1/2} \mu(y)(c_\epsilon \ln y \cdot k_1 f_1(e) k_2 f_2(w) + O(1)), & \text{(I) } \mu^2 = 1. \end{array} \right.$$ 

Here $R_\eta(y; \cdot)$ are holomorphic functionals over $(0, 1)$; when $f_1, f_2$ are given, there exists a constant $c$ such that $|R_\eta(y; k_1 f_1, k_2 f_2)| \leq cy$ for all $k_i \in K (i = 1, 2)$ when $y$ is sufficiently close to 0.
4.2. MATRIX COEFFICIENTS OF $\widetilde{SL}_2(F)$-REPRESENTATIONS. We describe the asymptotic formulas for the matrix coefficients of principal series of $\widetilde{SL}_2(F)$ (cf. Lemmas 17, 18, 21, 22 in [12]). Denote

$$K' = \begin{cases} SL_2(O_F), & F \text{ is } p\text{-adic}, \\ SO_2(\mathbb{R}), & F = \mathbb{R}. \end{cases}$$

Let $\widetilde{M}$ be the intertwining operator on $\sigma$, which is formally defined by

$$\widetilde{M}\varphi(g) = \int_{F} f([w, 1] (1, x) g) dx.$$ 

$\widetilde{M}$ can be defined via meromorphic continuation for all $\mu$, but with a simple pole when $\mu^2 = 1$.

Let the $SL_2(F)$-invariant Hermitian pairing on $\tau$ be

$$(\varphi_1, \varphi_2) = \begin{cases} \int_{K'} \varphi_1(k') \varphi_2(k') dk', & \text{case (I)}, \\ \int_{K'} \varphi_1(k') \overline{\varphi_2(k')} dk', & \text{case (II)}. \end{cases}$$

When regarding an element $g \in SL_2(F)$ as an element of $\widetilde{SL}_2(F)$, we take it as $[g, 1]$; let $\delta_g$ be the evaluation functional $\varphi \to \varphi(g)$ and write $g \circ \varphi$ or $g \varphi$ for the action of $g$ on $\varphi$. We introduce functionals

$$\widetilde{\lambda}_+(\varphi_1, \varphi_2) = \delta_e \varphi_1 \cdot \delta_{w^{-1}}(\widetilde{M}\varphi_2), \quad \widetilde{\lambda}_-(\varphi_1, \varphi_2) = \delta_e(\widetilde{M}\varphi_1) \cdot \delta_w \overline{\varphi_2}.$$ 

In the following, $k'_i$, $e$, $w$ are all regarded as elements of $\widetilde{SL}_2(F)$.

(1) $F$ is $p$-adic. Let $\varpi$ be a uniformizer of $F$; then there exists $\ell > 0$ such that when $n \geq \ell$, the following asymptotic formula holds for all $a \in \varpi^n O_F^\times$, $k_i \in K$ ($i = 1, 2$):

$$\left(\sigma[\begin{smallmatrix} a & -1 \\ a & 1 \end{smallmatrix}], 1\right]^{k_1}_1 \varphi_1, k_2^2 \varphi_2) = \begin{cases} \frac{|a|m(K')\chi_{\varpi}(a)}{1 + |\varpi|} \sum_{\eta = \pm 1} \tilde{\lambda}_\eta(k_1^2 \varphi_1, k_2^2 \varphi_2) \mu^\eta(a), & \text{case (I) } \mu^2 \neq 1, \\ \frac{|a|m(K')\chi_{\varpi}(a)}{1 + |\varpi|} \sum_{\eta = \pm 1} \tilde{\lambda}_\eta(k_1^2 \varphi_1, k_2^2 \widetilde{M}\varphi_2) \mu^\eta(a), & \text{case (II)}, \\ \frac{|a|m(K')\mu(a)\chi_{\varpi}(a)}{1 + |\varpi|} \left(2^{1/2}(1 - |\varpi|)v(a)k_1^2 \varphi_1(e)k_2^2 \varphi_2(w) + O(1)\right), & \text{case (I) } \mu^2 = 1. \end{cases}$$
2) $F = \mathbb{R}$. There exists $\epsilon > 0$ such that when $y \in (0, \epsilon)$, the following asymptotic formula holds for all $k_i \in K$ ($i = 1, 2$).

$$\left( \sigma[(y_{y^{-1}}), 1], k_i \varphi_1, k'_i \varphi_2 \right) = \begin{cases} 2y \sum_{n=\pm 1} \left( \lambda_n(k'_i \varphi_1, k''_i \varphi_2) + \bar{R}_n(y; k'_i \varphi_1, k''_i \varphi_2) \right) \mu(y), & (I) \ \mu^2 \neq 1, \\ 2y \sum_{n=\pm 1} \left( \lambda_n(k'_i \varphi_1, k''_i \varphi_2) + \bar{R}_n(y; k'_i \varphi_1, k''_i \varphi_2) \right) \mu(y), & (II), \\ y\mu(y)(c_{\sigma} \ln y \cdot k'_i \varphi_1(e) \overline{k''_i \varphi_2(w)} + O(1)), & (I) \ \mu^2 = 1. \end{cases}$$

Here $\bar{R}_n(y; -)$ are holomorphic functionals over $(0, 1)$; when $\varphi_1, \varphi_2$ are given, there exists a constant $c$ such that $|\bar{R}_n(y; k'_i \varphi_1, k''_i \varphi_2)| \leq cy^2$ for all $k'_i \in K$ ($i = 1, 2$) when $y$ is sufficiently close to 0.

4.3. $\pi = \rho(\mu, \mu^{-1})$ with $\mu^2 \neq 1$. For brevity, we present here only the proof of Proposition 4 when $F$ is $p$-adic; when $F = \mathbb{R}$, the proof is similar and hence skipped. Our proof is carried out by building the following facts in order:

1) The decomposition of $G$ and $H$. Since $\mu^2 \neq 1$, Proposition 2 tells that $d_\sigma(s), d_\pi(s)$ both have a simple zero at $s = 0$ and hence $m_\sigma = m_\pi = 1$. One can choose a uniformizer $\varpi$ of $F$ such that $\mu^2(\varpi) \neq 1$. Put $K' = SL_2(O_F)$, $K = PGL_2(O_F)$, $g_n = (\zeta^n \varpi^{-n}) \in SL_2(F)$, $h_m = [(\varpi - 1)] \in PGL_2(F)$, $G_n = K'g_nK'$, $H_m = K'KmK'$. Then $G = \bigcup_{n \geq 0} G_n$ and $H = \bigcup_{m \geq 0} H_m$ and one can compute a $G$-integral or $H$-integral by summing the individual integrals over $G_n$ or $H_m$.

2) Matrix coefficients. For given $\phi_i, \varphi_i, f_i$ ($i = 1, 2$), there is $\ell \geq 0$ such that when $n, m \geq \ell$, $k_i \in K, k'_i \in K'$, the following asymptotic formulas of matrix
coefficients hold:

\[
\left( \pi \left( a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right)_{f_1, f_2} = \frac{|a|^{1/2} m(K)}{1 + |\varpi|} \sum_{\eta = \pm 1} C^k_{\eta} \mu^n(a),
\]

\[
\left( \sigma \left( a^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right)_{\varphi_1, \varphi_2} = \frac{|a|m(K')}{1 + |\varpi|} \sum_{\eta = \pm 1} \tilde{C}^k_{\eta} \mu^n(a),
\]

where \( C^k_{\eta} = \lambda \eta(k_1 f_1, k_2 f_2) \), \( \tilde{C}^k_{\eta} = \tilde{\lambda} \eta(k'_1 \varphi_1, k'_2 \varphi_2) \) when \( \mu \) is unitary and \( C^k_{\eta} = \lambda \eta(k_1 f_1, k_2 M f_2) \), \( \tilde{C}^k_{\eta} = \tilde{\lambda} \eta(k'_1 \varphi_1, k'_2 M \varphi_2) \) when \( \mu \) is not unitary.

3) The integral on \( G_n \times H_m \). Put

\[
J_{n,m}^{\varphi_1, \varphi_2}(s; \phi_1, \phi_2)
\]

\[
= \left| \varpi \right|^{n-m+(n+m)s} \mu(\varpi)^n \int_{V} \phi_1 \left( \begin{pmatrix} \varpi^n x_1 & \varpi^{n-m} x_2 \\ \varpi^{n+m} x_3 & -\varpi^n x_1 \end{pmatrix} \right) \overline{\phi_2(x)} dX.
\]

Then when \( n, m \geq \ell \), we have

\[
\int_{G_n \times H_m} (\omega \psi(g, h) \phi_1, \phi_2) (\sigma(g) \varphi_1, \varphi_2) (\pi(h) f_1, f_2) \Delta(g)^s \Delta(h)^s dgdh
\]

\[
= \sum_{\eta_1, \eta_2 \in \{\pm 1\}} \int_{k'_{1,2}} \tilde{C}^{k'_{1,2}}_{\eta_1} C^{k_{1,2}}_{\eta_2} J_{n,m}^{\varphi_1, \varphi_2}(s; k'_1 \phi_1, k'_2 \phi_2).
\]

4) Integral operators. We introduce the following functionals, where \( \phi, \phi_1, \phi_2 \in S(V) \):

\[
\Phi_{\phi_1, \phi_2}(t) := \Phi(t; \phi_1, \phi_2) := |t|^{1/2} \int_{V} \phi_1 \left( \begin{pmatrix} 0 & tx_2 \\ 0 & 0 \end{pmatrix} \right) \overline{\phi_2(x)} dX,
\]

\[
Z(\Phi, \chi, s) = \int_{F \times F} \Phi(t) \chi(t)|t|^s d^X t,
\]

\[
T_\chi(s; \phi) = \int_{F \times F} \chi(x_2)|x_2|^s + \frac{1}{2} \omega \psi(g, h) \phi \left( \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix} \right) d^X x_2,
\]

\[
S_\chi(s; \phi) = \int_{F \times F \times F} \chi(x_2)|x_2|^{-s + \frac{1}{2}} \phi \left( \begin{pmatrix} x_1 x_2 \\ x_3 - x_1 \end{pmatrix} \right) dx_1 dx_2^X dx_3.
\]
LEMMA 4: Suppose that $|\chi^{\pm 1}(\varpi)||\varpi|^{1/2} < 1$. Let the Haar measure on $F^\times$ be such that $m(O_F^\times) = 1$. Define functionals

$$Z(\Phi, \chi, s) = \int_{F^\times} \Phi(t) \chi(t)|t|^s d^\times t,$$

$$\Phi_{\phi_1, \phi_2}(t) := \phi(t; \phi_1, \phi_2) := |t|^{1/2} \int_{V} \phi_1 \left( \begin{array}{c} 0 \\ t x_2 \\ 0 \end{array} \right) \phi_2(X) dX,$$

$$T_\chi(s; \phi) = \int_{F^\times} \chi(x_2)|x_2|^{s + \frac{1}{2}} \omega_\psi(g, h) \phi \left( \begin{array}{c} 0 \\ x_2 \\ 0 \end{array} \right) d^\times x_2,$$

$$S_\chi(s; \phi) = \int_{F^\times F^\times F} \chi(x_2)|x_2|^{-s + \frac{1}{2}} \phi_2 \left( \frac{x_1}{x_3 - x_4} \right) dx_1 d^\times x_2 d \times x_3.$$

Then $Z(\Phi_{\phi_1, \phi_2}, \chi, s) = \zeta_F(1)^{-1} T_\chi(s; \phi_1) S_{\chi^{-1}}(s; \phi_2)$.

**Proof.** Because $|\chi^{\pm 1}(\varpi)||\varpi|^{1/2} < 1$, we can introduce the following transformation when $s$ is in an open neighborhood of 0:

$$Z(\Phi_{\phi_1, \phi_2}, \chi, s)$$

$$= \int_{F^\times} \chi(t)|t|^{s + \frac{1}{2}} \left( \int_{V} \phi_1 \left( \begin{array}{c} 0 \\ t x_2 \\ 0 \end{array} \right) \phi_2(X) dX \right) d^\times t$$

$$= \zeta(F)^{-1} \int_{F^\times} \int_{F^\times} \chi(t)|t|^{s + \frac{1}{2}} \left( \int_{F^2} \phi_1 \left( \begin{array}{c} 0 \\ t x_2 \\ 0 \end{array} \right) \phi_2 \left( \frac{x_1}{x_3 - x_4} \right) dx_1 dx_2 \right) d^\times x_2 d^\times t$$

$$= \zeta(F)^{-1} \int_{F^\times F^\times F} \chi(t)|t|^{s + \frac{1}{2}} \left( \int_{F^2} \phi_1 \left( \begin{array}{c} 0 \\ t x_2 \\ 0 \end{array} \right) \phi_2 \left( \frac{x_1}{x_3 - x_4} \right) dx_1 dx_2 \right) d^\times x_2 d^\times t.$$

The reason is that the double integral in the end is absolutely convergent near $s = 0$ when $|\chi^{\pm 1}(\varpi)||\varpi|^{1/2} < 1$. One then makes a change of variable $t \rightarrow tx_2^{-1}$, which yields

$$Z(\Phi_{\phi_1, \phi_2}, \chi, s)$$

$$= \zeta(F)^{-1} \int_{F^\times F^\times F} \chi(t)|t|^{s + \frac{1}{2}} \phi_1 \left( \begin{array}{c} 0 \\ t \end{array} \right)$$

$$\times \left( \int_{F^2} \chi^{-1}(x_2)|x_2|^{s + \frac{1}{2}} \phi_2 \left( \frac{x_1}{x_3 - x_4} \right) dx_1 dx_2 \right) d^\times x_2 d^\times t$$

$$= \zeta(F)^{-1} \left( \int_{F^\times} \chi(t)|t|^{s + \frac{1}{2}} \phi_1 \left( \begin{array}{c} 0 \\ t \end{array} \right) d^\times t \right)$$

$$\times \left( \int_{F^\times F^\times F} \chi^{-1}(x_2)|x_2|^{s + \frac{1}{2}} \phi_2 \left( \frac{x_1}{x_3 - x_4} \right) dx_1 d^\times x_2 d \times x_3 \right)$$

$$= \zeta(F)^{-1} T_\chi(s; \phi_1) S_{\chi^{-1}}(s; \phi_2).$$
Now we start to prove Proposition 4 when $F$ is $p$-adic.

**Lemma 5:** $n_{\sigma, \pi} = 1$ and $J_{\sigma, \pi, \omega_\psi}$ is $\tilde{G} \times \tilde{G} \times H \times H$-invariant.

**Proof.** For given $\phi_i, \varphi_i, f_i$ ($i = 1, 2$), write $J(s) = J_{\sigma, \pi, \omega_\psi}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)$.

**Step 1:** Our first observation is that the behavior of $J(s)$ at $s = 0$ is determined by the integration near $\infty$, namely,

$$J(s) \sim \sum_{n, m \geq \ell} \int_{G_n \times H_m} (\omega_\psi(g, h)\phi_1, \phi_2)(\sigma(g)\varphi_1, \varphi_2)(\pi(h)f_1, f_2) \Delta(g)^s \Delta(h)^s dg dh,$$

$$= \sum_{n, m \geq \ell} \sum_{\eta_1, \eta_2} \int_{k'_i, k_i} \overline{\mathcal{C}_{n_1}^{k'_1, k'_2} C_{n_2}^{k_1, k_2} J_{n, m}(s; k'_1, k_1 \phi_1, k'_2, k_2 \phi_2)} \sim \sum_{n, m \geq 0} \int_{k'_i, k_i} \overline{\mathcal{C}_{n_1}^{k'_1, k'_2} C_{n_2}^{k_1, k_2} J_{n, m}(s; k'_1, k_1 \phi_1, k'_2, k_2 \phi_2)}.$$

One easily sees that $\sum_{n, m \geq 0} J_{n, m}^{\eta, \eta}(s; k'_1, k_1 \phi_1, k'_2, k_2 \phi_2) \sim 0$ when $\eta_1 \neq \eta_2$ and that

$$\sum_{n, m \geq 0} J_{n, m}^{\eta, \eta}(s; k'_1, k_1 \phi_1, k'_2, k_2 \phi_2) = \left( \sum_{n \geq m \geq 0} + \sum_{m > n \geq 0} \right)$$

$$\sim \zeta_F(2s) \sum_{n_1 \geq 0} |\omega|^{(1/2 + s)n_1} \mu(\omega)^{n_1 \eta} \int_{V} k'_1, k_1 \phi_1 \begin{pmatrix} 0 & \omega^{n_1}x_2 \\ 0 & 1 \end{pmatrix} k'_2, k_2 \phi_2 (X) dX$$

$$+ \zeta_F(2s) \sum_{n_1 < 0} |\omega|^{(1/2 - s)n_1} \mu(\omega)^{n_1 \eta} \int_{V} k'_1, k_1 \phi_1 \begin{pmatrix} 0 & \omega^{n_1}x_2 \\ 0 & 1 \end{pmatrix} k'_2, k_2 \phi_2 (X) dX$$

$$\sim \zeta_F(2s) \sum_{n_1 \in \mathbb{Z}} \Phi(\omega^{n_1}; k'_1, k_1 \phi_1, k'_2, k_2 \phi_2) \mu(\omega^{n_1})|\omega^{n_1}|^s.$$
STEP 2: We notice that \( \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \Phi(t; k_1', k_1, k_2', k_2, \phi_2) \overline{\mu}^{\eta_1}(t) |t|^s \) is a constant function of \( t \) on \( \omega^{n_1} \mathcal{O}_F^\times \) for each \( n_1 \). Hence

\[
\sum_{n_1 \in \mathbb{Z}} \left( \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \Phi(\omega^{n_1}; k_1', k_1, k_2', k_2, \phi_2) \overline{\mu}^{\eta}(\omega^{n_1}) |\omega^{n_1}|^s \right)
\]

\[
= \sum_{n_1 \in \mathbb{Z}} \int_{\omega^{n_1} \mathcal{O}_F^\times} \left( \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \Phi(t; k_1', k_1, k_2', k_2, \phi_2) \overline{\mu}^{\eta}(t) |t|^s \right) d^\times t
\]

\[
= \int_{\mathcal{F}^\times} \left( \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \Phi(t; k_1', k_1, k_2', k_2, \phi_2) \overline{\mu}^{\eta}(t) |t|^s \right) d^\times t
\]

\[
= \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \zeta(\Phi(k_1', k_1, k_2', k_2, \phi_2), \overline{\mu}^{\eta}, s).
\]

It is easy to see that \( \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \zeta(\Phi(k_1', k_1, k_2', k_2, \phi_2), \overline{\mu}^{\eta}, 0) \) is independent of \( \eta \). Therefore, the final output is

\[
J(s) \sim \zeta_F(2s) \sum_{\eta} \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \zeta(\Phi(k_1', k_1, k_2', k_2, \phi_2), \overline{\mu}^{\eta}, s)
\]

\[
= \zeta_F(2s) \sum_{\eta} \int_{k_1', k_i} \overline{C}_{\eta}^{k_1', k_2'} C_{\eta}^{k_1, k_2} \zeta(\Phi(k_1', k_1, k_2', k_2, \phi_2), \overline{\mu}^{\eta}, 0)
\]

\[
= 2 \zeta_F(2s) \int_{k_1', k_i} \overline{C}_{1}^{k_1', k_2'} C_{-1}^{k_1, k_2} \zeta(\Phi(k_1', k_1, k_2', k_2, \phi_2), \overline{\mu}, 0)
\]

\[
= \frac{\zeta_F(s)}{\zeta_F(1)} \int_{k_1', k_i} \overline{C}_{1}^{k_1', k_2'} C_{-1}^{k_1, k_2} \zeta(\Phi(k_1', k_1, k_2', k_2, \phi_2), \overline{\mu}^{\eta}, 0)
\]

\[
= \frac{\zeta_F(s)}{\zeta_F(1)} \int_{k_1', k_i} \zeta(\Phi(k_1', k_1, k_2', k_2, \phi_2), \overline{\mu}^{\eta}, 0)
\]

\[
= \frac{\zeta_F(s)}{\zeta_F(1)} \int_{k_1', k_i} \lambda^+(k_1', k_2', \phi_2) \lambda^-(k_1, k_2, \phi_2) T_{\overline{\mu}^{\eta}}(0; k_1', k_1, k_2', k_2, \phi_2) S_{\mu^{-1}}(0; k_1', k_2', k_2, \phi_2).
\]

STEP 3: One checks easily that the functional

\[
(\phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)
\]

\[
\sim L \int_{k_1', k_i} \lambda^+(k_1', k_2', \varphi_2) \lambda^-(k_1, k_2, \varphi_2) T_{\overline{\mu}^{\eta}}(0; k_1', k_1, k_2', k_2, \phi_2) S_{\mu^{-1}}(0; k_1', k_2', k_2, \phi_2).
\]
is a non-zero element in \((\omega_{\psi}^\vee \otimes \omega_{\psi}^* \otimes \pi^* \otimes \pi^\vee) \tilde{G} \times \tilde{G} \times H \times H\). For example, when \(\mu\) is unitary, the right-hand side is a non-zero scalar multiple of

\[
\frac{T_{\mu-1,\phi_1}(k_1',k_1)\overline{\varphi_1(k_1')}Mf_1(k_1)}{T_{\mu,\phi_2}(k_2',k_2)\overline{M\varphi_2(k_2')}f_2(k_2)} = (T_{\mu-1,\phi_1}(\phi_1 \otimes Mf_1)(T_{\mu,\phi_2}, \overline{M\varphi_2} \otimes f_2),
\]

where \(T_{\chi,\phi}(g,h) := T_{\chi}(0,\omega(g,h)\phi) \in \overline{\rho}(\chi^{-1}) \otimes \rho(\chi,\chi^{-1})\).

Hence

\[
J_{\sigma,\pi,\omega,\psi}(s) \sim \zeta_L(s)
\]

So \(J_{\sigma,\pi,\omega,\psi}(s)\) has a simple pole at \(s = 0\) and its leading coefficient at \(s = 0\) is \(\tilde{G} \times \tilde{G} \times H \times H\)-invariant.

In the current situation, \(J_{\sigma,\omega,\psi}(s)\) and \(J_{\pi,\omega,\psi}(s)\) are holomorphic at \(s = 0\) and it is easy to verify that

\[
\begin{align*}
J_{\sigma,\omega,\psi}(0) \in (\omega_{\psi}^\vee \otimes \omega_{\psi}^* \otimes \sigma^* \otimes \tau^\vee) \tilde{G} \times \tilde{G}, & \quad J_{\pi,\omega,\psi}(0) \in (\omega_{\psi}^\vee \otimes \omega_{\psi}^* \otimes \pi^* \otimes \pi^\vee) \tilde{G} \times H \times H,
\end{align*}
\]

because the integrals defining \(J_{\sigma,\omega,\psi}(0)\) and \(J_{\pi,\omega,\psi}(0)\) are absolutely convergent.

We shall show that \(J_{\sigma,\omega,\psi}(0), J_{\pi,\omega,\psi}(0)\) are non-zero by estimating the integrals of the following remainder terms:

\[
R_\sigma(s; -) = J_{\sigma,\omega,\psi}(s; -) - J_{\sigma,\omega,\psi}(0; -),
\]

\[
R_\pi(s; -) = J_{\pi,\omega,\psi}(s; -) - J_{\pi,\omega,\psi}(0; -).
\]

**Lemma 6:**

\[
\int_H R_\sigma(\omega(h)\phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(h)f_1, f_2)\Delta(h)^s dh \sim -J_{\sigma,\pi,\omega,\psi}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2).
\]

**Proof.** The left-hand side is absolutely convergent when \(\text{Re}(s) \gg 0\) and defines a holomorphic function on a right half plane. It can be extended to a meromorphic function on \(\mathbb{C}\) whose behavior at \(s = 0\) is determined by the integrand’s
behavior near $\infty$:

$$L.H.S \sim \sum_{m \geq \ell} \int_{H_m} R_{\sigma}(\omega_\psi(h)\phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(g)f_1, f_2)\Delta(h)^s dh$$

$$\sim \sum_{m \geq \ell} \int_{H_m} \left( \sum_{n \geq \ell} \int_{G_n} (\omega_\psi(g, h)\phi_1, \phi_2)(\sigma(g)\varphi_1, \varphi_2)\Delta(g)^s dg \right) \frac{1}{\pi(g)f_1, f_2}\Delta(h)^s dh$$

$$= \sum_{m \geq \ell} \left( \sum_{n \geq \ell} \int_{G_n \times H_m} (\omega_\psi(g, h)\phi_1, \phi_2)(\sigma(g)\varphi_1, \varphi_2)(\pi(g)f_1, f_2)\Delta(g)^s \Delta(h)^s dg dh \right)^s$$

$$= \sum_{m \geq \ell} \left( \sum_{n \geq \ell} \sum_{\eta_1, \eta_2} \int_{k_i'k_i} \frac{C_{\eta_1}^{k_1'}C_{\eta_2}^{k_2'}C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)}{C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)} \right)^s$$

$$= \sum_{n \geq \ell} \sum_{k_i'k_i} \left( \sum_{m \geq \ell} \sum_{\eta_1, \eta_2} \frac{C_{\eta_1}^{k_1'}C_{\eta_2}^{k_2'}C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)}{C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)} \right)^s.$$ 

One may suppose that $\ell$ is big enough such that when $X \in \text{supp}(\phi_2)$ and $v(a_1), v(a_3) \geq \ell$, we have

$$\phi_1 \begin{pmatrix} a_1x_2 & x_2 \\ a_3x_3 & -a_1x_1 \end{pmatrix} = \phi_1 \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix}.$$ 

Then we can estimate

$$\sum_{m \geq \ell} \left( \sum_{n \geq \ell} \frac{C_{\eta_1}^{k_1'}C_{\eta_2}^{k_2'}C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)}{C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)} \right)^s$$

just as in step 1 of the proof of Proposition 5. We have

$$\sum_{m \geq \ell} \left( \sum_{n \geq \ell} \frac{C_{\eta_1}^{k_1'}C_{\eta_2}^{k_2'}C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)}{C_{\eta_1}^{k_1}C_{\eta_2}^{k_2}J_{n,m}(s; k_1', k_1\phi_1, k_2', k_2\phi_2)} \right)^s \Bigg|_0^s \sim \begin{cases} 0, & \eta_1 = \eta_2, \\
(\zeta_F(2s) - \zeta_F(s)) \sum_{n_1 \in \mathbb{Z}} \Phi(\omega^{n_1}; \phi_1, \phi_2)\overline{\omega^{n_1}} \omega^{n_1}|s^s, & \eta_1 = -\eta_2 = \eta. \end{cases}$$
It follows that
\[ L.H.S. = \sum_{\eta_1, \eta_2} \int_{k'_1, k_1} \sum_{m \geq \ell} \left( \sum_{n \geq \ell} C_{n_1}^{k'_1, k'_2} C_{n_2}^{k_1, k_2} J_{n_1, n_2}^{m} (s, k'_1, k_1, k'_2, k_2) \right) \bigg|_{0}^{s} \]
\[ \sim (\zeta_F(2s) - \zeta_F(s)) \sum_{\eta} \int_{k'_1, k_1} \tilde{C}_{\eta}^{k'_1, k'_2} C_{-\eta}^{k_1, k_2} Z(\Phi_{k'_1, k_1, k'_2, k_2, \phi_2}^{\eta}, 0) \]
\[ \sim - \zeta_F(2s) \sum_{\eta} \int_{k'_1, k_1} \tilde{C}_{\eta}^{k'_1, k'_2} C_{-\eta}^{k_1, k_2} Z(\Phi_{k'_1, k_1, k'_2, k_2, \phi_2}^{\eta}, 0) \]
\[ \sim - J_{\sigma, \pi, \omega_1}^{\eta}(s, \phi, \phi_2, \varphi_1, \varphi_2, f_1, f_2). \]

**Corollary 3:** \( n_{\sigma} = 0 \) and \( J_{\sigma, \omega} = J_{\sigma, \omega_1}(0) \) is \( \tilde{G} \times \tilde{G} \times H \)-invariant.

**Proof.** One only needs to show that \( J_{\sigma, \omega_1}(0) \neq 0 \). We choose a local theta correspondence \( \theta : \omega_1 \rightarrow \sigma \otimes \pi \); because \( J_{\sigma, \omega_1}(0) \) is \( G \times G \times H \)-invariant, we have \( J_{\sigma, \omega_1}(0) = cJ_{\theta, \sigma} \) for a certain number \( c \). It follows that
\[
J_{\sigma, \pi, \omega_1}^{\eta}(s, \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)
\]
\[ = \int_{H} J_{\sigma, \omega_1}^{\eta}(0; \omega_1^{\eta}(h) \phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(h) f_1, f_2) \Delta(h)^s dh \]
\[ + \int_{H} R_{\sigma}(\omega_1^{\eta}(h) \phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(h) f_1, f_2) \Delta(h)^s dh \]
\[ \sim c I_{\pi}(s; \theta(\phi, \varphi_1, \phi_1, \varphi_2), f_1, f_2) - J_{\sigma, \pi, \omega_1}^{\eta}(s, \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2) \]
\[ \sim c d_{\pi}(s)^{-1} J_{\theta}(\phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2) - J_{\sigma, \pi, \omega_1}^{\eta}(s, \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2). \]

Hence
\[
J_{\sigma, \pi, \omega_1}^{\eta}(s) \sim \frac{c}{2d_{\pi}(s)} J_{\theta}. \]

Because \( J_{\sigma, \pi, \omega}(s) \) has a simple pole at \( s = 0 \), it is forced that \( c \neq 0 \), that is, \( J_{\sigma, \omega_1}(0) \neq 0 \).  

We can prove the following lemma and its corollary in a similar way.

**Lemma 7:**
\[
\int_{H} R_{\pi}(\omega_1^{\eta}(g) \phi_1, \phi_2, f_1, f_2)(\varphi_1, \varphi_2) \Delta(g)^s dg \sim -J_{\sigma, \pi, \omega}(s, \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2). \]

**Corollary 4:** \( n_{\pi} = 0 \) and \( J_{\pi, \omega_1} = J_{\pi, \omega_1}(0) \) is \( \tilde{G} \times H \times H \)-invariant.
Lemma 8: Let \( \theta : \omega_\psi \to \sigma \otimes \pi \) be a local theta correspondence. If

\[
\lim_{s \to 0} \frac{J_{\sigma, \omega_\psi}(s)}{L_{\sigma}(s)} = J_{\theta, \sigma}, \quad \text{and} \quad \lim_{s \to 0} \frac{J_{\pi, \omega_\psi}(s)}{L_{\pi}(s)} = J_{\theta, \pi},
\]

then

\[
d_{\sigma, \pi} := \lim_{s \to 0} \frac{d_{\sigma}(s)L_{\sigma}(s)}{d_{\pi}(s)L_{\pi}(s)} = 1.
\]

Proof. Lemmas 6 and 7 and Corollary 2 lead to this lemma.

Combining Lemma 5, Corollaries 3 and 4 and Lemma 8, we obtain Proposition 4 when \( F \) is \( p \)-adic.

4.4. \( \pi = \rho(\mu, \mu^{-1}) \) with \( \mu^2 = 1 \). We shall prove Proposition 5 when \( F \) is \( p \)-adic and skip the proof when \( F = \mathbb{R} \); the proof of the current case actually follows the same line as in the previous section and only differs in a few details.

Now let \( F \) be a \( p \)-adic field. Proposition 2 tells that \( d_{\sigma}(s), d_{\pi}(s) \) both have a triple zero at \( s = 0 \) and hence \( m_\sigma = m_\pi = 3 \). Let \( K', K, g_n, h_m, G_n, H_m \) be as in Section 4.3.

For given \( \phi_i, \varphi_i, f_i \) \( (i = 1, 2) \), there is \( \ell \geq 0 \) such that when \( n, m \geq \ell, k_i \in K, k_i' \in K' \), the following asymptotic formulas of matrix coefficients hold:

\[
\left( \pi \begin{pmatrix} a & 1 \\ 1 & k_1 f_1, k_2 f_2 \end{pmatrix} \right) = \frac{|a|^{1/2} m(K) \mu(a)}{1 + |\varpi|} \left( (1 - |\varpi|)v(a) C_{k_1, k_2}^{k_1, k_2} + R_{k_1, k_2}^{k_1, k_2} \right),
\]

\[
\left( \sigma \begin{pmatrix} a & 1 \\ a^{-1} & k_1' \varphi_1, k_2' \varphi_2 \end{pmatrix} \right) = \frac{|a| m(K') \mu(a) \chi_{\psi}(a)}{1 + |\varpi|} \left( (1 - |\varpi|)v(a) \tilde{C}_{k_1, k_2}^{k_1', k_2'} + \tilde{R}_{k_1, k_2}^{k_1', k_2'} \right),
\]

where \( R_{k_1, k_2}^{k_1, k_2}, \tilde{R}_{k_1, k_2}^{k_1, k_2} \) are bounded on \( K \times K, K' \times K' \) respectively and \( C_{k_1, k_2}^{k_1, k_2} = f_1(k_1)f_2(wk_2), \tilde{C}_{k_1, k_2}^{k_1', k_2'} = \varphi_1(k_1')\varphi_2(wk_2) \).

Put

\[
J_{n,m}^{\phi_1, \phi_2}(s; \phi_1, \phi_2) = \mu(\varpi)^{n+m}|\varpi|^{n-m+(n+m)s}nm \int_V \phi_1 \left( \frac{\varpi^{x_1}}{\varpi^{x_3}}, \frac{\varpi^{-m x_2}}{-\varpi^{x_1}} \right) \phi_2(X) dX
\]

and let \( T_\chi(s; \phi), S_\chi(s; \phi), \Phi_{\phi_1, \phi_2}(t), Z(\Phi, \chi, s) \) be as in Section 4.3.

Lemma 9: \( n_{\sigma, \pi} = 3 \) and \( J_{\sigma, \pi, \omega_\psi} \) is \( G \times \tilde{G} \times H \times H \)-invariant.
Proof. Step 1: Given \( \phi_i, \varphi_i, f_i \) \((i = 1, 2)\), we choose \( \ell \) such that the asymptotic formulas for matrix coefficients of \( \sigma, \pi \) hold when \( n, m \geq \ell \). Then

\[
J_{\sigma,\pi,\omega_\psi}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)
\]

\[
\sim \sum_{n, m \geq \ell} \int_{G_n \times H_m} (\omega_\psi(g, h)\phi_1, \phi_2)(\sigma(g)\varphi_1, \varphi_2)(\pi(h)f_1, f_2) \Delta(g)^* \Delta(h)^* \, dg \, dh
\]

\[
\sim (1 - |\omega|)^2 \sum_{n, m \geq \ell} \int_{k_i', k_i} C^{k_1', k_2'} C^{k_1, k_2} \mu(\omega)^{n + m} nm |\omega|^{(n + m)s} \Phi_{k_1', k_1 \phi_1, k_2', k_2 \phi_2} (\omega^{n - m})
\]

\[
\sim (1 - |\omega|)^2 \sum_{n, m \geq 0} \int_{k_i', k_i} C^{k_1', k_2'} C^{k_1, k_2} \mu(\omega)^{n + m} nm |\omega|^{(n + m)s} \Phi_{k_1', k_1 \phi_1, k_2', k_2 \phi_2} (\omega^{n - m})
\]

\[
= (1 - |\omega|)^2 \int_{k_i', k_i} C^{k_1', k_2'} C^{k_1, k_2} \sum_{n, m \geq 0} \mu(\omega)^{n + m} nm |\omega|^{(n + m)s} \Phi_{k_1', k_1 \phi_1, k_2', k_2 \phi_2} (\omega^{n - m}).
\]

Put \( n = n_1 + m \). It is not hard to see that

\[
\sum_{n, m \geq 0} \mu(\omega)^{n + m} nm |\omega|^{(n + m)s} \Phi_{k_1', k_1 \phi_1, k_2', k_2 \phi_2} (\omega^{n - m})
\]

\[
\sim \left( \sum_{m \geq 0} m^2 |\omega|^{2m s} \right) \left( \sum_{n_1 \in \mathbb{Z}} \mu(\omega)^{n_1} |\omega|^{n_1 s} \Phi_{k_1', k_1 \phi_1, k_2', k_2 \phi_2} (\omega^{n_1}) \right)
\]

\[
= \frac{|\omega|^{2s} (1 + |\omega|^{2s})}{(1 - |\omega|^{2s})^3} \left( \sum_{n_1 \in \mathbb{Z}} \mu(\omega^{n_1}) \Phi_{k_1', k_1 \phi_1, k_2', k_2 \phi_2} (\omega^{n_1}) \right).
\]

Hence

\[
J_{\sigma,\pi,\omega_\psi}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)
\]

\[
\sim \frac{2\zeta_F(2s)^3}{\zeta_F(1)^2} \int_{k_i', k_i} C^{k_1', k_2'} C^{k_1, k_2} \left( \sum_{n_1 \in \mathbb{Z}} \mu(\omega^{n_1}) \Phi_{k_1', k_1 \phi_1, k_2', k_2 \phi_2} (\omega^{n_1}) \right).
\]
Step 2: One checks easily that
\[
\int_{k',k_i} \overline{C_{n}^{k_i,k}k_2} C_{n}^{k_1,k_2} \mu(t) \Phi_{k',k_1,k_2} \mu(t) \text{ is a non-zero element in } n \otimes J = 0 \text{ and } L \text{ follows that}
\]
\[
\frac{3}{\zeta_F(1)^3} \int_{k',k_i} \overline{C_{n}^{k_i,k}k_2} C_{n}^{k_1,k_2} Z(\Phi_{k',k_1,k_2} \mu, 0)
\]
\[
= \frac{2}{\zeta_F(1)^3} \int_{k',k_i} k_1 o_1(e) k_2 o_2(w) k_1 f_1(e) k_2 f_2(w) T_x(0; k_1, k_2) S_\mu(0; k_2, k_2) \phi \text{ and the lemma is proved.}
\]

where \( L(\cdot) \) is the functional
\[
(\phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)
\]
\[
\frac{L}{\zeta_F(1)^3} \int_{k',k_i} k_1 o_1(e) k_2 o_2(w) k_1 f_1(e) k_2 f_2(w) T_x(0; k_1, k_2) S_\mu(0; k_2, k_2).
\]

It is easy to see that \( L \) is a non-zero element in
\[
(\omega_\psi^\vee \otimes \omega_\psi^* \otimes \sigma^* \otimes \pi^* \otimes \pi^\vee) \tilde{G} \times G \times H \times H.
\]

Actually, it is a non-zero scalar multiple of \((T_{\omega,\psi}, \varphi_1 \otimes M f_1)(T_{\omega,\psi}, \tilde{M} \varphi_2 \otimes f_2)\), where \( T_{\omega,\psi}(g, h) := T_x(0, \omega_\psi(g, h) \phi) \in \tilde{\rho}(\chi^{-1}) \otimes \rho(\chi, \chi^{-1}). \)

Therefore
\[
J_{\sigma,\pi,\omega,\psi}(s) \sim \frac{\zeta_F(s)^3}{4\zeta_F(1)^3} L
\]

and the lemma is proved.

It is easy to see that \( J_{\sigma,\omega,\psi}(s) \) and \( J_{\pi,\omega,\psi}(s) \) are holomorphic at \( s = 0 \) and
\[
J_{\sigma,\omega,\psi}(0) \in (\omega_\psi^\vee \otimes \omega_\psi^* \otimes \sigma^* \otimes \pi^\vee) \tilde{G} \times G \times H, \quad J_{\pi,\omega,\psi}(0) \in (\omega_\psi^\vee \otimes \omega_\psi^* \otimes \pi^* \otimes \pi^\vee) \tilde{G} \times G \times H.
\]

We put
\[
R_\sigma(s; -) = J_{\sigma,\omega,\psi}(s; -) - J_{\sigma,\omega,\psi}(0; -),
\]
\[
R_\pi(s; -) = J_{\pi,\omega,\psi}(s; -) - J_{\pi,\omega,\psi}(0; -).
\]

Then we can prove the following lemma and its corollary just as in Section 4.3.
**Lemma 10:**
\[
\int_R \sigma(h) \phi_1(h, \varphi_1, \varphi_2) (f_1, f_2) \Delta(h)^s dh \sim 7J_{\sigma, \omega}(\phi_1, \varphi_1, \varphi_2, f_1, f_2),
\]
\[
\int_R \sigma(h) \varphi_1(h, \varphi_1, \varphi_2) (f_1, f_2) \Delta(g)^s dg \sim 7J_{\sigma, \omega}(\phi_1, \varphi_1, \varphi_2, f_1, f_2).
\]

**Corollary 5:** $n_\omega = 0$ and $J_{\sigma, \omega}(0)$ is $G \times G \times H$-invariant; $n_\omega = 0$ and $J_{\sigma, \omega}(0)$ is $G \times H \times H$-invariant; if
\[
\lim_{s \to 0} \frac{J_{\sigma, \omega}(s)}{L_\sigma(s)} = J_{\theta, \sigma} \quad \text{and} \quad \lim_{s \to 0} \frac{J_{\sigma, \omega}(s)}{L_\omega(s)} = J_{\theta, \omega},
\]
then
\[
d_{\sigma, \omega} := \lim_{s \to 0} \frac{d_\sigma(s) L_\sigma(s)}{d_\omega(s) L_\omega(s)} = 1.
\]

### 4.5. Global Theory

Let $\sigma$ and $\pi$ be irreducible unitary cuspidal representations of $SL_2(\mathbb{A}_Q)$ and $PGL_2(\mathbb{A}_Q)$ that are in global theta correspondence with respect to $\omega$. Let the pairings of $\sigma$ and $\pi$ be the standard Petersson inner product pairings:

\[
(\varphi_1, \varphi_2) = \int_{SL_2(\mathbb{Q}) \setminus SL_2(\mathbb{A}_Q)} \varphi_1(g) \overline{\varphi_2(g)} dg, \quad (f_1, f_2) = \int_{PGL_2(\mathbb{Q}) \setminus PGL_2(\mathbb{A}_Q)} f_1(h) \overline{f_2(h)} dh.
\]

Here the measures $dg, dh$ are Tamagawa measures on $SL_2(\mathbb{A}_Q)$ and $PGL_2(\mathbb{A}_Q)$ respectively.

There is a global inner product formula concerning the lifting $\sigma \to \pi$ (cf. Theorem 6.2 in [13]).

\[
\Theta(\phi_1, \varphi_1), \Theta(\phi_2, \varphi_2) = L(\frac{1}{2}, \pi) \prod_v \frac{\zeta_{Q_v}(2)}{L(\frac{1}{2}, \pi_v)} \int_{SL_2(\mathbb{Q}_v)} (\omega_{\psi_v}(g_v) \phi_1, v, \phi_2, v) \cdot (\pi_v(h_v) f_1, v, \phi_2, v) dh_v.
\]

Our theory easily yields a dual inner product formula concerning the lifting $\Pi \to \tau$.

**Proposition 6:** Suppose that $\phi_i \in \omega$, $f_i \in \pi$ are decomposable. Then

\[
\Theta(\phi_1, f_1), \Theta(\phi_2, f_2) = L(\frac{1}{2}, \pi) \prod_v \frac{\zeta_{Q_v}(2)}{L(\frac{1}{2}, \pi_v)} \int_{PGL_2(\mathbb{Q}_v)} (\omega_{\psi_v}(h_v) \phi_1, v, f_1) \cdot (\pi_v(h_v) f_2, v, f_2) dh_v.
\]
The local pairing on $\pi_v$ is chosen so that their product is the global pairing and that local spherical vectors are paired to 1 at almost all places.

Proof. Because Conjectures A, B and 1, and the Multiplicity Assumption are true for $(\sigma_v, \pi_v)$ at all places, we have a dual inner product formula by the global duality principle (cf. Proposition 3). Note that $d_{\sigma_v, \pi_v} = 1$; the only thing needed to compute is the ratio $d_{\pi_v}(s)/d_{\sigma_v}(s)$, which is $|2|_{Q_v}$ as shown in [12]. Because $\prod_v |2|_{Q_v} = 1$, we obtain the dual inner product formula as stated.

5. The pair $(SL_2, SO(2, 2))$

In this subsection, $F$ denotes a totally real number field or its local completion, $V = M_{2 \times 2}(F)$, $q(X) = \det(X)$. There is an isomorphism $(GL_2 \times GL_2)/\Delta G_m \sim GSO(V, q)$, where $\Delta G_m$ is the diagonally embedded multiplicative group and $GL_2 \times GL_2$ acts on $V$ by $(h_1, h_2) \circ X = h_1 X h_2^{-1}$. The similitude character $\nu(\cdot)$ of $GSO(V)$ corresponds to $(h_1, h_2) \to \det h_1 \det h_2^{-1}$. We have $SO(V) \cong \{(g_1, g_2) | g_i \in GL_2, \det g_1 = \det g_2\}/\Delta G_m$ with respect to this identification; $GSO(2, 2), SO(2, 2)$ is realized as $GSO(V), SO(V)$.

5.1. Local theory. In this subsection, $F$ will be a $p$-adic field or $\mathbb{R}$. Let $\psi$ be a non-trivial character of $F$ and $w = \left( \begin{array}{c} \gamma \\ 1 \end{array} \right)$; the oscillator representation $\omega_\psi$ of $SL_2 \times O(V)$ on $S(V)$ is given by

$$\omega_\psi(h)\phi(X) = \phi(h^{-1} \circ X), \quad h \in O(V),$$
$$\omega_\psi\left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \phi(X) = |a|^2 \phi(aX),$$
$$\omega_\psi\left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \phi(X) = \psi(nq(X))\phi(X),$$
$$\omega_\psi(w)\phi(X) = \gamma(\psi, q) \int_V \phi(Y)\psi(q(X, Y))dY.$$

The oscillator representation can be extended to

$$R = \{(g, h) \in GL_2(F) \times GO(V) | \det g = \nu(h)\};$$

for $h \in GO(V)$, put $L(h)\phi(X) = |\nu(h)|^{-1} \phi(h^{-1} \circ X)$. Then for $(g, h) \in R$,

$$\omega_\psi(g, h)\phi(X) = L(h)\omega_\psi\left( \begin{array}{c} 1 \\ \det g^{-1} \end{array} \right) g\phi(X).$$

Put $G = SL_2(F), H = SO(V), \tilde{G} = GL_2(F), \tilde{H} = GSO(V)$ ($\tilde{G}, \tilde{H}$ refer to the preimages of $G, H$ in the metaplectic group). Let $\sigma \in \text{Irr}(G), \pi \in \text{Irr}(H)$ be
unitary and in local theta correspondence with respect to $\omega_\psi$. Let the height function $\Delta(g)$ on $G$ be the restriction of the height function on $GL_2(F)$. Let the height function $\Delta(h)$ on $H$ be the one induced from the height function on $GL_2(F) \times GL_2(F)$, that is, $\Delta([h_1, h_2]) = \Delta_{GL_2}(h_1)\Delta_{GL_2}(h_2)$.

Conjecture 1 is true for irreducible admissible unitary representations of $G$ and $H$ by Proposition 2, and hence is also true for irreducible admissible unitary representations of $\tilde{G}$ and $\tilde{H}$.

When $\sigma \in \text{Irr}(G), \pi \in \text{Irr}(H)$ are in local theta correspondence with respect to $\omega_\psi$, the Multiplicity Assumption is true for the pair $(\sigma, \pi)$ by Remark 5. In this section, we shall prove the following theorem.

**Theorem 2:** Conjectures A and B are true for $(\sigma, \pi)$ with

$$d_{\sigma, \pi} = \begin{cases} 1, & \text{\sigma is square-integrable,} \\ \frac{1}{2}, & \text{\sigma is trivial,} \\ 2, & \text{\sigma is a direct summand of a unitary principal series.} \end{cases}$$

Note that the meromorphic continuation of the functional-valued functions $J_{\sigma, \pi, \omega_\psi}(s), J_{\sigma, \omega_\psi}(s), J_{\pi, \omega_\psi}(s)$ can be easily checked (cf. Remark 1(a)). Hence one only needs to verify the remaining part of Conjecture A and follow the mechanism in Section 2 to prove Conjecture B; such a proof needs to distinguish three types of $\sigma$, and for convenience, we keep the notations in Section 2.

The representation $\sigma$ falls into one and only one of the following three types:

(i) $\sigma$ is square-integrable;

(ii) $\sigma$ is trivial;

(iii) $\sigma$ is a direct summand of a unitary principal series. (Here a unitary principal series refers to a principal series that is unitary.)

The argument of Theorem 2 for types (i) and (ii) of $\sigma$ is easy:

(i) If $\sigma$ is square-integrable, then $\pi$ is also square-integrable and the integrals defining $J_{\sigma, \pi, \omega_\psi}(s), J_{\sigma, \omega_\psi}(s), J_{\pi, \omega_\psi}(s)$ are both absolutely convergent at $s = 0$. It is easy to see that $J_{\sigma, \omega_\psi}(s) \neq 0$ (cf. the proof of Proposition 6.1 in [13]) and hence $J_{\sigma, \pi, \omega_\psi}(0) \neq 0, J_{\pi, \omega_\psi}(0) \neq 0$. It follows that $n_{\sigma, \pi} = n_\sigma = n_\pi = 0, m_\sigma = m_\pi = 0$ and that Conjectures A and B hold for $(\sigma, \pi)$. The equality $d_{\sigma, \pi} = 1$ is derived by changing the order of integration in the integral defining $J_{\sigma, \pi, \omega_\psi}(0)$. 
(ii) When $\sigma$ is trivial, direct computation easily shows that $n_{\sigma,\pi} = n_{\sigma} = n_{\pi} = 1$, $m_{\sigma} = m_{\pi} = 0$ and that $J_{\sigma,\pi,\omega,\psi}, J_{\sigma,\omega,\psi}, J_{\pi,\omega,\psi}$ are $G \times G \times H \times H$, $G \times G \times H$, $G \times H \times H$-invariant, respectively; hence Conjectures A and B hold in this case. One also computes easily that $J_{\sigma,\pi,\omega,\psi}, J_{\sigma,\omega,\psi}, J_{\pi,\omega,\psi}$ are $G \times G \times H \times H$-, $G \times G$-, $G \times H$-invariant, respectively; hence Conjectures A and B hold in this case. One also computes easily that $R_{\sigma,\pi}(s) \sim J_{\sigma,\pi,\omega,\psi}(s)$ and $R_{\pi,\sigma}(s) \sim J_{\sigma,\pi,\omega,\psi}(s)$; applying Corollary 2, we get that $d_{\sigma,\pi} = 1/2$.

When $\sigma$ is of type (iii), it is a direct summand of $\text{Ind}_{B}^{G}(\mu)$, where $B$ is the subgroup of upper triangular matrices in $\text{SL}_{2}(F)$ and $\mu$ is either (I) a unitary character or (II) of the form $|\cdot|^{2s_{0}}$ with $s_{0} \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. In this case, we shall prove the following statement, which is more explicit than Theorem 2.

**Proposition 7:** When $\sigma$ is a direct summand of a unitary principal series $\text{Ind}_{B}^{G}(\mu)$, Conjectures A and B are true with $d_{\sigma,\pi} = 2$. Particularly,

1. when $\mu \neq 1$, we have $m_{\sigma} = 1, m_{\pi} = 2, n_{\sigma,\pi} = 2, n_{\sigma} = 0, n_{\pi} = 1$;
2. when $\mu = 1$, we have $m_{\sigma} = 3, m_{\pi} = 6, n_{\sigma,\pi} = 6, n_{\sigma} = 0, n_{\pi} = 3$.

We make two comments here.

(a) To facilitate the computation necessary for the proof of Proposition 7, it is more convenient to consider $\sigma, \pi$ as the restriction of irreducible unitary admissible representations of $\widetilde{G}, \widetilde{H}$ to $G, H$ separately. Let $\mu_{1}, \mu_{2}$ be quasi-characters of $F^{\times}$ such that $\mu_{1}\mu_{2}^{-1} = \mu$ and write $\tau = \rho(\mu_{1}, \mu_{2})$ for the corresponding induced representation of $\widetilde{G} = \text{GL}_{2}(F)$; then $\sigma \subseteq \tau|_{G}$ and $\pi \subseteq \tau^{\vee} \boxtimes \tau|_{H}$. We make $\mu_{1}, \mu_{2}$ unitary or $\mu_{1} = \mu_{2}^{-1} = |\cdot|^{s_{0}}$ according to case (I) or (II).

(b) The proof of Proposition 7 follows the same line but differs in minor details for the follows four cases:

1. $F$ is $p$-adic and $\mu \neq 1$,
2. $F = \mathbb{R}$ and $\mu \neq 1$,
3. $F$ is $p$-adic and $\mu = 1$,
4. $F = \mathbb{R}$ and $\mu = 1$.

For demonstration purposes, we provide the proof of Proposition 7 in case (1) and skip the proof for the other three cases; for simplicity, we further assume that $\mu$ is unitary so that the pairing on $\tau$ can be expressed without using intertwining operators.

From now until the end of Section 5.1, we suppose that $\sigma$ is a direct summand of the unitary principal series $\text{Ind}_{B}^{G}(\mu)$ and that $\mu$ is a non-trivial unitary
character. We shall prove Proposition 7 in this case by building the following facts in order:

1. \( n_{\sigma,\pi} = 2 \) and \( J_{\sigma,\pi,\omega} \) is \( G \times G \times H \times H \)-invariant (cf. Lemma 11).
2. \( J_{\sigma,\omega}(s) \) is holomorphic at \( s = 0 \) and \( J_{\sigma,\omega}(0) \) is \( G \times G \times H \)-invariant (cf. Lemma 12).
3. \( J_{\pi,\omega}(s) \) has at most a simple pole at \( s = 0 \) and \( \tilde{J}_{\pi,\omega} := \lim_{s \to 0} sJ_{\pi,\omega}(s) \) is \( G \times H \times H \)-invariant (cf. Lemma 13).
4. \( J_{\sigma,\omega}(0) \neq 0 \) and hence \( J_{\sigma,\omega} = J_{\sigma,\omega}(0) \) (cf. Lemma 14, Corollary 6).
5. \( \tilde{J}_{\pi,\omega} \neq 0 \) and \( J_{\pi,\omega} = \tilde{J}_{\pi,\omega} \) (cf. Lemma 15, Corollary 7).
6. \( d_{\sigma,\pi} = 2 \) (cf. Lemma 16).

Now we introduce the following technical set-up so that the integrals involved in \( J_{\sigma,\pi,\omega}(s) \), \( J_{\sigma,\omega}(s) \), \( J_{\pi,\omega}(s) \) can be turned into an infinite sum, which is relatively easier to analyze.

1) The decomposition of \( G \) and \( H \). Let \( \varpi \) be a uniformizer of \( F \) such that \( \mu(\varpi) \neq 1 \). Write \( g_n = (\varpi^n, \varpi^{-n}) \in G \), \( h_{p,q}^\epsilon = \left[ (\varpi^{p+\epsilon}, \varpi^{-p}), (\varpi^{q+\epsilon}, \varpi^{-q}) \right] \in H \), where \( \epsilon = 0, 1; \) put \( K' = SL_2(O_F) \),

\[
K = SO_{2,2}(O_F) = \{(g_1, g_2) \in GL_2(O_F) \times GL_2(O_F) | \det g_1 = \det g_2\}/\Delta O_F^\times
\]
and \( G_n = K'g_nK' \), \( H_{p,q}^\epsilon = K h_{p,q}^\epsilon K \). Then

\[
G = \bigsqcup_{n \geq 0} G_n, \quad H = (\bigsqcup_{p,q \geq 0} H^0_{p,q}) \sqcup (\bigsqcup_{p,q \geq 0} H^1_{p,q}).
\]

The measures of these pieces are

\[
m(G_n) = c_n|\varpi|^{-2n}m(K'), \quad m(H_{p,q}^\epsilon) = c_{p,q}^\epsilon|\varpi|^{-2(p+q+\epsilon)}m(K),
\]

where \( c_n = 1 + (1 - \delta_{0,n})|\varpi| \), \( c_{p,q}^0 = (1 + (1 - \delta_{0,p})|\varpi|)(1 + (1 - \delta_{0,q})|\varpi|) \), \( c_{p,q}^1 = (1 + |\varpi|)^2 \) and \( \delta_{a,b} \) is the Dirac symbol.

2) The unitary pairings on \( \sigma \) and \( \pi \). Let the unitary pairing on \( \tau \) (or \( \tau^\vee \)) be

\[
(\varphi_1, \varphi_2) = \int_{GL_2(O_F)} \varphi_1(k)\varphi_2(k) dk, \quad \text{where } \varphi_i \in \tau \text{ (or } \tau^\vee \).
\]
Let the pairing on \( \sigma \) be

\[
(\varphi_1, \varphi_2)_{\sigma} = \frac{m(K')}{m(GL_2(O_F))} \cdot (\varphi_1, \varphi_2)_{\tau}.
\]
Let the pairing on \( \pi \) be given by

\[
(F'_{1} \boxtimes F_1, F'_{2} \boxtimes F_2)_{\pi} = \frac{m(K)}{m(GL_2(O_F))^2}(F'_{1}, F_{2})_{\tau^\vee} (F_{1}, F_{2})_{\tau},
\]

where \( F_i' \in \tau^\vee, F_i \in \tau \).
3) Matrix coefficients. Let $M$ be the intertwining operator on $\tau$ or $\tau^\vee$ whose formal definition is $M \varphi(g) = \int_F \varphi(w(1\frac{t}{1})) dw$; $M$ can be rigorously defined via meromorphic continuation when $\mu_1 \mu_2^{-1} \neq 1$. We define functionals $\lambda_\eta(\cdot, \cdot)$, where $\eta = \pm 1$:

$$
\lambda_+(\varphi_1, \varphi_2) = \varphi_1(e)M\varphi_2(w^{-1}), \quad \lambda_-(\varphi_1, \varphi_2) = M\varphi_1(e) \cdot \varphi_2(w).
$$

Given vectors $\phi_i \in \omega_\psi, \varphi_i \in \sigma \subseteq \rho(\mu_1, \mu_2)$ and

$$
f_i = F'_i \otimes F_i \in \pi \subseteq \rho(\mu_1^{-1}, \mu_2^{-1}) \otimes \rho(\mu_1, \mu_2),
$$

where $F'_i \in \rho(\mu_1^{-1}, \mu_2^{-1})$ and $F_i \in \rho(\mu_1, \mu_2)$, there exists $\ell$ such that when $n, p, q \geq \ell$ and $k'_i \in K', k_i \in K$, we have

\[
\left(\sigma(g_n)^{k'_1} \varphi_1, k'_2 \varphi_2\right) = \frac{\rho^{n|m(K')}}{1 + \rho} \sum_{\eta \in \{\pm 1\}} C_{\eta}^{k'_1, k'_2} \mu(\rho)^{\eta n},
\]

\[
\left(\pi(h_{p,q}^\epsilon)^{k_1 f_1, k_2 f_2}\right) = \frac{\rho^{p+q+\epsilon|m(K)}}{(1 + \rho)^2} \sum_{\eta_1, \eta_2 \in \{\pm 1\}} D_{\eta_1, \eta_2}^{k_1, k_2} \mu(\rho)^{-\eta_1 p + \eta_2 q + \frac{n+1+q}{2} \epsilon},
\]

where $C_{\eta}^{k'_1, k'_2} = \lambda_\eta(k'_1 \varphi_1, k'_2 \varphi_2)$ and $D_{\eta_1, \eta_2}^{k_1, k_2}(f_1, f_2) = \lambda_{\eta_1}(F'_1, F'_2) \lambda_{\eta_2}(F_1, F_2)$.

For $\phi \in S(X)$, $(g_n, h_{p,q}^\epsilon)$ acts on $\phi(X)$ by

$$
\omega_\psi(g_n, h_{p,q}^\epsilon)^\epsilon(\phi(X)) = |\rho|^{2n} \phi \left(\left(\frac{\rho^{n+p-q}x_1}{\rho^{n-p-q}x_3} \frac{\rho^{n+p+q}x_2}{\rho^{n-p+q}x_4}\right)\right).
$$

4) The integral on $G_n \times H_{p,q}^\epsilon$. We put

$$
J_{n,p,q,\epsilon}(s) = \int_{G_n \times H_{p,q}^\epsilon} \omega_\psi(g, h)^\epsilon(\phi_1, \phi_2)(\sigma(g)^\epsilon(\varphi_1, \varphi_2))(\pi(h)^\epsilon(f_1, f_2) \Delta(g)^s \Delta(h)^s) dg dh
$$

and

$$
J_{n,p,q,\epsilon}^{\eta_1, \eta_2}(s; k_1^i, k_1^i \phi_1, k_2^i, k_2^i \phi_2) = \mu(\rho)^{-\eta_1 p - \eta_2 q + \frac{n+1+q}{2} \epsilon} |\rho|^{(n-(p+q))+(n+p+q+\epsilon)2s}
\times \int_V \int_{K' \times K} \frac{C_{\eta_1, \eta_2}^{k_1^i, k_2^i \phi_1 \phi_2}(X) dX.}
$$

Then when $n, p, q \geq \ell$, we have

$$
J_{n,p,q,\epsilon}(s) = \sum_{\eta_1, \eta_2} \int_{K' \times K} C_{\eta_1, \eta_2}^{k_1^i, k_2^i \phi_1, k_2^i, k_2^i \phi_2}.
$$

5) Some integral operators. For convenience, we introduce nine functionals
\[ \Phi_{1,\phi_1,\phi_2}(t) = \int_V \phi_1 \left( \begin{pmatrix} t^{-1} & x_2 \\ 0 & 0 \end{pmatrix} \right) \phi_2 \left( \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \right) dX, \]

\[ \Phi_{2,\phi_1,\phi_2}(t) = |t|^{-1} \int_V \phi_1 \left( \begin{pmatrix} 0 & t^{-1} x_2 \\ 0 & 0 \end{pmatrix} \right) \phi_2 \left( \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \right) dX, \]

\[ \Phi_{4,\phi_1,\phi_2}(t) = \int_V \phi_1 \left( \begin{pmatrix} 0 & x_2 \\ 0 & t^{-1} x_4 \end{pmatrix} \right) \phi_2 \left( \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \right) dX; \]

\[ T_{1,\chi}(s; \phi) = \int_{F^\times} \chi(t)|t|^{-2s} \phi((t x_2 \; 0)) d^X t dx_2, \]

\[ S_{1,\chi}(s; \phi) = \int_{F^\times} \chi(t)|t|^{1+2s} \phi((t \; 0)) d^X t dx_3 dx_4; \]

\[ T_{4,\chi}(s; \phi) = \int_{F^\times} \chi(t)|t|^{-2s} \phi((0 \; x_2 \; 0)) d^X t dx_2, \]

\[ S_{4,\chi}(s; \phi) = \int_{F^\times} \chi(t)|t|^{1+2s} \phi((0 \; x_1 \; x_3)) d^X t dx_1 dx_3; \]

\[ T_{2,\chi}(s; \phi) = \int_{F^\times} \chi(t)|t|^{-2s} \phi((0 \; 0 \; t)) d^X t, \]

\[ S_{2,\chi}(s; \phi) = \int_{F^\times} \chi(t)|t|^{2s} \phi((x_1 \; x_4 \; t)) d^X t dx_1 dx_4. \]

Now we start to prove Proposition 7 in case (1).

**Lemma 11:** \( n_{\sigma, \pi} = 2 \) and \( J_{\sigma, \pi, \omega_\psi} \) is \( G \times G \times H \times H \)-invariant.

**Proof.** Given \( \phi_i \in \omega_\psi, \varphi_i \in \sigma, f_i \in \pi \) \((i = 1, 2)\), we write

\[ J(s) = J_{\sigma, \pi, \omega_\psi}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2). \]

**Step 1:** The behavior of \( J(s) \) is determined by the integration near \( \infty \), that is,

\[ J(s) = \sum_\epsilon \sum_{n,p,q \geq 0} J_{n,p,q,\epsilon}(s) \]

\[ \gtrsim \sum_\epsilon \sum_{n,p,q \geq \ell} J_{n,p,q,\epsilon}(s) \]

\[ = \sum_\epsilon \sum_{n,p,q \geq \ell} \sum_{\eta, \eta_1, \eta_2} \int_{K^1 \times K^2} C_{\eta}^{k_1,k_2} D_{\eta_1,\eta_2}^{k_1,k_2} J_{\eta,\eta_1,\eta_2}(s; k_1, k_1, k_2, k_2) \]

\[ \gtrsim \sum_\epsilon \sum_{n,p,q \geq \ell} \sum_{\eta, \eta_1, \eta_2} \int_{K^1 \times K^2} C_{\eta}^{k_1,k_2} D_{\eta_1,\eta_2}^{k_1,k_2} J_{\eta,\eta_1,\eta_2}(s; k_1, k_1, k_2, k_2) \]

\[ \gtrsim \sum_\epsilon \sum_{n,p,q \geq \ell} \sum_{\eta, \eta_1, \eta_2} \int_{K^1 \times K^2} C_{\eta}^{k_1,k_2} D_{\eta_1,\eta_2}^{k_1,k_2} \left( \sum_\epsilon \sum_{n,p,q \geq \ell} J_{\eta,\eta_1,\eta_2}(s; k_1, k_1, k_2, k_2) \right) \]

\[ = \sum_{\eta, \eta_1, \eta_2} \int_{K^1 \times K^2} C_{\eta}^{k_1,k_2} D_{\eta_1,\eta_2}^{k_1,k_2} J_{\eta,\eta_1,\eta_2}(s; k_1, k_1, k_2, k_2), \]

where \( J_{\eta,\eta_1,\eta_2}(s; k_1, k_1, k_2, k_2) = \sum_\epsilon \sum_{n,p,q \geq \ell} J_{n,p,q,\epsilon}(s; k_1, k_1, k_2, k_2). \)
STEP 2: We now estimate $J_{n,\eta_1,\eta_2}(s; -)$; the idea is to decompose the set $$S := \{n, p, q, \epsilon | n, p, q \geq 0\}$$ into six subsets $S_{ij}$ and then estimate the sum of $J_{n,\eta_1,\eta_2}(s; -)$ over each $S_{ij}$. Here

\begin{align*}
S_{11} &= \{0 \leq n < (p - q) \leq (p + q + \epsilon)\} , \quad S_{21} = \{0 \leq n < (q - p) \leq (p + q + \epsilon)\} , \\
S_{12} &= \{0 \leq (p - q) \leq n < (p + q + \epsilon)\} , \quad S_{22} = \{0 < (q - p) \leq n < (p + q + \epsilon)\} , \\
S_{13} &= \{0 \leq (p - q) \leq (p + q + \epsilon) \leq n\} , \quad S_{23} = \{0 < (q - p) \leq (p + q + \epsilon) \leq n\} .
\end{align*}

We put $J_{i j}^{\eta_1,\eta_2}(s; -) = \sum_{n, p, q, \epsilon} J_{n,\eta_1,\eta_2}(s; -)$; then $J_{n,\eta_1,\eta_2}(s; -) = \sum_{i j} J_{i j}^{\eta_1,\eta_2}(s; -)$.

Let us estimate $J_{11}^{\eta_1,\eta_2}(s)$ first. Put $a = n, b = (p - q) - n, c = (p + q + \epsilon) - n$.

Then

\begin{align*}
J_{11}^{\eta_1,\eta_2}(s; k_1,k_2,\phi_1, k'_2,\phi_2) &= \sum_{a \geq 0} \sum_{b > 0} \sum_{c \geq 0} \mu(\omega) (-\eta+\eta_1)a+\frac{a_1+\eta_2}{2}b+\frac{a_1-\eta_2}{2}c |\omega|^{-(b+c)} |\omega|^{(2a+b+c)2s} \\
& \quad \cdot \int_V k'_1 k_1 \phi_1 \left( \left( \frac{\omega^{-b} x_1}{\omega^{2a+b+c} x_3} - \frac{\omega^{-\eta}}{\omega^{2a+b+c} x_4} \right) \right) k_2 k_2 \phi_2 (X) dX \\
& = \sum_{a \geq 0} \sum_{b > 0} \sum_{c \geq 0} \mu(\omega) (-\eta+\eta_1)a+\frac{a_1+\eta_2}{2}b+\frac{a_1-\eta_2}{2}c |\omega|^{b} |\omega|^{(2a+b+c)2s} \\
& \quad \cdot \int_V k'_1 k_1 \phi_1 \left( \left( \frac{x_1}{\omega^{2a+b+c} x_3} - \frac{x_2}{\omega^{2a+b+c} x_4} \right) \right) k_2 k_2 \phi_2 \left( \left( \frac{\omega^{b} x_1}{x_3} - \frac{\omega^{b} x_2}{x_4} \right) \right) dX \\
& \geq \sum_{a, b, c \geq 0} \mu(\omega) (-\eta+\eta_1)a+\frac{a_1+\eta_2}{2}b+\frac{a_1-\eta_2}{2}c |\omega|^{b} |\omega|^{(2a+b+c)2s} \\
& \quad \cdot \int_V k'_1 k_1 \phi_1 \left( \left( \frac{x_1}{x_3} \frac{x_2}{0} \right) \right) k_2 k_2 \phi_2 \left( \left( \frac{\omega^{b} x_1}{x_3} - \frac{\omega^{b} x_2}{x_4} \right) \right) dX.
\end{align*}

It is easy to see that

\begin{align*}
J_{11}^{\eta_1,\eta_2}(s; k_1,k_2,\phi_1, k'_2,\phi_2) & \geq 2 \left\{ \begin{array}{ll}
\zeta_F(2s) \zeta_F(4s) \sum_{b \geq 0} \mu(\omega^b) |\omega^b|^{2s} \Phi_{1, k'_1, k_1, k'_2, k_2, \phi_1, \phi_2} (\omega^b), & \eta = \eta_1 = \eta_2 , \\
0, & \text{otherwise.}
\end{array} \right.
\end{align*}
One can estimate the other $J_{ij}^{\eta_1,\eta_2}(s)$ in a similar way; the result is

$$J_{12}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(2s) \zeta_F(4s) \sum_{b \leq 0} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b),$$

$$J_{12}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(4s) \zeta_F(2s) \sum_{b \geq 0} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b),$$

$$J_{13}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(2s) \zeta_F(4s) \sum_{b \leq 0} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b),$$

$$J_{13}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(4s) \zeta_F(2s) \sum_{b \geq 0} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b),$$

and all other $J_{ij}^{\eta_1,\eta_2}(s)$ are $\sim 0$.

It follows that

$$J_{12}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(2s) \zeta_F(4s) \sum_{b \in \mathbb{Z}} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b),$$

$$J_{12}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(4s) \zeta_F(2s) \sum_{b \in \mathbb{Z}} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b),$$

$$J_{13}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(2s) \zeta_F(4s) \sum_{b \in \mathbb{Z}} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b),$$

$$J_{13}^{\eta_1,\eta_2}(s, k_1, k_2, k_3, k_4) \sim_2 \zeta_F(4s) \zeta_F(2s) \sum_{b \in \mathbb{Z}} \mu^{\eta}(\varpi^b) |\varpi^b|^{2s} \Phi_{1, k_1, k_2, k_3, k_4}(\varpi^b).$$

**Step 3:** It is now time to estimate

$$\int_{K^2} \frac{C_{k_1, k_2}^{k_1, k_2} D_{\eta_1, \eta_2}^{k_1, k_2} J_{12, \eta_1, \eta_2}(s, k_1, k_2) \eta_1, \eta_2}{K^2}$$

by the result in Step 2, such an integral is $\sim 0$ when $(\eta, \eta_1, \eta_2)$ is not of the form $(\eta, \eta, \eta), (\eta, \eta_1, \eta_2)$. (\eta, \eta, \eta).$
is constant on $\omega^b \mathcal{O}_F^\times$ for each $b \in \mathbb{Z}$. Hence
\[
\int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} J_{\eta, \eta, \eta}(s, k_1', k_1, k_2, k_2) = 2 \zeta_F(2s) \zeta_F(4s) \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} \sum_{b \in \mathbb{Z}} \mu^n(\omega^b) |\omega^b|^{2s} \Phi_{1, k_1', k_1, k_2, k_2} \omega^b(\omega^b) \\
= \zeta_F(2s) \zeta_F(4s) \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} \mu^n(\omega^b) |\omega^b|^{2s} \Phi_{1, k_1', k_1, k_2, k_2} \omega^b(\omega^b) \\
= \zeta_F(2s) \zeta_F(4s) \sum_{b \in \mathbb{Z}} \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} \mu^n(\omega^b) |\omega^b|^{2s} \Phi_{1, k_1', k_1, k_2, k_2} \omega^b(\omega^b) \\
= \zeta_F(2s) \zeta_F(4s) \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} \mu^n(\omega^b) |\omega^b|^{2s} \Phi_{1, k_1', k_1, k_2, k_2} \omega^b(\omega^b) \\
= \zeta_F(2s) \zeta_F(4s) \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} Z(\mu^n, 2s, \Phi_{1, k_1', k_1, k_2, k_2}).
\]

Similarly
\[
\int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} J_{\eta, \eta, \eta}(s, k_2', k_1, k_2', k_2) = \zeta_F(2s) \zeta_F(4s) \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} Z(\mu^n, 2s, \Phi_{k_2', k_1, k_2', k_2}).
\]

and
\[
\int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} J_{\eta, \eta, \eta}(s, k_1', k_1, k_2, k_2') = \zeta_F(2s) \zeta_F(4s) \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} Z(\mu^n, 2s, \Phi_{k_2', k_1, k_2', k_2}).
\]

**Step 4:** Combining the formulas in Steps 1 and 3, we get that
\[
J(s) \sim \zeta_F(2s) \zeta_F(4s) \sum_{\eta} \left( \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2} D_{\eta, n, \eta}^{k_1, k_2} Z(\mu^n, 2s, \Phi_{1, k_1', k_1, k_2, k_2}) \\
+ \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2'} D_{\eta, n, \eta}^{k_1, k_2} Z(\mu^n, 2s, \Phi_{4, k_1', k_1, k_2, k_2}) \\
+ \int_{K'^2 \times K^2} C_{\eta, n, \eta}^{k_1', k_2'} D_{\eta, n, \eta}^{k_1, k_2} Z(\mu^n, 2s, \Phi_{k_2', k_1, k_2', k_2}).\right)
\]
We remark that the inner three integrals are equal and independent of $\eta$; this is because of the inner symmetry of the integral expression.

It follows that

$$J_{\sigma, \pi, \omega, \psi}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)$$

$$\sim 6 \zeta_F(2s) \zeta_F(4s) \int_{K_1^{12} \times K^2} \frac{C_{+}^{k_1', k_2'} D_{k_1, k_2}^{k_1, k_2} Z(\mu, 2s, \Phi_1, k_1', k_2') \Phi_1}{K'_{2}^2 \times K_2^2}$$

$$\sim 6 \zeta_F(2s) \zeta_F(4s) \int_{K_1^{12} \times K^2} \frac{C_{+}^{k_1', k_2'} D_{k_1, k_2}^{k_1, k_2} Z(\mu, 0, \Phi_1, k_1', k_2') \Phi_1}{K'_{2}^2 \times K_2^2}$$

$$= \frac{3}{4} \zeta_F(s)^2 L(\phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2),$$

where $L(\cdot)$ is the functional

$$(\phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2)$$

$$\rightarrow \int_{K_1^{12} \times K^2} \varphi_1(k_1') M \varphi_2(w^{-1}k_2') (F_1' \boxtimes F_1)(k_1)(MF_2' \boxtimes MF_2)([w^{-1}, w^{-1}]k_2)$$

$$\cdot T_{1, \mu-1}(s; k_1', k_2') S_{1, \mu-1}(s; k_2', k_2') dk_1' dk_2' dk_1 dk_2.$$
Lemma 12: \( J_{\sigma, \omega_\psi}(s) \) is holomorphic at \( s = 0 \) and \( J_{\sigma, \omega_\psi}(0) \) is \( G \times G \times H \)-invariant.

Proof. Given \( \phi_i, \varphi_i \) (\( i = 1, 2 \)), we put \( J_\sigma(s) = J_{\sigma, \omega_\psi}(s; \phi_1, \phi_2, \varphi_1, \varphi_2) \). Then

\[
J_\sigma(s) = \sum_{n \geq 0} \int_{G_n} \left( \omega_\psi(g)\phi_1, \phi_2 \right) (\sigma(g)\varphi_1, \varphi_2) \Delta(g)^s dg \\
= \sum_{\eta} \int_{K' \times K'} \left( \sum_{n \geq \ell} \mu(\omega) - \eta n \right) |\omega|^{n(1+2s)} \int_V k_1^{n} \phi_1(\omega^n X) k_2^{n} \varphi_2(X) dX \right) dk_1^{n} dk_2^{n} \\
+ \sum_{0 \leq n < \ell} \int_{G_n} \left( \omega_\psi(g)\phi_1, \phi_2 \right) (\sigma(g)\varphi_1, \varphi_2) \Delta(g)^s dg.
\]

The second part is an integral on a compact set and hence a polynomial function of \(|\omega|^s\). The first part defines a rational function that is holomorphic at \( s = 0 \). Hence \( J_\sigma(s) \) is holomorphic at \( s = 0 \) and the lemma is proved.

The \( G \times G \times H \)-invariance of \( J_{\sigma, \omega_\psi}(0) \) is easy to verify. Actually it is obviously \( H \)-invariant and \( K' \times K' \)-invariant; since \( G \) is generated by \( K' \) and \( (\omega^{-1}, \omega) \), one only needs to check that \( J_{\sigma, \omega_\psi}(0) \) is invariant by \( (\omega^{-1}, \omega) \), which is easy. 

For convenience, we introduce a functional

\[
\Phi_{\pi, \phi_1, \phi_2}(t) = \int_V \phi_1 \left( t^{-1} x_1, x_2, 0 \right) \phi_2 \left( x_3, 0, x_4 \right) dX.
\]

Lemma 13: \( J_{\pi, \omega_\psi}(s) \) has at most a simple pole at \( s = 0 \) and \( \tilde{J}_{\pi, \omega_\psi} := \lim_{s \to 0} s J_{\pi, \omega_\psi}(s) \) is \( G \times H \times H \)-invariant.

Proof. For given \( \phi_i, \varphi_i \) (\( i = 1, 2 \)), we put \( J_\pi(s) = J_{\pi, \omega_\psi}(s; \phi_1, \phi_2, f_1, f_2) \) and shall compute it up to the relation \( \sim \).

The asymptotic formula of matrix coefficients of \( \pi \) tells that

\[
\int_{H_{p,q}^s} (\omega_\psi(h)\phi_1, \phi_2)(\sigma(h)f_1, f_2) \Delta(h)^s dh \\
= \sum_{\eta_1, \eta_2} \int_{K \times K} D^{k_1, k_2}_{\eta_1, \eta_2} B_{p,q,\epsilon}^{\eta_1, \eta_2}(s; k_1^{n} \phi_1, k_2^{n} \phi_2) dk_1 dk_2,
\]

where

\[
B_{p,q,\epsilon}^{\eta_1, \eta_2}(s; \phi_1, \phi_2) = \mu(\omega)^{n_1 p - n_2 q + \frac{n_1 - n_2}{2}} |\omega|^{(p+q+\epsilon)(-1+2s)} \int_V \phi_1 \left( x_1, x_3, x_4, x_2 \right) \phi_2 \left( x_1, x_3, x_4, x_2 \right) dX.
\]
It follows that

\[ J_\pi(s) \sim \sum_{\epsilon} \sum_{p,q \geq 0} \int_{H_{p,q}} (\omega_\psi(h) \phi_1, \phi_2)(\sigma(h)f_1, f_2) \Delta(h)^s dh \]

\[ = \sum_{\epsilon} \sum_{p,q \geq 0} \sum_{\eta_1, \eta_2} \int_{K \times K} D_{\eta_1, \eta_2}^{k_1, k_2} B_{p,q, \epsilon}^{\eta_1, \eta_2}(s, k_1 \phi_1, k_2 \phi_2) dk_1 dk_2 \]

\[ = \sum_{p,q \geq 0} \int_{K \times K} D_{\eta_1, \eta_2}^{k_1, k_2} \sum_{\eta} B_{p,q, \epsilon}^{\eta_1, \eta_2}(s, k_1 \phi_1, k_2 \phi_2) dk_1 dk_2. \]

It is easy to see that

\[ \sum_{p,q \geq 0} B_{p,q, \epsilon}^{\eta_1, \eta_2}(s, k_1 \phi_1, k_2 \phi_2) \]

\[ = \begin{cases} 0, & \eta_1 \neq \eta_2, \\ \zeta_F(2s) \sum_{b \in \mathbb{Z}} \mu_\epsilon(b) |b|^{2s} \Phi_{\pi, k_1 \phi_1, k_2 \phi_2}(b), & \eta_1 = \eta_2 = \eta, \end{cases} \]

and that \( \int_{K \times K} D_{\eta_1, \eta_2}^{k_1, k_2} \mu_\epsilon(t)|t|^{2s} \Phi_{\pi, k_1 \phi_1, k_2 \phi_2}(t) \) on \( b \mathcal{O}_F^\times \) for each \( b \in \mathbb{Z} \). Hence

\[ J_\pi(s) \sim \zeta_F(2s) \sum_{\eta} \int_{K \times K} D_{\eta_1, \eta_2}^{k_1, k_2} \left( \int_{\mathbb{F}^\times} \mu_\epsilon(t)|t|^{2s} \Phi_{\pi, k_1 \phi_1, k_2 \phi_2}(t) dt \right) dk_1 dk_2 \]

\[ = \zeta_F(2s) \sum_{\eta} \int_{K \times K} D_{\eta_1, \eta_2}^{k_1, k_2} Z(\mu_\epsilon, 2s, \Phi_{\pi, k_1 \phi_1, k_2 \phi_2}) dk_1 dk_2 \]

\[ \sim \zeta_F(2s) \sum_{\eta} \int_{K \times K} D_{\eta_1, \eta_2}^{k_1, k_2} Z(\mu_\epsilon, 0, \Phi_{\pi, k_1 \phi_1, k_2 \phi_2}) dk_1 dk_2. \]

This formula tells that

\[ \lim_{s \to 0} s J_\pi(s) = 2 \text{Re} \epsilon_0 \zeta_F(s) \sum_{\eta} \int_{K \times K} D_{\eta_1, \eta_2}^{k_1, k_2} Z(\mu_\epsilon, 0, \Phi_{\pi, k_1 \phi_1, k_2 \phi_2}) dk_1 dk_2 \]

is a finite number and hence \( J_{\pi, \omega_\psi}(s) \) has at most a simple pole at \( s = 0 \). It is not hard to derive from the above formula that \( \widetilde{J}_{\pi, \omega_\psi} \) is \( G \times H \times H \)-invariant. 

It remains to show that \( J_{\sigma, \omega_\psi}(0) \neq 0 \) and \( \widetilde{J}_{\pi, \omega_\psi} \neq 0 \) so that \( n_\sigma = 0, n_\pi = 1. \)
To provide a more delicate analysis, we introduce two “remainder terms”:

\[ R_\sigma(s) = J_{\sigma,\omega}(s) - J_{\sigma,\omega}(0) \quad R_\pi(s) = J_{\pi,\omega}(s) - \frac{\lim_{s \to 0} s J_{\pi,\omega}(s)}{s}. \]

**Lemma 14:**

\[
\int_H R_\sigma(\omega(h) \phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(h)f_1, f_2)\Delta(h)^s dh
\]

\[ \sim - \frac{5}{3} J_{\sigma,\pi,\omega}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2). \]

**Proof.** The left hand side is absolutely convergent when Re(s) >> 0 and defines a holomorphic function on a right half plane. It can be extended to a meromorphic function on \( \mathbb{C} \) whose behavior at \( s = 0 \) is determined by the integrand’s behavior near \( \infty \).

L.H.S \[ \sim \sum_{\epsilon} \sum_{p,q \geq \ell} \int_{H_{p,q}} R_\sigma(\omega(h) \phi_1, \phi_2, \varphi_1, \varphi_2)(\pi(g)f_1, f_2)\Delta(h)^s dh \]

\[ \sim \sum_{\epsilon} \sum_{p,q \geq \ell} \left( \sum_{n \geq \ell} \int_{G_n \times H_{p,q}} (\omega(h) \phi_1, \phi_2)(\pi(g)\varphi_1, \varphi_2)(\pi(g)f_1, f_2) \right)^s \]

\[ \times \Delta(g)^{s-1} dh \]

\[ = \sum_{\epsilon} \sum_{p,q \geq \ell} \left( \sum_{n \geq \ell} \int_{G_n \times H_{p,q}} \frac{\Delta(g)^{s-1} dh}{\Delta(g)} \right)^s \]

One can estimate \( \sum_{\epsilon} \sum_{p,q \geq \ell} \left( \sum_{n \geq \ell} J_{\epsilon,n,p,q}(s; k_1, k_2, k'_1, k'_2) \right)^s \) and its \( k'_i, k_i \)-integral with \( \overline{C}_{\eta}^{k_1, k'_1} D_{\eta_1, \eta_2}^{k_2, k'_2} \) up to the relation \( \sim \) just as in Step 2 of the
proof of Lemma 11; the output is
\[
L.H.S. \sim 2(\zeta_F(4s) - \zeta_F(2s))\zeta_F(2s)
\]
\[
\times \sum_{\eta} \int_{K^2} C_{\eta}^{k_1', k_2'} D_{\eta}^{k_1, k_2} Z(\mu^\eta, 2s, \Phi_{1, k_1', k_1, k_2, k_2})
\]
\[
+ 2(\zeta_F(4s)^2 - \zeta_F(2s)^2)
\]
\[
\times \sum_{\eta} \int_{K^2} C_{\eta}^{k_1', k_2'} D_{-\eta}^{k_1, k_2} Z(\mu^\eta, 2s, \Phi_{2, k_1', k_1, k_2, k_2})
\]
\[
+ (\zeta_F(4s) - \zeta_F(2s))\zeta_F(2s)
\]
\[
\times \sum_{\eta} \int_{K^2} C_{\eta}^{k_1', k_2'} D_{\eta}^{k_1, k_2} Z(\mu^\eta, 2s, \Phi_{1, k_1', k_1, k_2, k_2})
\]
\[
\sim -\frac{5}{3} J_{\sigma, \pi, \omega}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2).
\]

The above lemma immediately leads to

**Corollary 6:** $n_\sigma = 0$ and $J_{\sigma, \omega} = J_{\sigma, \omega}(0)$ is $G \times G \times H$-invariant.

We can prove the following lemma and its corollary in a similar way.

**Lemma 15:**

\[
\int_H R_\pi(\omega_\psi(g)\phi_1, \phi_2, f_1, f_2)(\sigma(g)\varphi_1, \varphi_2)\Delta(g)^s dg
\]
\[
\sim -\frac{1}{3} J_{\sigma, \pi, \omega}(s; \phi_1, \phi_2, \varphi_1, \varphi_2, f_1, f_2).
\]

**Corollary 7:** $n_\pi = 1$ and $J_{\pi, \omega} = J_{\pi, \omega}(0)$ is $G \times H \times H$-invariant.

**Lemma 16:** Let $\theta : \omega_\psi \to \sigma \otimes \pi$ be a local theta correspondence. If

\[
\lim_{s \to 0} \frac{J_{\sigma, \omega_\psi}(s)}{L_{\sigma}(s)} = J_{\theta, \sigma} \quad \text{and} \quad \lim_{s \to 0} \frac{J_{\pi, \omega_\psi}(s)}{L_{\pi}(s)} = J_{\theta, \pi},
\]

then

\[
d_{\sigma, \pi} := \lim_{s \to 0} \frac{d_{\sigma}(s)L_{\sigma}(s)}{d_{\pi}(s)L_{\pi}(s)} = 2.
\]

**Proof.** One just needs to apply Corollary 2 to the current situation.
When combining Lemma 11, Corollaries 6 and 7, and Lemma 16, we obtain Proposition 7 in the case that \( \sigma \) is a direct summand of \( \text{Ind}_B^G(\mu) \) with \( \mu \) being non-trivial and unitary.

5.2. Global theory. Let \( F \) be a total real number field and \( \mathbb{A}_F \) the ring of \( F \)-adels. Let \( \psi \) be a character of \( \mathbb{A}_F/F \) and \( \omega_\psi \) the global oscillator representation of \( SL_2(\mathbb{A}_F) \times O(V)_{\mathbb{A}_F} \) on \( S(V_{\mathbb{A}_F}) \). Just as in the local case, \( \omega_\psi \) can be extended to \( R(\mathbb{A}_F) := \{(g, h) \in GL_2(\mathbb{A}_F) \times GO(V_{\mathbb{A}_F}) | \det g = \nu(h)\} \) for \( \phi \in S(V_{\mathbb{A}_F}) \), we associate a theta function on \( R(\mathbb{A}_F) \), that is, \( \Theta(\phi, \varphi)(h) = \sum_{\xi \in V_F} \omega_\psi(g, h) \phi(\xi) \).

For an irreducible cuspidal representation \( \tau \) of \( GL_2(\mathbb{A}_F) \) and a function \( \varphi \in \tau \), we define its global theta lift via \( \phi \in S(V_{\mathbb{A}_F}) \) as

\[
\Theta(\phi, \varphi)(h) = \int_{SL_2(\mathbb{A}_F)} \Theta(g_1g, h)\varphi(g_1g)dg_1,
\]

where \( g \) is any element in \( GL_2(\mathbb{A}_F) \) such that \( \det g = \nu(h) \). The global theta lift of \( \tau \) to \( GSO(V_{\mathbb{A}_F}) \) is \( \Theta(\tau) := \{\Theta(\phi, \varphi) : \phi \in S(V_{\mathbb{A}_F}), \varphi \in \tau\} \).

For an irreducible cuspidal representation \( \Pi \) of \( GSO(V_{\mathbb{A}_F}) \) and a function \( f \in \Pi \), we define its global theta lift via \( \phi \in S(V_{\mathbb{A}_F}) \) as

\[
\Theta(\phi, f)(g) = \int_{SO(V_{\mathbb{A}_F})} \Theta(g, h_1h)f(h_1h)dh_1,
\]

where \( h \) is any element in \( GSO(V_{\mathbb{A}_F}) \) such that \( \nu(h) = \det g \). The global theta lift of \( \Pi \) to \( GL_2(\mathbb{A}_F) \) is \( \Theta(\pi) := \{\Theta(\phi, f) : \phi \in S(V_{\mathbb{A}_F}), f \in \Pi\} \).

It is known that \( \Theta(\tau) = \tau^\vee \boxtimes \tau \) and that

\[
\Theta(\Pi) = \begin{cases} 
0, & \text{if } \Pi \text{ is not of the form } \tau^\vee \boxtimes \tau, \\
\tau, & \text{if } \Pi = \tau^\vee \boxtimes \tau.
\end{cases}
\]

Let the pairings of \( \tau \) and \( \Pi = \tau^\vee \otimes \tau \) be the standard Petersson inner product pairing:

\[
(\varphi_1, \varphi_2) = \int_{PGL_2(F) \backslash PGL_2(\mathbb{A}_F)} \varphi_1(g)\overline{\varphi_2(g)}dg,
\]

\[
(f_1, f_2) = \int_{PGSO(V_F) \backslash PGSO(V_{\mathbb{A}_F})} f_1(h)\overline{f_2(h)}dh,
\]
where \( dg, dh \) are the Tamagawa measures on \( PGL_2(\mathbb{A}_F) \) and \( PGSO(V_{\mathbb{A}_F}) \) respectively.

Proposition 6.10 in [3] (which concerns theta lifting from \( GL_2 \) to \( GO(2, 2) \) and \( GO(3, 1) \)) easily leads to a global inner product formula concerning the lifting \( \tau \rightarrow \Pi \),

\[
\left( \Theta(\phi_1, \varphi_1), \Theta(\phi_2, \varphi_2) \right) = \frac{2L(1, \tau, \text{ad})}{\zeta_F(2)^2} \prod_v \frac{\zeta_{F_v}(2)^2}{L(1, \tau_v, \text{ad})} \int_{SL_2(F_v)} (\omega_{\psi_v}(g_v)\phi_{1,v}, \phi_{2,v}) \cdot (\tau_v(g_v)\varphi_{1,v}, \varphi_{2,v}) dg_v.
\]

Our quantitative duality theory then provides a dual inner product formula concerning the lifting \( \Pi \rightarrow \tau \).

**Proposition 8:** Suppose that \( \tau \) is of trivial central character and \( \phi_i \in \omega_\psi, f_i \in \Pi = \tau^\vee \otimes \tau \) \((i = 1, 2)\) are decomposable. Then

\[
\left( \Theta(\phi_1, f_1), \Theta(\phi_2, f_2) \right) = \frac{L(0, \tau, \text{ad})}{2^{\beta - 1} \zeta_F(2)} \prod_v \lim_{s \rightarrow 0} \frac{\zeta_{F_v}(2) \int_{H(F_v)} (\omega_{\psi_v}(h_v)\phi_{1,v}, \phi_{2,v}) (\Pi(h_v)f_{1,v}, f_{2,v}) \Delta(h_v)^s dh_v}{L(s, \tau_v, \text{ad})}.
\]

Here \( \beta \) is the number of local places \( v \) such that \( \tau_v \) is square-integrable and \( H = SO(V) \).

**Proof.** The theorem is about global theta lifting between groups of similitudes, but it is easy to prove it using the duality of local inner product formulas for isometry groups. We proceed as follows.

1. (Local decomposition of global representations) Let \( \tau = \otimes_v \tau_v \) and \( \Pi = \otimes_v \Pi_v \) be the local decompositions of \( \tau \) and \( \Pi \). At each place \( v \), we further restrict the local representations to isometry groups and do complete decompositions, that is, \( \tau_v|_{SL_2,v} = \bigoplus \sigma_{v,i} \) and \( \Pi_v|_{SO(V_{F_v})} = \bigoplus \pi_{v,i} \). The decompositions are finite and multiplicity free, and the index \( i \) can be numbered such that \( \sigma_{v,i} \) and \( \pi_{v,i} \) are in local theta correspondence with respect to \( \psi_v \). Note that for almost all \( v \), the index set \( \{i\} \) is of size 1, that is, \( \tau_v \) is \( SL_2,v \)-irreducible and \( \Pi_v \) is \( SO(V_{F_v}) \)-irreducible.

2. The global theta lifting \( \omega_\psi \otimes \Pi \rightarrow \tau \) induces a global \( SL_2(\mathbb{A}_F) \times SO(V_{\mathbb{A}_F}) \)-invariant homomorphism \( \Theta : \omega_\psi \rightarrow \Pi \otimes \tau \) such that \( \Theta(\phi, f) = (\Theta(\phi), f)_{\Pi} \); such a \( \Theta \) admits a local decomposition \( \Theta = \otimes_v \theta_v \), where \( \theta_v = \bigoplus_i \theta_{v,i} \) with \( \theta_{v,i} : \omega_{\psi_v} \rightarrow \pi_{v,i} \otimes \sigma_{v,i} \) being local theta correspondences.
(3) The global inner product formula in the direction $GL_2 \to GSO(V)$ tells that one can choose $\theta_v$ such that
\[
(\theta_v(\phi_{1,v}, \varphi_{1,v}), \theta_v(\phi_{2,v}, \varphi_{2,v}))_{\Pi_v} = \frac{\zeta_{F_v}(2)^2}{L(1, \tau_v, \text{ad})} \int_{SL_2(F_v)} (\omega_{\psi_v}(g_v)\phi_{1,v}, \phi_{2,v}) \cdot (\tau_v(g_v)\varphi_{1,v}, \varphi_{2,v}) dg_v,
\]
and that $\Theta = C_0 \otimes_v \theta_v$ with
\[
|C_0|^2 = \frac{2L(1, \tau, \text{ad})}{\zeta_F(2)^2}.
\]

(4) Note that $d_{\sigma_v,i}(s) = d_{\tau_v}(s)$, $d_{\pi_v,i}(s) = d_{\Pi_v}(s)$, and that $d_{\sigma_v,i}, d_{\pi_v,i}$ are independent of $i$. Hence the local duality principle for $\theta_{v,i}$ immediately leads to
\[
(\theta_v(\phi_{1,v}, f_{1,v}), \theta_v(\phi_{2,v}, f_{2,v}))_{\tau_v} = \lim_{s \to 0} \frac{d_{\sigma_v,i, \pi_v,i} d_{\Pi_v}(s) \zeta_{F_v}(2)^2}{d_{\tau_v}(s) L(1, \tau_v, \text{ad})} \int_{SO(V_{F_v})} (\omega_{\psi_v}(h_v)\phi_{1,v}, \phi_{2,v}) (\Pi_v(h_v)f_{1,v}, f_{2,v}) dh_v.
\]

(5) Since $\tau$ is of trivial central character, Proposition 2 yields that
\[
\frac{d_{\Pi_v}(s)}{d_{\tau_v}(s)} = \frac{\gamma(s, \tau_v, \text{ad})}{2}
\]
for all places $v$; Theorem 2 tells that
\[
d_{\sigma_v,i, \Pi_v,i} = \begin{cases} 2, & \tau_v \text{ is a unitary principal series}, \\ 1, & \tau_v \text{ is square-integrable}. \end{cases}
\]
It follows that
\[
(\theta_v(\phi_{1,v}, f_{1,v}), \theta_v(\phi_{2,v}, f_{2,v}))_{\tau_v} = \lim_{s \to 0} \frac{\gamma(s, \tau_v, \text{ad}) \zeta_{F_v}(2)^2}{2^s L(1, \tau_v, \text{ad})} \int_{SO(V_{F_v})} (\omega_{\psi_v}(h_v)\phi_{1,v}, \phi_{2,v}) (\Pi_v(h_v)f_{1,v}, f_{2,v}) dh_v
\]
\[
= \lim_{s \to 0} \frac{\epsilon(s, \tau_v, \text{ad}) \zeta_{F_v}(2)^2}{2^s L(s, \tau_v, \text{ad})} \int_{SO(V_{F_v})} (\omega_{\psi_v}(h_v)\phi_{1,v}, \phi_{2,v}) (\Pi_v(h_v)f_{1,v}, f_{2,v}) dh_v,
\]
where $t_v = 1$ when $\tau_v$ is a square-integrable representation and 0 when $\tau_v$ is a direct summand of a unitary principal series.
(6) We now combine the local inner products and eventually get
\[
\left( \Theta(\phi_1, f_1), \Theta(\phi_2, f_2) \right) = |C_0|^2 \prod_v (\theta_v(\phi_{1,v}, f_{1,v}), \theta_v(\phi_{2,v}, f_{2,v})) \Pi_v
\]
\[
= \frac{2L(1, \tau, \text{ad})}{2^\beta \zeta_F(2)^2} \frac{\zeta_{F_v}(2)^2}{L(s, \pi_v, \text{ad})} \int_{H_1(F_v)} (\omega_{\psi_v}(h_v)^{\phi_{1,v}}, \phi_{2,v})(\Pi(h_v) f_{1,v}, f_{2,v}) \Delta(h_v)^s dh_v
\]
\[
\times \prod_v \lim_{s \to 0} \frac{\zeta_{F_v}(2)^2}{L(s, \pi_v, \text{ad})} \int_{H_1(F_v)} (\omega_{\psi_v}(h_v)^{\phi_{1,v}}, \phi_{2,v})(\Pi(h_v) f_{1,v}, f_{2,v}) \Delta(h_v)^s dh_v
\]
\[
= \frac{L(0, \tau, \text{ad})}{2^{\beta-1} \zeta_F(2)} \prod_v \lim_{s \to 0} \frac{\zeta_{F_v}(2)^2}{L(s, \pi_v, \text{ad})} \int_{H_1(F_v)} (\omega_{\psi_v}(h_v)^{\phi_{1,v}}, \phi_{2,v})(\Pi(h_v) f_{1,v}, f_{2,v}) \Delta(h_v)^s dh_v
\]
\[
\times \prod_v \lim_{s \to 0} \frac{\zeta_{F_v}(2)^2}{L(s, \pi_v, \text{ad})} \int_{H_1(F_v)} (\omega_{\psi_v}(h_v)^{\phi_{1,v}}, \phi_{2,v})(\Pi(h_v) f_{1,v}, f_{2,v}) \Delta(h_v)^s dh_v.
\]

References


