COMPLEX VARIABLES: HOMEWORK 1

Notations: A complex number $z$ is written as $x + yi$ where $x, y$ are real numbers. Its real and imaginary parts are denoted by $\text{Re}(z)$ and $\text{Im}(z)$ respectively.

(1) (5 points) Find the real and imaginary parts of the $\frac{1 - i}{2 + 3i}$.

Solution. Multiply and divide by the conjugate of the denominator.

$$\frac{1 - i}{2 + 3i} = \frac{1 - i}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i} = \frac{-1 - 5i}{13}$$

Hence the real part is $\frac{-1}{13}$ and the imaginary part is $\frac{-5}{13}$.

(2) (5 points) Write $z = -1 + i$ in its polar form. Use this to compute $z^{20}$.

Solution. $|z| = \sqrt{2}$ and $\arg(z) = \frac{3\pi}{4}$. Thus

$$z = \sqrt{2} \left( \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) \right)$$

So, $|z^{20}| = (\sqrt{2})^{20} = 2^{10} = 1024$. And $\arg(z^{20}) = 20 \cdot \frac{3\pi}{4} = 15\pi$. Hence we get

$$z^{20} = 1024 (\cos(15\pi) + \sin(15\pi)i) = -1024$$

(3) (10 points) (2+3+5) Recall that the conjugate of $z = x + yi$ is defined as $\overline{z} = x - yi$. Prove the following, for any two complex number $z_1$ and $z_2$.

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Proof. Let us write $z_1 = a + bi$ and $z_2 = c + di$. Then

$$\overline{z_1 + z_2} = (a + c) - (b + d)i = (a - bi) + (c - di) = \overline{z_1} + \overline{z_2}$$

as required.

(b) $\overline{z_1z_2} = \overline{z_1} \overline{z_2}$

Proof. Again we can compute the left–hand side as

$$\overline{z_1z_2} = \overline{ac - bd + (ad + bc)i} = (ac - bd) - (ad + bc)i = (a - bi)(c - di) = \overline{z_1} \overline{z_2}$$

(c) $|z_1 + z_2|^2 = |z_1|^2 + 2 \text{Re}(z_1\overline{z_2}) + |z_2|^2$.

Proof. Let us start by writing $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$. (This is because, for any complex number $w$, we know $|w|^2 = w\overline{w}$). Hence the left–hand side can be written as

L.H.S. = $z_1\overline{z_1} + z_2\overline{z_2} + (z_1\overline{z_2} + \overline{z_1}z_2)$

= $|z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1}z_2)$

= $|z_1|^2 + |z_2|^2 + 2 \text{Re}(z_1\overline{z_2})$

(since for any complex number $w$, we know that $w + \overline{w} = 2 \text{Re}(w)$).
(4) **(15 points)** Let \( z_1, \ldots, z_n \) be \( n \) distinct solutions of the equation \( z^n = 1 \).

This means that
\[
(*) \quad z^n - 1 = (z - z_1)(z - z_2) \cdots (z - z_n)
\]

(a) **Product of** \( z_1, \ldots, z_n \) **is** \((-1)^{n-1}\).

**Proof.** Take the constant term on both sides of equation \((*)\). The left–hand side gives \(-1\) and the right–hand side gives \((-1)^n\) times product of all \( z_1, \ldots, z_n \). Hence, we get
\[
(-1)^n(z_1z_2 \cdots z_n) = -1 \Rightarrow (z_1z_2 \cdots z_n) = (-1)^{n-1}
\]

(b) **Sum of** \( z_1, \ldots, z_n \) **is** 0.

**Proof.** Take the coefficient of \( z^1 \) on both sides of equation \((*)\). The left–hand side gives 0 (since \( n \geq 2 \)) and the right–hand side gives the sum of all \( z_1, \ldots, z_n \).

(c) If \( z_1 = 1 \), then \((1 - z_2)(1 - z_3) \cdots (1 - z_n) = n\).

**Proof.** Let us divide both sides of equation \((*)\) by \( z - 1 \):
\[
\frac{z^n - 1}{z - 1} = (z - z_2)(z - z_3) \cdots (z - z_n)
\]

Now use the following identity
\[
\frac{z^n - 1}{z - 1} = z^{n-1} + z^{n-2} + \cdots + z + 1
\]
Then we get
\[
(z - z_2)(z - z_3) \cdots (z - z_n) = z^{n-1} + z^{n-2} + \cdots + z + 1
\]
Set \( z = 1 \) in this equation, to get
\[
(1 - z_2)(1 - z_3) \cdots (1 - z_n) = n
\]

(5) **(5 points)** Let \( f : D \to \mathbb{C} \) be a function, where \( D \subset \mathbb{C} \) is an open set. Let \( u(x, y) \) and \( v(x, y) \) be its real and imaginary parts respectively:
\[
f(x + yi) = u(x, y) + v(x, y)i
\]
(Assume that partial derivatives of \( u(x, y) \) and \( v(x, y) \) exist and are continuous). Prove that, if \( f \) is \( \mathbb{C} \)–differentiable, then \( u_{xx} + u_{yy} = 0 \). Use this to show that there is no \( \mathbb{C} \)–differentiable function whose real part is \( e^x \).

**Proof.** By Cauchy–Riemann equations, we have \( u_x = v_y \) and \( u_y = -v_x \). Hence
\[
u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0
\]
Since for \( u(x, y) = e^x \), we get \( u_{xx} + u_{yy} = e^x + 0 \neq 0 \), there cannot be any \( \mathbb{C} \)–differentiable function \( f \) whose real part is \( e^x \).

(6) **(10 points)** Prove that for any two complex numbers \( z_1, z_2 \) the following inequality holds
\[
||z_1| - |z_2|| \leq |z_1 - z_2|
\]
Prove that this inequality is an equality if, and only if \( \arg(z_1) = \arg(z_2) \) (or one of them is zero).

**Proof.** Use (c) of Problem (3) above to see that
\[
|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \text{Re}(z_1 \overline{z_2})
\]
Now $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \arg(z_1) - \arg(z_2)$ (if neither of them is zero). Note that if one of $z_1$ or $z_2$ is zero, then the statement to prove is obviously true. Moreover $|z_1 z_2| = |z_1||z_2| = |z_1| |z_2|$. Hence we get

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \Re(z_1 \overline{z_2})$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\arg(z_1) - \arg(z_2))$$

$$\geq |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$= (|z_1| - |z_2|)^2$$

Thus taking square-root gives $|z_1 - z_2| \geq ||z_1| - |z_2||$. Also, the inequality in the third line above is equality, if and only if $\cos(\arg(z_1) - \arg(z_2)) = 1$, that is, $\arg(z_1) = \arg(z_2)$. 