(4.0) Recall: we defined \( \int f(z) \, dz \), for a \( C \)-differentiable function \( f: D \to \mathbb{C} \), where \( D \subset \mathbb{C} \) is an open set and \( \gamma: [a, b] \to D \) a path, as follows.

- \( \int f(z) \, dz = \int_a^b f(\gamma(t)) \gamma'(t) \, dt \) if \( \gamma \) is smooth.
- If \( \gamma \) is piecewise smooth, we take sum over the smooth parts of \( \gamma \).

**Cauchy's Theorem I.** Let \( R \subset D \) be a rectangle. Then

\[
\oint_{\partial R} f(z) \, dz = 0 \quad (\text{boundary of } R)
\]

In this lecture, we generalize this theorem to other closed curves \( \gamma \).

(4.1) **Definition.** A subset \( U \subset \mathbb{C} \) is said to be (path-)connected if for any two points \( p, q \in U \), there exists a path \( \gamma: [a, b] \to U \) such that \( \gamma(a) = p \) and \( \gamma(b) = q \).

**Example:** \( \mathbb{C} \setminus \{0\} \) is connected. \( \mathbb{C} \setminus R \) is not connected.

\( R \subset \mathbb{C} \) is connected.
Fact: If $U \subset \mathbb{C}$ is open and connected, then any two points of $U$ can be connected by a "zig-zag path". That is, a path that consists entirely of horizontal or vertical line segments. (A proof of this fact is given at the end - optional reading).

Example of a zig-zag path:

(4.2) Morera's Theorem. Let $U \subset \mathbb{C}$ be an open and connected subset. Let $f: U \rightarrow \mathbb{C}$ be a continuous function such that for any rectangular closed path $\gamma: [a,b] \rightarrow U$, we have

$$\oint_{\gamma} f(z) \, dz = 0 \quad \text{in } U$$

Then there exists a $C^1$-differentiable function $F(z), F: U \rightarrow \mathbb{C}$ such that $F'(z) = f(z)$. (That is, $f(z)$ admits a $C^1$-differentiable antiderivative).
Proof of Morera’s Theorem.

Fix a point \( p \in U \) and for any \( w \in U \), define

\[
F(w) = \int_{\gamma} f(z) \, dz \quad \text{where } \gamma \text{ is a zig-zag path joining } p \text{ and } w.
\]

Since \( \int f(z) \, dz = 0 \) over any closed rectangular path, the definition of \( F(w) \) does not depend on the choice of \( \gamma \).

Ex.: Convince yourself of this statement.

\[
\begin{align*}
\int_{\gamma_1} f(z) \, dz - \int_{\gamma_2} f(z) \, dz &= \int_{\gamma_1} f(z) \, dz - \int_{\gamma_2} f(z) \, dz \\
&= \int f(z) \, dz = 0
\end{align*}
\]

(\( \gamma_1, \gamma_2 \) are two zig-zag paths joining \( p \) and \( q \))

Take the one when dotted lines are still in \( U \).
Let us prove that $F$ is $C$-differentiable, and $F' = f$.

By Cauchy-Riemann equations, we need to check that
\[
\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} = f \quad \text{(see Theorem (1.4))}
\]

Let us compute $\frac{\partial F}{\partial x}(w) = \lim_{h \to 0} \frac{1}{h} \left( F(w+h) - F(w) \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \int_{\gamma} f(z) \, dz
\]

where $\gamma : [0,1] \to U$

$\gamma(t) = w + th$ is straight line joining $w$ and $w+h$

\[
= \lim_{h \to 0} \frac{1}{h} \int_{0}^{1} f(w+th) \, dh \quad \text{(by definition)}
\]

\[
= \lim_{h \to 0} \int_{0}^{1} f(w+th) \, dh
\]

Claim. $\lim_{h \to 0} \int f(w+th) \, dt = f(w)$

Proof. We need to prove that given $\varepsilon > 0$, we can find $\delta > 0$ such that for $|h| < \delta$ we have

\[
|\int_{0}^{1} f(w+th) \, dt - f(w)| < \varepsilon
\]
Since $f$ is continuous, pick $\delta > 0$ such that $|k - w| < \delta$ implies $|f(w + k) - f(w)| < \epsilon$. Now for this $\delta$ we have, if $|h| < \delta$, then for every $t \in (0, 1)$, $|th| < \delta$ and hence $\left| \int_0^1 f(w + th) \, dt - f(w) \right|$

$$= \left| \int_0^1 (f(w + th) - f(w)) \, dt \right| < \epsilon \quad \text{(as claimed)}$$

Hence $\frac{\partial F}{\partial x}(w) = f(w)$. Similar computation shows that $\frac{\partial F}{\partial y}(w) = i f(w)$. Hence $\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y} = f$ as desired.

(4.3) Cauchy's Theorem (general version).

Let $D \subseteq \mathbb{C}$ be an open set and $f : D \rightarrow \mathbb{C}$ a $C$-differentiable function. Let $\gamma : [a, b] \rightarrow D$ be a simple closed path in $D$ whose interior lies in $D$. Then

$$\int_{\gamma} f(z) \, dz = 0$$
Remark 1. Every simple closed curve breaks the complex plane into two connected pieces: interior and exterior.

* We assume (always) that \( \gamma \) is oriented counterclockwise.

That is, interior is on the left, while travelling along \( \gamma \).

Remark 2. We have already seen an example when interior of a simple closed curve is not within the domain of the function and the conclusion \( \int_{\gamma} f(z) \, dz = 0 \) is false.

Namely, \( D = \mathbb{C} \setminus \{0\} \), \( f(z) = \frac{1}{z} \) and \( \gamma = r(\cos t + i \sin t) \), a circle of radius \( r \).

\( 0 \in \text{Interior of } \gamma \) but \( 0 \not\in D \).

Proof of Cauchy's Theorem.

Step 1. Replace \( D \) by a smaller open set \( U \) containing \( \gamma \) such that

(\( \star \)) for any simple closed curve \( \mu \) inside \( U \), interior of \( \mu \) is contained in \( U \).
Chain of implications:

1. $f$ is $C$-differentiable
2. Cauchy's Theorem I
   $\int f(z) \, dz = 0$ for any rectangle $R \subseteq D$
3. By property of $f$
4. $f$ admits an antiderivative on $U$
5. Morera's Theorem
   $\int f(z) \, dz = 0$ for any rectangular path $\gamma$ within $U \subseteq D$

\[ \int f(z) \, dz = 0 \]