Uniformly Distributed Dice Sums

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We will show that for m dice labeled from 1 to n, no assignment of probabilities to the faces (i.e. biasing) is possible that will leave the sum of the dice uniformly distributed over $\{m, m+1, \ldots, mn\}$.

1 The Two Dice Case

Let the probabilities assigned to the different faces of the two dice be represented by two n-component vectors p and q:

$$p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \qquad \text{and} \qquad q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

Consider the matrix P:

$$P = pq^T = \begin{pmatrix} p_1q_1 & \cdots & p_1q_n \\ \vdots & \ddots & \vdots \\ p_nq_1 & \cdots & p_nq_n \end{pmatrix}$$

Elements on the same north-east diagonal represent the probability of the getting the same sum, in different ways. The sums of elements on north-east diagonals are therefore the probabilities of getting a corresponding sum with the dice.

As the rows of pq^T are all linear combinations of the vector q, the rank of P is 1. We will now show that the condition of the sum being uniformly distributed, i.e. every diagonal having the same sum of $1/(2n-1)$, cannot give a matrix of rank 1.

If P had this property and was of rank 1, then the first row would be a multiple of the last, with a multiplicative factor $c \neq 0$ (note that no row can be the zero row). Then the following would be true:

$$p_1q_1 = cp_nq_1 \qquad \text{and} \qquad p_1q_n = cp_nq_n$$

As p_1q_n and p_nq_1 lie on the same diagonal, the uniform distribution condition gives

$$p_1q_n + p_nq_1 \leq \frac{1}{2n-1}$$

Also, $p_1q_1 = p_nq_n = 1/(2n-1)$, as they are the only elements along their respective diagonals.

Hence, after substituting we get

$$cp_nq_n + \frac{p_1q_1}{c} \leq \frac{1}{2n-1} \implies c + \frac{1}{c} \leq 1$$

which contradicts the AM-GM inequality.

Hence it is not possible for two n-dice to have their sum uniformly distributed.

2 The m Dice Case

Assume that it is possible for m dice with faces from 1 to n to be biased such that their sum is uniformly distributed from m to nm. Then let the vectors p (probabilities of last dice) and q (probabilities of sum of first $m-1$ dice) be defined as:

$$p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \qquad \text{and} \qquad q = \begin{pmatrix} q_{m-1} \\ q_m \\ \vdots \\ q_{n(m-1)} \end{pmatrix}$$

Similarly define p' to be the probabilities of the $(m-1)^{\text{st}}$ die and q' to be the probabilities sum of the first $(m-2)$ dice.

Then the following relation is true:

$$q_i = \sum_{j=1}^{i-(m-2)} p'_j q'_{i-j} \tag{1}$$

for $i = m-1, \ldots, n+m-2$.

Now if we look at the matrix pq^T as in section 1, we see that $p_n q_{m-1}$ and $p_1 q_{n(m-1)}$ do not lie on the same diagonal. Hence instead of bounding $c+1/c$ by 1, we can only manage a bound of 2 (by bounding each term by $1/(nm-m+1)$. But then the AM-GM inequality gives that $c=1$. Hence

$$p_n q_{m-1} = p_1 q_{n(m-1)} = \frac{1}{nm-m+1}$$

This implies that all other values on the diagonal corresponding to the sum $n+m-1$ are zero. In particular, $p_1 q_{n+m-2}$ is zero. Since $p_1 q_{m-1} \neq 0 \implies p_1 \neq 0$, we get $q_{n+m-2} = 0$.

Substituting $i = n+m-2$ in (1),

$$q_{n+m-2} = \sum_{j=1}^{n} p'_j q'_{n+m-2-j} = 0$$

This implies that for each $j = 1, 2, \ldots, n$, at least one of p'_j and $q'_{n+m-2-j}$ is zero.

Taking $j = n$ in particular, at least one of p'_n and q'_{m-2} is zero. This is a contradiction as if p'_n is zero, it is not possible for the sum of all m dice to be mn, and if q'_{m-2} is zero, it is not possible for the sum of all m dice to be m (as q' is the probabilities of sums of $m-2$ dice).

Hence the sum of m dice cannot be uniformly distributed, no matter the biasing.