For the characteristic of functions in $\mathfrak{N}$, we remark that for $x \le 0$ we have $\varphi(x; \lambda^2) = \varphi(0; \lambda^2) \cos \lambda x + \varphi'(0; \lambda^2) \sin \lambda x/\lambda$, and also that for a reduced string S we have by [6] $\varphi(0; \lambda^2) = (Q(\lambda) + Q(-\lambda))/2$, $\varphi'(0; \lambda^2)/\lambda = (Q(\lambda) - Q(-\lambda))/2i$. Therefore, for an arbitrary $F \in \mathfrak{N}$ $(F = \mathscr{F}f, f \in \mathfrak{M})$, Eq. (3) takes the form (1) if we put $h(x) = (f_1(-x) - if_2(-x))/2$ for $x > 0$ and $h(x) = (f_1(x) + f_2(x))/2$ for $x \le 0$, so that $h(x) \in L^2(-\infty, \infty)$. Conversely, for an arbitrary $h(x) \in L^2(-\infty, \infty)$, it is easily verified that the function (1) is the Fourier transform of some $f \in \mathfrak{M}$.

Since for a reduced string S, the vector functions $\Phi_j(x) = (\varphi(x; z_j^2), \varphi'(x; z_j^2)/z_j)$ $(s_j \in Z)$ form a system which is complete in $\mathfrak{H}_S$ (cf. [6]), $\mathfrak{L} = \mathscr{F}\mathfrak{H}_S$ coincides with the closure of the linear span of the functions $Q_j(\lambda) = \mathscr{F}\Phi_j(x)$. A simple calculation shows that up to a constant factor, $Q_j(\lambda)$ coincides with $Q(\lambda)/(\lambda - z_j)$.

We have thereby obtained a new justification and, at the same time, generalization (for the case of an even weight) of the result of Gurarii.

Akhiezer and Gurarii have also obtained a generalization of the Paley–Wiener theorem for the transforms which they considered – for an entire function $G \in \mathfrak{E}_\Omega$ of degree $\sigma(P) + \alpha$, the projection F of G onto $\mathfrak{N}$ has a representation (1) in which $h(x) = 0$ for $|x| > \alpha$. This result can be extended in the case when the string S is regular, i.e., $\mathscr{L} < \infty$. In this case, $Q(\lambda)$ turns out to be a function of finite degree $(\sigma = \sigma(Q) < \infty)$ and thus satisfies the conditions of Gurarii. A representation can now be obtained for $G \in \mathfrak{E}_\Omega$ also when $0 < \sigma(G) < \sigma(P)$. Under this condition, $G \in \mathfrak{L}$ and G can be recovered using a suitable generalization of the Paley–Wiener theorem in the theory of spectral functions of a regular string [9] (cf. also [10]).

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RATIONAL SOLUTIONS OF THE DUALITY EQUATIONS FOR YANG–MILLS FIELDS

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In this note we propose a construction for a large class of rational solutions of the duality equations [1]. These equations are equivalent to the condition that the operators (cf. [2])

$$L_1 = \lambda\partial_1 + \partial_2 + \lambda B_1 + B_2, \qquad L_2 = \lambda\bar{\partial}_2 - \bar{\partial}_1 + \lambda B_3 - B_4$$

commute, where λ is the spectral parameter, $B_i = B_i(z_1, z_2, \bar{z}_1, \bar{z}_2)$ the connection coefficients in the coordinates $z_1 = \frac{1}{2}(x_2 + ix_1)$; $z_2 = \frac{1}{2}(x_4 + ix_3)$; $\bar{z}_2$; $\bar{z}_1$ respectively; $(x_1, \ldots, x_4) \in \mathbf{R}^4$. For SU(n)-connections, $B_3 = -B_2^+$, $B_4 = -B_1^+$. We restrict ourselves below to the case $n = 2$.

I. Consider the linear space L^{2N+2} of pairs of functions $(f_1(\lambda), f_2(\lambda))$ rational in having poles at $\lambda_1, \ldots, \lambda_N$. Assume given functions $a_i(z, \bar{z}, \lambda) = a_{i0} + a_{i1}(\lambda - \lambda_i)$ and $b_i(z, \bar{z}, \lambda) = b_{i0} + b_{i1}(\lambda - \lambda_i)$ such that

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$$(\lambda\partial_1 + \partial_2)a_i = (\lambda\bar\partial_2 - \bar\partial_1)a_i = O(\lambda - \lambda_i)^2,$$
$$(\lambda\partial_1 + \partial_2)b_i = (\lambda\bar\partial_2 - \bar\partial_1)b_i = O(\lambda - \lambda_i)^2, \quad z = (z_1, z_2). \tag{1}$$

The system of 2N linear equations depending on z, $\bar z$ as parameters is equivalent to the conditions

$$a_i(z, \bar z, \lambda)f_1(z, \bar z, \lambda) - b_i(z, \bar z, \lambda)f_2(z, \bar z, \lambda) = 0\,|_{\lambda=\lambda_i}, \tag{2}$$

and defines a two-dimensional subbundle $\mathscr{L}$ in $\mathbf{R}^4 \times L^{2N+2}$.

We write the vector (f_1, f_2) in the form

$$(f_1, f_2) = \sum_{i=1}^{N} (b_{i0}, a_{i0}) \frac{\varphi_i}{\lambda - \lambda_i} + (\varphi_{N+1}, \varphi_{N+2}),$$

so that for each i, one of the equations in (2) holds. The remaining equations for the φ_i take the form

$$\sum_{j \neq i} \frac{a_{i0}b_{j0} - b_{i0}a_{j0}}{\lambda_i - \lambda_j} \varphi_j + d_i\varphi_i = b_{i0}\varphi_{N+2} - a_{i0}\varphi_{N+1}, \quad d_i = b_{i0}a_{i1} - a_{i0}b_{i1}, \quad i = 1, \ldots, N. \tag{3}$$

We denote by $\Psi(z, \bar z, \lambda)$ the (2×2) matrix with rows formed by the basis solutions of Eqs. (2). Ψ is defined uniquely up to multiplication by a nonsingular matrix $g(z, \bar z)$ not depending on λ.

THEOREM 1. Let $R_1 = \Psi(z, \bar z, \infty)$, $R_2 = \Psi(z, \bar z, 0)$ and $B_1 = -\partial_1 R_1 \cdot R_1^{-1}$, $B_2 = -\partial_2 R_2 \cdot R_2^{-1}$, $B_3 = -\bar\partial_2 R_1 \cdot R_1^{-1}$, $B_4 = -\bar\partial_1 R_2 \cdot R_2^{-1}$. Then Ψ satisfies the equations $L_1\Psi(z, \bar z, \lambda) = 0$, $L_2\Psi(z, \bar z, \lambda) = 0$.

COROLLARY. The operators L_1 and L_2 commute. Their coefficients B_i define a self-dual connection.

Remark 1. Changing the basis sections, i.e., the replacement $\Psi'(z, \bar z, \lambda) = g(z, \bar z)\Psi(z, \bar z, \lambda)$, is equivalent to a gauge transformation of the fields.

II. A very simple set of functions a_i, b_i satisfying Eqs. (1) is given by

$$(a_{i0}, b_{i0}) = (z_1 - \lambda_i z_2 + \alpha_i, \bar z_2 + \lambda_i \bar z_1 + \beta_i) A_i,$$
$$(a_{i1}, b_{i1}) = \left(-z_2 + \frac{\alpha_i\bar\lambda_i - \bar\beta_i}{1 + |\lambda_i|^2}, \bar z_1 + \frac{\bar\alpha_i + \bar\lambda_i\beta_i}{1 + |\lambda_i|^2}\right) A_i + \left(\frac{c_i}{b_{i0}}, 0\right).$$

Here α_i, β_i, c_i are constants, $A_i \in SL(2, \mathbf{C})$ a constant matrix. (These equalities mean that the row vectors are equal.)

In this case, the matrix elements of $\Psi(z, \bar z, \lambda)$ are rational functions of degree 2N in the variables z_1, z_2, $\bar z_1$, $\bar z_2$. The fields B_i are also rational functions of z, $\bar z$.

Example. Let $N = 1$, $A_1 = \hat 1$, $\lambda_1 = 0$. We then obtain a one-instanton solution of the duality equations, for which α, β define the center, and the absolute value of the real negative constant c in the dimension of the instanton.

Remark 2. It can be shown that for points in general position, the above construction coincides with the "multiplication" procedure (cf. [2]) and therefore contains at a minimum the (5N + 6)-parameter family of N-instanton solutions obtained in [3].

Remark 3. An algebrogeometric construction of all instanton solutions was obtained in [4]. However, so far it has not given an explicit parametrization of the manifold of "moduli" of N-instanton fields. The possibility of obtaining all N-instanton fields, i.e., smooth, asymptotically longitudinal fields in the framework of the above construction and its generalizations (one of which will be given below) remains unanswered. In order to obtain smooth fields on $\mathbf{R}^4$, it is necessary and sufficient that the matrix in the left-hand side of Eq. (3) be nonsingular for all z, $\bar z$. In this case, the determinant of $\Psi(z, \bar z, \lambda)$ (which does not depend on λ, as follows from the definition of Ψ) is nonzero. The last condition determines a submanifold in the space of parameters λ_i, α_i, β_i, c_i, A_i, $i = 1, \ldots, N$, the dimension of which is unknown.

Remark 4. In order to obtain SU(2)-connections, the dimensionality of the problem can be doubled by adding the points $\bar\lambda_{i+N} = -1/\bar\lambda_i$, for which we put $a_{i+N}(\lambda) = \bar b_i(-1/\bar\lambda)$, $b_{i+N}(\lambda) = -\bar a_i(-1/\bar\lambda)$. In this case a_{i+N} and b_{i+N} automatically satisfy Eqs. (1).

III. The construction proposed admits a natural generalization which permits construction of "eigenfunctions" of the operators L_1 and L_2 which are not rational in λ as before, but are rather defined on hyperelliptic curves.

The simplest generalization of a one-instanton solution (cf. Example 1) can be obtained as follows. Consider the hyperelliptic curve $\mathfrak{R}$, defined by $w^2 = \prod_{j=1}^{2N+2} (\lambda - \mu_j)$. It is a twofold covering of the λ plane. The dimension of the space L of rational functions on $\mathfrak{R}$, having poles of multiplicity N + 1 at two points $P^{\pm}$ on different sheets over a fixed point λ_0 is equal to 2N + 2 − N + 1 = N + 3 by the Riemann–Roch theorem.

Let $a(z, \bar{z}, \lambda) = \sum_{s=0} a_s(z, \bar{z})(\lambda - \lambda_0)^s$, $b(z, \bar{z}, \lambda) = \sum_{s=0}^{N} b_s(z, \bar{z})(\lambda - \lambda_0)^s$ satisfy Eqs. (1) in which the right-hand side $O(\lambda - \lambda_0)^2$ is replaced by $O(\lambda - \lambda_0)^{N+1}$. The solutions of the equations

$$a(z, \bar{z}, \lambda)\psi(z, \bar{z}, \lambda^+) - b(z, \bar{z}, \lambda)\psi(z, \bar{z}, \lambda^-) = O(1)|_{\lambda=\lambda_0},$$

where $\lambda^{\pm}$ are the inverse images on different sheets of λ, $\psi(z, \bar{z}, P) \in L$, $P \in \mathfrak{R}$, form a two-dimensional subbundle $\mathscr{L}$ in $\mathbf{R}^4 \times$ L. We denote by $\psi_1(z, \bar{z}, P)$ and $\psi_2(z, \bar{z}, P)$ basis sections of $\mathscr{L}$.

THEOREM 2. Let the $B_i(z, \bar{z})$ be given by the formulas in Theorem 1, where

$$R_1 = \begin{pmatrix} \psi_1(z, \bar{z}, P_\infty^+) & \psi_1(z, \bar{z}, P_\infty^-) \\ \psi_2(z, \bar{z}, P_\infty^+) & \psi_2(z, \bar{z}, P_\infty^-) \end{pmatrix}, \quad R_2 = \begin{pmatrix} \psi_1(z, \bar{z}, P_0^+) & \psi_1(z, \bar{z}, P_0^-) \\ \psi_2(z, \bar{z}, P_0^+) & \psi_2(z, \bar{z}, P_0^-) \end{pmatrix}.$$

Then the fields $B_i(z, \bar{z})$ satisfy the duality equations.

Here, $P_\infty^{\pm}$, $P_0^{\pm}$ are the inverse images on $\mathfrak{R}$ of the points $\lambda = \infty$ and $\lambda = 0$, respectively.

Remark 5. If the functions $a_s(z, \bar{z})$ and $b_s(z, \bar{z})$ are chosen to be rational, the fields B_i will also be rational in the variables z, $\bar{z}$.

Remark 6. In [5], the inverse problem method was used to construct finite-zone solutions of the duality equations. In this case, the eigenfunctions of the operators L_1 and L_2 are defined on a hyperelliptic curve, as in this section, but in contrast to our construction, they have essential singularities on the curve.

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CLASSIFICATION OF VON NEUMANN J-ALGEBRAS

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In various works on quantum field theory (cf. [1, 2]), the problem is discussed of the utility of introducing an indefinite metric (J-metric) into an initial Hilbert space $\hat{\mathfrak{H}}$ and considering, in addition to the usual duality operation, duality with respect to the J-metric. The concept of a von Neumann J-algebra is therefore natural, i.e., a von Neumann algebra of operators on the space $\hat{\mathfrak{H}}$ (cf. [3, 4]) with an additional involution induced by the J-metric.

The purpose of this note is to describe von Neumann J-algebras.

Let $\hat{\mathfrak{H}}$ be a Hilbert space with a J-metric (cf. [5, 6]), $\hat{\mathfrak{H}} = \mathfrak{H}_+ \dotplus \mathfrak{H}_-$, $\dim \mathfrak{H}_+ = \dim \mathfrak{H}_-$, $\mathfrak{H}$ a Hilbert space isomorphic to $\mathfrak{H}_+$ and $\mathfrak{H}_-$, and $J_{\pm}$, isometries of $\mathfrak{H}_{\pm}$ onto $\mathfrak{H}$. The J-metric in $\hat{\mathfrak{H}}$ is defined with the aid of an operator $\hat{\mathfrak{J}}$ given in matrix form by $\hat{\jmath} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, where I is the identity operator in $\mathfrak{H}$. Namely, $[\hat{x}, \hat{y}] = (\hat{J}\hat{x}, \hat{y})$. The operator $\hat{A}^0$ defined by $[\hat{A}\hat{x}, \hat{y}] = [\hat{x}, \hat{A}^0\hat{y}]$, $\hat{x}, \hat{y} \in \hat{\mathfrak{H}}$ (cf. [5, 6]) is called the J-dual of $\hat{A}$. An operator

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