Physica 3D (1981) 1 & 2, 267-293

HOLOMORPHIC BUNDLES AND NONLINEAR EQUATIONS.

I.M. Krichever
S.P. Novikov
Landau Institute of the Theoretical Physics
the USSR Academi of Sinces

I. In the theory of nonlinear equations of Korteveg -de Vries type, which can be represented in the Lax form

$$-\frac{\partial L}{\partial t} = [L, A] , \tag{1}$$

where

$$L = \sum_{i=0}^{n} u_i(x,t) \frac{\partial^i}{\partial x^i} \quad ; \quad A = \sum_{i=0}^{m} v_i(x,t) \frac{\partial^i}{\partial x^i}$$

the most interesting families of exact solutions (multisoliton and finitegap) are specified by the following conditions: there exists operator B commuting with L at the time t=o

$$[L, B] = 0 = \left[\sum_{i=0}^{n} u_i(x,0) \frac{\partial^i}{\partial x^i} , \sum_{i=0}^{N} w_i(x) \frac{\partial^i}{\partial x^i} \right] . \tag{2}$$

If it is so, then such operator B(t) exists at all times t.

In case of "rank 1" (see below), if, for example, the degrees of operators L and B are co-prime (and in case of matrix coefficients, if eigenvalues of higher coefficients of operators L and B are different), the "typical" solutions of equations (1), which satisfy the restriction (2), are periodic and quasiperiodic functions of x and t. They can be represented by the Θ-functions of Riemann surfaces. Periodic operator L has some remarkable spectral properties - its Bloch spectrum is finitegap .

Rapidly decreasing multisolition solutions (corresponding to reflectless potentials) and rational solutions of equations (1) are obtained from periodic solutions by means of different limiting processes (see surveys [1], [2], book [3]).

Let's recall the lemma by Burchnal and Chaundy ([4],). For any pair of commuting ordinary differential operators (2) there exists algebraic relation

$$R(\lambda, \mu) = 0, \tag{3}$$

where R(z,w) - is a polynominal with constant coefficients. For the common eigenfunctions of the operators L and B

$$L\psi = \lambda\psi \ ; \ B\psi = \mu\psi \quad ; \quad \psi = \psi(x, \lambda, \mu) \tag{4}$$

the relation (3) is valid,

$$R(\lambda, \mu) = 0. \tag{3}$$

This relation determines the algebraic curve Γ. The pair (λ, μ), satisfying (3) is point P of Γ. Common eigenfunction $\psi(x, P)$ is defined on the surface Γ.

Definition. The multiplicity of the pair (λ, μ) eigen values of the operators L and B (i.e. l is the dimension of the space of solutions ψ of (4) for a fixed point $P \in \Gamma$) is called the "rank" of the commuting pair of the operators L and B.

The common eigenfunctions ψ of L and B determine the l-dimensional holomorphic bundle over the surface Γ.

All the results concerning the equations (2) and exact solutions of the KdV-type equations, obtained before 1978, refer to the case of rank 1.

It is noteworthy, that in the theory of "one-dimensional" systems of KdV type (1), condition (2) includes the operator L from the Lax-pair.

For some physicaly important "two-dimensional" systems of the KdV type the analog of the algebraic representation (1) was found in the papers [5], [6]. In this representation operator L has the form of:

$$L = \frac{\partial}{\partial y} - M ;$$

$$\frac{\partial L}{\partial t} = [A, L] \longleftrightarrow \left[\frac{\partial}{\partial y} - M, \frac{\partial}{\partial t} - A\right] = 0 \tag{5}$$

where M and A are ordinary linear differential operators related to x, with coefficients depending on (x,y,t).

To obtain the exact solutions of the "two-dimensional" systems (5) the authors have introduced the following constraints including the auxiliary pair of the operators L_1 and L_2:

$$[L, L_i] = 0; \quad i = 1,2 \quad ; \left[\frac{\partial}{\partial t} - A, L_i\right] = 0 ;$$
$$[L_1, L_2] = 0 ; \quad L = \frac{\partial}{\partial y} - M. \tag{6}$$

Here L_1 and L_2 are ordinary linear differential operator related to x. Unlike the theory of the onedimensional systems (1) the degrees of operators L_1 and L_2 are arbitrary !

This class of solutions in case of the commuting pair L_1 and L_2 of rank 1 was found in work [7] and in case of the commuting pair of any rank in the work [8], [9]. The solutions of rank l > 1 depend on the arbitrary functions of one variable.

The most significant example of the systems (5) is a well-known two-dimensional KdV equation (or KP equation), where

$$M = \frac{\partial^2}{\partial x^2} - U(x, y, t) \quad , \quad A = \frac{\partial^3}{\partial x^3} - \frac{3}{2} U \frac{\partial}{\partial x} + W(x, y, t)$$

$$\begin{cases} W_x = \frac{3}{4} U_y - \frac{3}{4} U_{xx} \\ W_y = U_t - \frac{3}{4} U_{xy} - \frac{1}{4} U_{xxx} + \frac{3}{2} U U_x \end{cases} \tag{7}$$

or

$$\frac{3}{4} U_{yy} = \frac{\partial}{\partial x} \left\{ U_t + \frac{1}{4} (6 U U_x - U_{xxx} \right\}$$

The solutions of rank 1 (i.e. pair of L_1 and L_2 has rank 1) have the form of:

$$U(x,y,t) = \text{const} + 2\frac{\partial^2}{\partial x^2} \ln \theta(\vec{U}x + \vec{V}y + \vec{Z}t + \vec{W}), \tag{8}$$

according to [10], and where $\theta(v_1, \dots, v_g)$ - is the theta-function by Riemann, corresponding to the Riemann surface Γ (4).

In case of $l > 1$ even the investigations of the equations of commutativity $[L_1, L_2]=0$ is very complicated. The solution of the problem of classifying such pair L_1, L_2 of any rank $l > 1$ was found in the work [11]. The determination of the coefficients of these operators is reduced to a certain Riemann problem. The method, which permits to eliminate the Rieman's problem and to obtain exact formula for coefficients of the operators L_1 and L_2 of rank $l > 1$ has been developed in work [9], [12].

II. Multipoints vector analog of the Baker-Akhiezer function.

Let's consider the set of the matrix (lxl) functions, $\Psi_s(\vec{x},k)$, $s = 1, \dots, n$, $\vec{x} = (x_1, \dots, x_n)$, such that $\Psi_s(0,k)=1$ and the matrixes:

$$A_j^s(\vec{x},k) = \left(\frac{\partial}{\partial x_j} \Psi_s(\vec{x},k)\right) \Psi_s^{-1}(\vec{x},k) \tag{9}$$

are polynomial on k.

Matrix functions $A_j^s(\vec{x},k)$ must satisfy the relations:

$$\frac{\partial A_j^s}{\partial x_i} - \frac{\partial A_i^s}{\partial x_j} = [A_i^s, A_j^s] \tag{10}$$

Any set of matrixes polynomial of k A_i^s, satisfying(10), uniquely determine the functions $\Psi_s(\vec{x},k)$.

Let $(\Gamma, P_1, \dots, P_m, k_s)$ - is any nonsingular Riemann surface of the genus g with fixed points $P_1, \dots, P_m$ and local parameters $z_s = k_s^{-1}(P)$ in their neighbourhood.

Consider now the unordered set of points $(\gamma) = (\gamma_1, \dots, \gamma_{gl})$ and set (α) of the complex (l-1) - vectors $\vec{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,l-1})$.

Remark. The set (γ, α) is called "Tiurin parameters" for the stable (in the Mamford's sense) l-dimensional holomorphic vector bundle of the lg degree over Γ with fixed framing, i.e. with a fixed set of holomorphic sections $\eta_1, \ldots, \eta_l$ ([13]).
The points $\gamma_1, \ldots, \gamma_{lg}$ are the points of the linear dependence of the sections η_i and $\alpha_{i,j}$ are coefficients of linear dependence

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{i,j}\, \eta_j(\gamma_i). \tag{11}$$

Let's set up the problem: to find vector-function $\psi(\vec{x}, P)$ which is meromorphic on Γ except for the points $P_1, \ldots, P_m$, and such that:

1°. a) the poles of $\psi(\vec{x}, P) = (\psi_1, \ldots, \psi_l)$ lie in the points γ_i
b), for the residues of $\psi_j(x, P)$ (the coordinates of $\psi(\vec{x}, P)$), the following relations are true

$$\operatorname{res}_{\gamma_i} \psi_j(\vec{x}, P) = \alpha_{i,j} \cdot \operatorname{res}_{\gamma_i} \psi_l(\vec{x}, P), \tag{12}$$

$\alpha_{i,j}$ and γ_i do not depend on x.

2°. In the small neighbourhood of the point P_s the vector-function $\psi(\vec{x}, P)$ must have the representation:

$$\psi(\vec{x}, P) = \left(\sum_{i=0}^{\infty} \xi_i(\vec{x})\, k_s^{-i}\right) \Psi_s(\vec{x}, k_s). \tag{13}$$

In case of l=1 the assimptotic functions Ψ_s are the exponents; in this case ψ is n-point scalar analog of the classical Baker-Akhiezer function ([10]).

Following the scheme of the work [11] , which is based on the methods of [14], [15] , one can obtain the general statement.

Theorem. The dimension of the linear space of the functions, which satisfy above-mentioned restrictions with fixed x, is equal to l. For the unique determination of ψ it is enough to fix its value at any point. The construction of ψ is equivalent to the system of the linear singular integral equations on the small circles - the points $P_1, \ldots, P_m$ neighbourhoods' boundaries.

The integral equations are solved separately for each x. The relations (12) and the value $\psi(x, P_0)$ gives us unique solution of the singular integral equations.

The matrix $\Psi(\vec{x}, P)$ whose rows are linearly independent solutions of the problem (12-13), is called the matrix function of the Baker-Akhiezer type. It follows that $\Psi(\vec{x}, P)$ is determined uniquely up to the multiplying by the invertible matrix function

$$\tilde{\Psi}(\vec{x}, P) = G(\vec{x})\, \Psi(\vec{x}, P). \tag{14}$$

Except for the Turin parameters the construction of Ψ depends on the of matrixes Ψ_s.

Example 1. KP - Equation; commuting ordinary operators. (see [8], [9]).

Let's consider one-point vector-function $\psi(x,y,t,P)$ with essential singular point P_0 on the Riemannian surface Γ of the genus g. It is determined by the Tiurin's parameters and the assimptotic matrix $\Psi_0(x,y,t, P_0)$. In case l=1 it is the classical Gordon-Klebsh - Baker's function [16].

a) We should choose the functions $A_i(x,y,t,k)$, i=1,2,3, which determine the function $\Psi_0(x,y,t,P)$ according to (9), in case l=2, in the form

$$A_1 = \begin{pmatrix} 0 & 1 \\ k-u & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -\frac{u_x}{4}, & k+\frac{u}{2} \\ k^2-\frac{ku}{2}-\frac{u^2}{2}-\frac{u_{xx}}{4}, & \frac{u_x}{4} \end{pmatrix},$$

where $u=u(x,y,t)$.

From the compatibility equations (10) it follows that $u=u(x,t)$ doesn't depend on y and satisfies the KdV equation

$$4u_t = 6uu_x - u_{xxx}$$

b) Case l=3. Let's choose A_i in the form:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k-w & -\frac{3}{2}u & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix};$$

$$A_2 = \begin{pmatrix} u & 0 & 1 \\ k-w+u_x & -\frac{u}{2} & 0 \\ -w_x+u_{xx} & k-w+\frac{u_x}{2} & -\frac{u}{2} \end{pmatrix}$$

From (10) it follows, that $u=u(x,y)$ doesn't depend on t and is the solution of the Bussinesque equation

$$3u_{yy} + u_{xxxx} - 6(uu_x)_x = 0.$$

c) In case $l>3$ the matrixes $A_i(x,y,t)$ should be chosen in the form

$$A_1 = \begin{pmatrix} 0, & \cdots & 0, 0 \\ \cdots & & \cdots \\ 0, & \cdots & 0, 0 \\ u_0, & \cdots & u_{l-2}, 0 \end{pmatrix} + \hat{\varkappa} ; \quad \hat{\varkappa} = \begin{pmatrix} 0 & 1 & 0, & \cdots & 0, 0 \\ 0 & 0 & 1, & \cdots & 0, 0 \\ \cdots & & \cdots & & \cdots \\ 0 & 0 & \cdots & & 0 \; 1 \\ k & 0 & \cdots & & 0 \; 0 \end{pmatrix},$$

$$A_2 = \hat{\varkappa}^2 + a_2 , \qquad A_3 = \hat{\varkappa}^3 + a_3 ,$$

where a_2 and a_3 are the $(l\times l)$ matrix, independent of k, whose elements are the differential polynominals of $u_0, \ldots, u_{l-2}$.

<u>Important statement.</u> In all before-mentioned cases the vector-function - of the Baker-Akhiezer type ψ, which has in the neighbourhood of the point p_0 the representation:

$$\psi(x,y,t,P) = \left(\sum_{i=0}^{\infty} \xi_i(x,y,t) k^{-i} \right) \Psi_0(x,y,t,k) , \tag{15}$$

$$\xi_0 = (1, 0, \ldots, 0), \quad \xi_i = \left(\xi_i^{(1)}, \ldots, \xi_i^{(l)}\right),$$

Satisfies to the pair of the scalar linear equations

$$\left(\frac{\partial}{\partial y} - M\right)\psi = 0 \quad ; \quad \left(\frac{\partial}{\partial t} - A\right)\psi = 0 ; \tag{16}$$

$$M = \frac{\partial^2}{\partial x^2} + U \quad ; \quad A = \frac{\partial^3}{\partial x^3} + \frac{3}{2} U \frac{\partial}{\partial x} + W .$$

The coefficients U and W don't depend on P. For U there are the formulas:

$$l=2: \; U = u(x,t) - 2\xi^{(2)}_{1x} \quad ; \quad l \geq 3 : \; U = -2\xi^{(l)}_{1x} .$$

<u>**Conclusion.**</u> The function $U(x,y,t)$ is the solution of the KP equation

$$\frac{3}{4} U_{yy} = \frac{\partial}{\partial x} \left\{ U_t - \frac{1}{4}\left(6UU_x - U_{xxx}\right)\right\}$$

Consequently the class of solutions of the KP equation, which depend on the following data $\{\Gamma, P_0, \gamma, \alpha, u_0, \ldots, u_{l-2}\}$ is obtained. In case $l=2$ the function $u_0(x,t)$ is the solution of the usual KdV equation.

The vector-function $\psi(x,o,o,P)=\psi(x,P)$, which depends on one variable x only, was introduced in the work [11]. The l coordinates of this function are the set of common eigenfunctions of the commuting pair of ordinary scalar differential operators:

$$L_1 \psi^q(x,P) = \lambda \psi^q(x,P),$$
$$L_2 \psi^q(x,P) = \mu \psi^q(x,P), \tag{17}$$
$$\psi = (\psi^1, \dots, \psi^l).$$

where λ, μ are arbitrary algebraic functions on the curve Γ, which have only one pole in the point P_0. If the degrees of the poles λ and μ are m and n, then the degrees of the operators L_1 and L_2 are ml and nl: That means, that the commutative ring of the operators of the rank l is determined by the surface Γ, the point P_0, with the local parameter, the Tiurin parameters $(\gamma_1, \dots, \gamma_{gl}, \alpha_1, \dots, \alpha_{gl})$ and the arbitrary functions $u_0, \dots, u_{l-2}$

Each operator of this ring is determined by the function $\lambda(P)$ with only one pole in the point P_0. Exact formulas for the coefficients of these operators will be obtained below in some special cases. All the relations (6) follow from the equations (16), (17).

III. Example 2. The two-dimensional Schrödinger operator; two-point Baker-Akhiezer type function with separate variables.

The natural generalization of the Lax type equations (1) in case of the operators L, which essentially depend on a few space variables, is not trivial. Let's note, that the operators, corresponding to the KP type equations, include the operator $\frac{\partial}{\partial y}$ only in the first degree.

It is known, that the nontrivial operator P, whose commutator with the operator $L=\Delta+u$, $[P, L]$, is the operator of multiplication by the function doesn't exist for the typical potentials u(x), $x=(x_1, \dots, x_n)$, $n>1$. This means, that nontrivial evolutional systems of the Lax form, preserving the full spectrum of L, do not exist. The eigenvalues of the operator L in the case $n>1$ have the infinite degree of the degeneration. It is enough for the reconstruction of the operator L to use the "inverse problem data" about the eigenfunctors of one energy level.

The equations of the form

$$\frac{\partial L}{\partial t} = [A, L] + BL, \tag{18}$$

where B is the differential operator, were introduced in [17] and stimulated our work [18].

We shall introduce a certain class of two-dimentional "funite-gap" Schrödinger operators; the inverse problem of reconstructing the operators from the data on one energy level was solved in the work [18].

Let's review the main ideas, on the formulation of the inverse problem for the operator

$$H=\left(i\frac{\partial}{\partial x}-A_1\right)^2+\left(i\frac{\partial}{\partial y}-A_2\right)^2+u(x,y).$$

Let the potential $u(x,y)$ and the vector-potential $A_1(x,y)$, $A_2(x,y)$ be the periodic functions of x and y with the periods T_1, T_2. Consider the equation $H\psi=E\psi$. It is natural to introduce the Bloch-functions as the eigenfunctions of monodromy operators

$$\psi(x+T_1,y)=e^{ip_1T_1}\psi(x,y);\ \psi(x,y+T_2)=e^{ip_2T_2}\psi(x,y);\quad \lambda_j=e^{ip_jT_j}.$$

"The numbers p_1 and p_2 are called the "quasi-momenta". The eigenvalues of the monodromy operators T_1, T_2 and Schrödinger operator H form the two-dimentional manifold" M^2 in the 3- space (λ_1,λ_2,E) The points of this submanifold are the triples of λ_1,λ_2,E such that there exists the solution of equation

$$H\psi=E\psi;\ \psi(x+T_1,y)=\lambda_1\psi(x,y);\ \psi(x,y+T_2)=\lambda_2\psi(x,y).$$

The operator H would be called the operator with a good analytical properties, if the manifold M^2, is the complete two-dimentional analytical submanifold in C^3. The intersection of M^2 with hyperplane $E=E_0$ is the analytical surface R (E_0), which is called "the complex Fermi-surface".

The operator H is called "finite-gap", if the genus of this surface $R(E_0)$ is finite for some E_0. Let's clarify the asymptotic behaviour of the Bloch functions at the complex values of p_1 and p_2, $E(p_1, p_2)=E_0$, $|p_i|\to\infty$, $p_1^2+p_2^2=O(1)$. This means, that the surface R is compactified by two "infinite" points P_1 and P_2, and the Bloch function has the representation:

$$\psi=e^{k_1(x+iy)}\left(\sum_{s=0}^{\infty}\xi_s(x,y)k_1^{-s}\right)\sim e^{k_1 z},$$

$$\psi=e^{k_2(x-iy)}\left(\sum_{s=0}^{\infty}\zeta_s(x,y)k_2^{-s}\right)\sim e^{k_2\bar z},$$

where k_1^{-1} and K_2^{-1} are the local parameters in the neighbourhoods of the points P_1 and P_2. Outside these points P_1, P_2, the function $\psi(x,y,P)$, $P\in R$, is meromorphic and has g poles $\gamma_1,\dots,\gamma_g$. The problem of the reconstruction of the operator H from the curve R with two fixed points P_1, P_2 and the set of $\gamma_1,\dots,\gamma_g$ was solved in [18].

Let's pay attention to the important fact: the asymptotics of ψ near the points P_1 and P_2 depend on different variables z and $\bar z$. The functions of the Baker-Akhiezer type with such properties will be called "two-point" functions with separate variables".

For the operators H of the rank 1 the following formulae are valid:

$$u(x,y)=\frac{\partial^2}{\partial z\partial\bar z}\ln\Theta\left(\vec U_1 z+\vec U_2\bar z+\vec W\right),$$

$$A_{\bar z} = A_1 + i A_2 = -\frac{\partial}{\partial z} \ln \frac{\theta(\vec{U}_1 z + \vec{U}_2 \bar z + \vec{V}_1 + \vec{W})}{\theta(\vec{U}_1 z + \vec{U}_2 \bar z + \vec{V}_2 + \vec{W})},$$

$$A_z = A_1 - i A_2 = 0, \quad z = x + iy, \quad \bar z = x - iy.$$

The vectors U_i, V_i are independent of z, $\bar z$ and are determined only by the points P_1, P_2. The vector W is determined by the set $\gamma_1, \dots, \gamma_g$. In the given gauge the operator H is not Hermitian. The conditions on the parameters of our construction $\{R, P_1, P_2, \gamma_1, \dots, \gamma_g\}$ which lead to Hermitian operators H were found in [19]. The condition of the finiteness of genus $g < \infty$ for the operator H is not resistant to variation of energy level. This means that, if for one value $E=E_0$ the genus of "complex Fermi-surface" is finite $g < \infty$ then it becomes infinite for values E close to E_0.

The natural generalization of the θ-functions is determined sometimes for the curve of the infinite genus [20]. For complete construction of the theory of the two-dimensional Schrödinger operator the generalization of our construction for the ease of the infinite genus is therefore necessary. Primarily it is necessary to find the assimptotics and the disposition of the poles of the Bloch functions on the surface of the quasi-momenta at the fixed energy level. Let's note that corresponding assimptotics must be considered in the unphysical region of the complex values of the quasi-momenta.

The following algebraic requirements for the two-dimensional Schrödinger operators are analogous to the equations (2), specifying the finitegap solution of Lax-type equations [18]:

There exist the linear operators L_1 and L_2, such that the commutators have the form:

$$[H, L_i] = B_i H; \quad [L_1, L_2] = B_3 H, \tag{19}$$

where B_1, B_2, B_3 are the differential operators.
The eigenvalues of the operators:

$$H\psi = 0; \quad L_1\psi = \lambda\psi; \quad L_2\psi = \mu\psi \tag{20}$$

satisfy to the algebraic relation

$$Q(\lambda, \mu) = 0. \tag{21}$$

Here $Q(\lambda, \mu)$ is the polynominal.

The important concept of the "rank" for algebra of the operators (19) is introduced: the number of the linearly independent solutions of the equations (20) would be called the rank of the algebra (19). For the algebra of the rank 1 the eigenfunctions form the 1-dimensional holomorphic bundle over the curve Γ, which is determined by the equation (21).

The before-mentioned construction of the operators H has rank 1.

It would be of interest to analyse the relation between the concept of the rank of the operator and the concept of "typical" position of the operator H with periodic coefficients. For the finitegap operators such relation is as follows. At the fixed degrees of the operators L_1 and L_2 the number of the parameters, which determine the algebraic relation (21) for the rank-1 algebras is greater then the number of these parameters for the $l > 1$ rank operators. However, except for these parameters the algebra of the rank 1 is determined by the $2(l-1)$ arbitrary functions of one variable. That's why in general the rank $l > 1$ algebras isn't the degeneration of the rank 1 algebras.

Let's present the construction of the finitegap operators H of the rank l.

Let $\Psi_1(z,k)$ and $\Psi_2(z,k)$ be the matrix functions, determining the equations

$$\frac{\partial}{\partial z}\Psi_1(z,k) = A^1(z,k)\,\Psi_1(z,k);\quad \frac{\partial}{\partial \bar z}\Psi_2(\bar z,k) = A^2(\bar z,k)\,\Psi_2(\bar z,k), \tag{22}$$

where

$$A^1 = \begin{pmatrix} 0, 1, 0 & \cdots & , 0 & , 0 \\ 0, 0, 1 & & 0 & 0 \\ \cdots & & \cdots & \\ & & & \\ 0, \ 0, \ \cdots & & , 0, & 1 \\ k+u_0, u_1, \cdots & & , u_{l-2}, & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & k+v_0 \\ 1 & 0 & & 0 & v_1 \\ 0 & 1 & & & \\ \vdots & \vdots & & \vdots & \vdots \\ & & & 0 & v_{l-2} \\ 0 & 0 & & 1 & 0 \end{pmatrix} \tag{23}$$

$\Psi_i(0,k)=1$ and $u_i=u_i(z)$, $v_i=v_i(\bar z)$ are arbitrary functions. Consider the vector-function of the Baker-Akhiezer type $\psi(z,\bar z,P)$ on the Riemannian surface Γ of the genus g, corresponding to the Tiurin parameters (γ, α), which has the following representation in the neighbourhoods of the two fixed points P_1 and P_2.

$$\psi(z,\bar z,P) = \Big(\sum_{s=0}^{\infty} \xi_s(z,\bar z)\,k_1^{-s}\Big)\,\Psi_1(z,k_1), \tag{24}$$

$$\psi(z,\bar z,P) = \Big(\sum_{s=0}^{\infty} \zeta_s(z,\bar z)\,k_2^{-s}\Big)\,\Psi_2(\bar z,k_2). \tag{25}$$

The function ψ is normalized by the condition, $\xi_0 = (1, 0, \ldots, 0)$. Here ξ_s, ζ_s are vectors $\xi_s = (\xi_s^{(1)}, \ldots, \xi_s^{(l)})$; $\zeta_s = (\zeta_s^{(1)}, \ldots, \zeta_s^{(l)})$ and $k_j^{-1}(P)$ are the local parameters in the neighbourhoods of P_j.

<u>Statement</u>. The vector-function ψ of the Baker-Akhiezer type statisfies the equation $H\psi = 0$, where

$$H = \frac{\partial^2}{\partial z \partial \bar{z}} + v(z,\bar{z})\frac{\partial}{\partial \bar{z}} + u(z,\bar{z})$$

is two-dimensional Shrödinger operator with scalar coefficients.

$$v(z,\bar{z}) = -\frac{\partial}{\partial z} \ln \zeta_0^{(1)}(z,\bar{z}),$$
$$u(z,\bar{z}) = -\frac{\partial}{\partial \bar{z}} \xi_1^{(\ell)}(z,\bar{z}). \tag{26}$$

Only Hermitian operators H may have the physical sense.
Is it was mentioned above, for the rank 1 operators H the restrictions on the parameters, corresponding to the Hermitian operators, were found in the work [19] . Following the idea of this work we shall find here the analogous conditions in the case l=2.

Let us consider the curves Γ with antiholomorphic involutions : $\sigma: \Gamma \to \Gamma$, which transpose the fixed points $\sigma(P_1)=P_2$ and the local parameters, $\sigma(k_1)=-\bar{k}_2$. For any two points there exists the Abelian differential of the third kind with the simple poles in these points and with the residues ± 1, correspondingly. In our case, for points P_1, P_2 we shall consider the odd differential of the third kind $\omega(P) \equiv -\bar{\omega}(\sigma(P))$. The difference between any two differentials of such type is the odd holomorphic differential. This means, that the real dimension of the space of the odd differentials of the third kind equals g. The set of the zeroes $(\gamma_1, \dots, \gamma_{2g})$ of the $\omega(P)$ is invariant with respect to the anti-involution σ, $\sigma(\gamma_i) = \gamma_{\sigma(i)}$, where $\sigma(i)$ is the corresponding permutation of the indices.

Example. Let Γ be the hyperelliptic curve, given in C^2 by the equation

$$y^2 = \lambda \prod_{i=1}^{2g} (\lambda - \lambda_i).$$

The set of the complex numbers $\lambda_1, \dots, \lambda_{2g}$ satisfies the conditions $\lambda_i = \bar{\lambda}_{2g-i}$, $\prod \lambda_i = 1$.
The antiholomorphic involution σ on Γ, transposing points $P_1=0$ and $P_2=\infty$ has the form:

$$P=(y,\lambda) \longrightarrow \sigma(P) = (-\bar{y}\,\bar{\lambda}^{-(g+1)}, \bar{\lambda}^{-1}).$$

The Abelian differentials with their poles in the points P_1, P_2 have the form:

$$\omega = \lambda^{-1} d\lambda + \sum_{i=0}^{g-1} c_i \lambda^i y^{-1} d\lambda.$$

If the constants c_i satisfy the conditions $c_i = -\bar{c}_{g-1-i}$, the differential ω is odd.
The zeroes $\gamma_1, \dots, \gamma_{2g}$ of ω are the zeroes of the function $\lambda^{-1} + \sum c_i \lambda^i y^{-1}$ on the curve Γ.
In the case of rank l=2 the Tiurin parameters are the sets of γ_i and the complex numbers α_i. Let $\bar{\alpha}_i = -\alpha^{-1}_{\sigma(i)}$.
Besides these parameters the vector-function $\psi(z,\bar{z},P)$ is determined by the two functions $u_0(z)$ and $v_0(z)$ (23). Let $u_0(z) = -\bar{v}(\bar{z})$.

Statement. The above-mentioned restrictions on the parameters of construction correspond to the Hermitian operators H.

Sketch of the proof: let us consider the scalar function

$$\varphi(z,\bar z,P) = \psi(z,\bar z,P)\,\psi^+(z,\bar z,P).$$

(the cross denotes the Hermitian conjugation)

In case $l=2$ and $u_0(z)=-\bar v_0(\bar z)$, it follows from (23) that $\Psi_1(z,k)\Psi_2^+(\bar z,-\bar k)=1$.

This means that $\varphi(z,\bar z,P)$ is the meromorphic function on the whole curve Γ. It is easy to check that, if $\bar\alpha_i = -\alpha_{\sigma(i)}^{-1}$, the poles φ in the points γ_i are simple. The differential $\varphi(z,\bar z,P)\omega$ has the only two simple poles in the points P_1 and P_2, because in the poles φ the ω equals zero. The sum of the residues of the differential $\varphi\omega$ equals zero.

Consequently, $\varphi(z,\bar z,P^+) = \varphi(z,z,P^-)$. These values are equal to $\varphi(z,\bar z,P^+)=\zeta_0^{(1)}$, $\varphi(z,\bar z,P^-)=\zeta_0^{(2)}$, by definition. From (26) it follows, that $v(z,\bar z)$ is real and the operator H is Hermitian.

IV. The deformations of the holomorphic vector bundles.

In general the problem of the reconstruction of the vector analog of the Baker-Akhiezer function ψ is reduced to the Riemann problem, which is equivalent to the system of the singular integral equations. But for the calculation of the linear operator and corresponding solutions of the nonlinear equation the Rieman problem can be sometimes eliminated. This possibility is based on the investigation of the equations on the Tiurin parameters (γ,α).

Let, Γ be a nonsingular algebraic curve of the genus g with fixed points $P_1, \ldots, P_m$ and local parameters $k_s^{-1}(P)$ in their neighbourhoods. The logarithmic derivatives $\chi_i(\vec x,P)$ of the Baker-Akhiezer function $\Psi(\vec x,P)$ will be considered.

The matrix function $\Psi(\vec x,P)$ was determined earlier according to the Tiurin parameters and the "assimptotic functions" $\Psi_s(x,k)$. We have by definition:

$$\left(\frac{\partial}{\partial x_i} - \chi_i(\vec x,P)\right)\Psi(\vec x,P)=0 \tag{27}$$

The functions $\chi_i(\vec x,P)$ are meromorphic functions on the curve Γ, which have the poles in the points $P_1, \ldots, P_m$. Besides P_j, the functions $\chi(x,P)$ have lg simple poles in the points $\gamma_1, \ldots, \gamma_{gl}$. The rank of the matrix-residues χ_i at the points γ_s equals 1.

Consequently, there is (l-1)- vector $\alpha_s=(\alpha_{s,1},\ldots,\alpha_{s,l-1})$ in each points γ_s such that for matrix elements the following relations are valid:

$$\operatorname{res}_{\gamma_s}\chi_i^{a\ell} = \alpha_{s\ell}\operatorname{res}_{\gamma_s}\chi_i^{a\ell}. \tag{28}$$

The parameters $\gamma(\vec x), \vec\alpha(\vec x)$ satisfy the equations:

$$\frac{\partial}{\partial x_i}\gamma = -\operatorname{Sp}\chi_{i,0}, \tag{29}$$

$$\frac{\partial}{\partial x_i}\alpha_j = -\sum_{a=1}^{\ell}\alpha_a\chi_{i,1}^{aj} + \alpha_j\left(\sum_{a=1}^{\ell}\alpha_a\chi_{i,1}^{a\ell}\right), \tag{30}$$

$\gamma_s(0)=\gamma_s^0$, $\vec\alpha_s(0)=\vec\alpha_s^0$, where $\chi_{i,0}$ and $\chi_{i,1}$

are coefficients of the expansions $\mathcal{X}_i(\vec{x}, P)$ in the neighbourhood of the poles $\gamma_s(\vec{x})$ (the index S is omitted).

$$\mathcal{X}_i(\vec{x}, P) = \frac{\mathcal{X}_{i,0}}{k-\gamma} + \mathcal{X}_{i,1} + O(k-\gamma). \tag{31}$$

Let $u_{is}(x, k)$ be the matrices, which are polynominal on k and such, that all differences

$$\mathcal{X}_i(\vec{x}, P) - u_{is}(\vec{x}, k_s^-(P)) \tag{32}$$

are regular functions in the neighbourhood of P_s.

Statement. For arbitrary functions $u_{is}(x, k)$, (polynominal on k), and arbitrary $\gamma(\vec{x})$, $\alpha(\vec{x})$ there exists matrix function $\mathcal{X}_i(\vec{x}, P)$, satisfying the conditions (23), (32). This function is uniquely determined by its value at some point $P_0 \mathcal{X}_i(x, P_0) = u_{i,0}(x)$.

The ambiquity of the determination of $\mathcal{X}_i$ is connected with the ambiguity of the determination of $\Psi(x, P)$ which is determined up to the multiplying by the invertible matrix function G(x).

The proof of this statement directly follows from the Riemann-Roh theorem for the dimension of the functions' space which have simple poles at the points γ_s and n_i-fold poles at the points P_i.

This dimension equals the number of inhomogeneous linear equations, which are equivalent to the conditions (23), (32) and the condition $\mathcal{X}_i(\vec{x}, P_0) = u_{i0}(\vec{x})$.

Let the function $\mathcal{X}_i(x, P)$ be determined by the parameters

$\{u_{is}(\vec{x}, k), u_{io}(\vec{x}), \gamma(\vec{x}), \alpha(\vec{x})\}$.

Statement. The solution of the equations (27), $\Psi(o, P)=1$, is the Baker-Akhiezer type function iff the equations (29,30) are valid.

For brevity, the index i will be omitted, i.e. it will be assumed that Ψ depends on one parameter x.

First of all we shall prove that the equations (29), (30) are equivalent to the absence of the singularity Ψ at the points $\gamma_j(x)$ (i.e. Ψ is holomorphic function in these points).

Let $\Psi(x,P)$ be holomorphic in $\gamma = \gamma_s(x)$, then for each column $\{\psi_i\}$ of the matrix Ψ the equality:

$$\sum_{i=1}^{\ell} \alpha_i \psi_i = 0 \qquad , \alpha_\ell = 1 \tag{33}$$

is valid (here i is the index in the column)

This equality means that the coefficient at $(k-\gamma)^{-1}$ on the left of (27) is equal to zero. Except for this

$$\frac{d}{dx}\psi_i = \sum_\ell \left(\mathcal{X}_1^{i\ell} \psi_\ell + \mathcal{X}_0^{i\ell} \frac{\partial \psi_\ell}{\partial k} \right). \tag{34}$$

By derivation of the equality (33) we shall obtain, that

$$\sum_{a=1}^{\ell} \left(\alpha_{ax}\psi_a + \alpha_a \psi_{ax} + \alpha_a \gamma_x \frac{\partial \psi_a}{\partial k} \right) = 0 .$$

From (33) and (34) it follows:

$$\sum_a \Big[\alpha_{ax}\psi_a + \alpha_a\Big(\sum_b \chi_1^{ab}\psi_b + \chi_0^{ab}\frac{\partial\psi_b}{\partial k}\Big) + \gamma_x\alpha_a\frac{\partial\psi_a}{\partial k}\Big] = \sum_a\Big(\alpha_{ax} + \sum_b \alpha_b\chi_1^{ba}\Big)\psi_a = 0. \tag{35}$$

The equality (29) is the simple consequence of the equality between the logarithmic derivative of the det-erminant Ψ and the trace of $\chi(x,P)$. The coefficient at ψ_a in the equalities (33) and (35) must be proportional, thats is why the equation (30) is valid.

Let us prove the inverse **part** of theorem. We shall consider the matrix $\tilde\chi$ which is gauge equivalent to χ

$$\tilde\chi = \partial_x g\, g^{-1} + g\chi g^{-1},$$

where

$$g = \begin{pmatrix} \frac{\alpha_1}{k-\gamma}, & \frac{\alpha_2}{k-\gamma}, & \dots & , \frac{\alpha_{l-1}}{k-\gamma}, & \frac{1}{k-\gamma} \\ 0, & 0, & \dots & 0, 1, & 0 \\ \vdots & & & & \\ 0, & 1, 0, & \dots & , 0, & 0 \\ 1, & 0, 0, & \dots & , 0, & 0 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} 0, & 0, & \dots & , 0, 0, 1 \\ 0, & 0, & \dots & , 0, 1, 0 \\ & \dots & & \dots \\ 0, & 1, 0, & \dots & \dots 0, 0, 0 \\ k-\gamma, & -\alpha_{l-1}, & \dots & \dots, -\alpha_2, -\alpha_1 \end{pmatrix}$$

The direct consequence of the equations (29), (30) is the absence of the singularity $\tilde\chi$ at the point $k=\gamma$. That means that the solution of the equation

$$\frac{d}{dx}\tilde\Psi = \tilde\chi\tilde\Psi$$

has no singularity. Then the function $\Psi = g^{-1}\tilde\Psi$, satisfying the equation (27), is nonsingular at $k=\gamma$ too.

Now the form of Ψ in the neighbourhood of P_s will be found. Let us formulate the following Riemann problem: to find the matrix function $\Psi_s(x, k)$, which is holomorphic on k except for only one point $k=\infty$ and in the neighbourhood:

$$\Psi_s(x,k) = R(x,k)\,\Psi(x,k)\ ;\ R = \sum_{i=0}^{\infty}\xi_i(x)k^{-i} \quad . \tag{36}$$

This problem has only one solution, such that $\Psi_s(x,0)=1$.

Lemma. The logarithmic derivative of Ψ_s is the polynominal function on k

$$\left(\frac{d}{dx}\Psi_s\right)\Psi_s^{-1} = \sum_{i=1}^{n_s} u_{si}(x)k^i .$$

It is valid, because $\left(\frac{d}{dx}\Psi_s\right)\Psi_s^{-1}$ has no singularities on k except $k=\infty$ and has the n_s-fold pole in the infinity $k=\infty$

From (36) it follows the expression (13) for $\Psi(x,P)$ in the neighbourhoods of the points P_s, i.e. Ψ is the Baker-Akhiezer type function.

V. Finitegap KP equation solutions of the rank 2 and genus 1.

In this paragraph the equations on the Tiurin parameters, corresponding to the "finite-gap" KP equation solutions of the rank 2 genus 1, will be considered. These solutions correspond to the commutative pair of the operators L_4, L_6, whose degrees are equal to 4 and 6. In the nondegenerate case such operators satisfy the relations:

$$L_6^2 = 4L_4^3 + g_1 L_4 + g_2 \tag{37}$$

and are determined by the algebraic curve Γ (the constants g_1, g_2), the Tiurin parameters (γ,α) on the elliptic curve Γ and one arbitrary function $u_0(x)$ ([11]). The elliptic curve Γ is determined by the equation (37).

In this case the Tiurin parameters are the points γ_1, γ_2 and complex numbers $\alpha_{11}=\alpha_1$, $\alpha_{21}=\alpha_2$ corresponding to these points.

According to the example 1a §1 the solution of the KP equation, corresponding to the pair L_4 and L_6, is determined by the set (γ,α) and by the solution of the KdV equation $u_0(x, t)$.

The logarithmic derivative of matrix analog of the Baker-Akhiezer function $\Psi(x,y,t,P)$, has the assimptotic form:

$$\left(\frac{\partial}{\partial x}\Psi\right)\Psi^{-1} = \chi_1(x,y,t,\lambda) = \begin{pmatrix} 0 & 1 \\ k-u & 0 \end{pmatrix} + O(\lambda) \tag{38}$$

where $\lambda = k^{-1}$ is the parameter on the elliptic curve.

The form of the singularity χ at $\lambda=0$ and parameter (γ,α) determine the function χ . Any elliptic function may be represented as the sum of the ζ-functions [21] . Let us find χ_1 in the form:

$$\chi_1 = A\,\zeta(\lambda-\gamma_1) + B\,\zeta(\lambda-\gamma_2) + C\,\zeta(\lambda) + \mathcal{D},$$

where A,B,C,D are matrices independent of λ . The Weierstrass ζ- function is determined by the series

$$\zeta(\lambda) = \lambda^{-1} + \sum \left[(\lambda-\omega_{mn})^{-1} + \omega_{mn}^{-1} + \lambda\,\omega_{mn}^{-2}\right];\ \omega_{mn} = m\omega + n\omega',$$

and the relation $\zeta'(z) = -\wp(z)$ is valid.

The ζ-function is not two-periodic in contrast to $\wp$- function. The function is an elliptic function iff the equality:

$$A + B + C = 0 \tag{39}$$

is valid. The expansion (38) of $\mathcal{X}_1$ means that $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. According to the definition, the residues $\mathcal{X}_1$ at the points γ_1 and γ_2 have the rank 1,

i.e. $A = \begin{pmatrix} \alpha_1 a, & a \\ \alpha_1 b, & b \end{pmatrix}$; $B = \begin{pmatrix} \alpha_2 c & c \\ \alpha_2 d & d \end{pmatrix}$

Therefore

$$A = \frac{1}{\alpha_2 - \alpha_1} \begin{pmatrix} 0 & 0 \\ \alpha_1 & 1 \end{pmatrix}, \quad B = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} 0 & 0 \\ \alpha_2 & 1 \end{pmatrix}.$$

Free term in the expansion (38) equals $\begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}$. Consequently,

$$\mathcal{D} - A\zeta(\gamma_1) - B\zeta(\gamma_2) = \begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix}. \tag{40}$$

(Recall that $\zeta(-\lambda) = -\zeta(\lambda)$)

The following expression has been finally obtained:

$$\mathcal{X}_1 = \frac{1}{\alpha_2 - \alpha_1} \begin{pmatrix} 0 & 0 \\ \alpha_1 & 1 \end{pmatrix} \zeta(\lambda - \gamma_1) + \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} 0 & 0 \\ \alpha_2 & 1 \end{pmatrix} \zeta(\lambda - \gamma_2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \zeta(\lambda) + \mathcal{D}, \tag{41}$$

where $\mathcal{D}$ is determined from the equality (40).

According to (29),

$$\gamma_{1x} = -\operatorname{Sp} A = \frac{1}{\alpha_1 - \alpha_2} \quad ; \quad \gamma_{2x} = -\operatorname{Sp} B = \frac{1}{\alpha_2 - 1} \tag{42}$$

The matrix $\mathcal{X}_{1,1}$, which according to (30), determine the dynamics of α_1 is equal to

$$\frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} 0 & 0 \\ \alpha_2 & 1 \end{pmatrix} \zeta(\gamma_1 - \gamma_2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \zeta(\gamma_1) + \mathcal{D}.$$

Consequently,

$$\alpha_{1x} = \alpha_1^2 + u - \varphi(\gamma_1, \gamma_2). \tag{43}$$

Similarly,

$$\alpha_{2x} = \alpha_2^2 + u + \varphi(\gamma_1, \gamma_2). \tag{44}$$

Here

$$\varphi(\gamma_1, \gamma_2) = \zeta(\gamma_2 - \gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2).$$

The expansions of the logarithmic derivatives $\Psi_y \Psi^{-1}$ and $\Psi_t \Psi^{-1}$ in the neighbourhood of $\lambda=0$ have the form:

$$\chi_2 = \Psi_y \Psi^{-1} = \begin{pmatrix} k & 0 \\ v & k \end{pmatrix} + O(\lambda) \quad , \lambda = k^{-1} \tag{45}$$

$$\chi_3 = \Psi_t \Psi^{-1} = \begin{pmatrix} \omega_1 & , & k+\frac{u}{2} \\ k^2 - \frac{uk}{2} + \omega_2 & , & -\omega_1 \end{pmatrix} + O(\lambda). \tag{46}$$

The expansions (45), (46) uniquely determine the functions χ_2 and χ_3 and their representations in form of the ζ-functions' sum as well as χ_1.

The equations on the Tiurin parameters will be as follows:

$$\gamma_{iy} = 1 \quad ; \quad \alpha_{iy} = -v(x,y,t) \; ; \tag{47}$$

$$\gamma_{it} = (-1)^i \left(\alpha_1 \alpha_2 + \frac{u}{2}\right) \frac{1}{\alpha_1 - \alpha_2} \; ; \tag{48}$$

$$\alpha_{it} = -2\alpha_i \omega_1 + \alpha_i^2 \frac{u}{2} - \omega_2 - (-1)^i \left(\frac{u}{2} + \alpha_i^2\right)\Phi - \wp(\gamma_i) \, . \tag{49}$$

Let us define $\gamma_1 = y+c(x,t)$, $\gamma_2 = y-c(x,t)+c_0$ c_0=const, $\alpha_1-\alpha_2=z(x,t)$, $\alpha_1+\alpha_2= w(x,y,t)$, $\Phi = \Phi(y,c,c_0)$.

The compatibility conditions of the equations (43,44,47-49) leads to the following:

$$\begin{aligned} v &= (\alpha_2-\alpha_1)^{-1}\left(\wp(\gamma_2) - \wp(\gamma_1)\right) \, , \\ w_1 &= \tfrac{1}{2}(\alpha_1-\alpha_2)^{-1}\left(\wp(\gamma_1) - \wp(\gamma_2)\right) - \frac{u_x}{4} \, , \\ w_2 &= w_{1x} - \frac{u^2}{2} + \wp(\gamma_1) + \wp(\gamma_2) \, . \end{aligned} \tag{50}$$

The equations on the Tiurin parameters in new variable have the form

$$\begin{aligned} & c_x = z^{-1} \; ; \; z_x = zw - 2\Phi(y,c,c_0) \; ; \; c_y = z_y = 0 \, , \\ & c_t = 2z^{-1}(z^2 - \varphi) \, ; \\ & u(x,y,t) = -\alpha_1^2 - \alpha_2^2 + \varphi(x,t) = -\frac{z^2+w^2}{2} + \varphi(x,t) \, ; \\ & w_x = -\frac{z^2+w^2}{2} + 2\varphi(x,t) \, . \end{aligned} \tag{51}$$

The substitution of the expression $W = (\ln z)_x + 2\Phi z^{-1}$ into the equation for W_x yields:

$$\varphi(x,t) = \frac{1+3c_{xx}^2}{4c_x^2} + Q c_x^2 - \frac{1}{2}\frac{c_{xxx}}{c_x}; \qquad (52)$$

$$u(x,y,t) = \frac{c_{xx}^2 - 1}{c_x^2} + 2\Phi c_{xx} + c_x^2(\Phi_c - \Phi^2) - \frac{1}{2}\frac{c_{xxx}}{c_x}; \qquad (53)$$

$$c_t = \frac{3}{8c_x}(1 - c_{xx}^2) - \frac{1}{2} Q c_x^3 + \frac{1}{4} c_{xxx} \qquad (54)$$

__Statement.__ Every solution c(x,t) of the equation (54) determine, according to (53), the solution of KP equation. This solution u(x,y,t) is the periodic function of y. It has not any singularity and is the boundary function on x , if $c_x = z^{-1} \neq 0$, $z \neq 0$.

The comparison of the constructions of the KP equation solutions, one of which uses the vector analog of the Baker-Akhiezer function and the other was mentioned above, shows that the equation (54) is latently isomorphic to the KdV equation. However this isomorphism is difficult to trace.

The equation (54) is the integrable system, which admits the zero curvature representation. The operators in this representation algebraically depend on the auxiliary "spectral parameter" λ, which is determined on the elliptic curve, differing from all the known cases, which contain the rational parameter λ .

This representation, having the form:

$$\mathcal{X}_{1t} - \mathcal{X}_{3x} + [\mathcal{X}_1, \mathcal{X}_3] = 0, \qquad (55)$$

$\mathcal{X}_i = \mathcal{X}_i(x,y,t)$, permits, as usual, to obtain the integrals of the equation (54) from the expansion of $\mathcal{X}_1$. The investigation of the general system, which have the form (55), will be undertaken in the next paragraph.

Let us consider the stationary solutions of the equation (54), which have the form u(x + at, y). They correspond to the solutions of the Bussinesque equation. The simple substitution ([3] , p.301) permits to obtain the more generaly solutions of the KP equation, which have "knoidal" wave type $u(x+a_1t, y+b_1t)$.

The substitution of $z = h^{-2}(c)$ into (54) ($c_t = 0$) leads to the Hamiltonian equation

$$\frac{d^2h}{dc^2} = -\frac{\partial W(h,c)}{\partial h}, \qquad (56)$$

$$W = -\frac{1}{2}Q(c,c_0)h^2 + a h^{-2} - \frac{1}{8}h^{-6}, \quad \text{where } Q = \Phi_c + \Phi^2$$

is the elliptic function.

This system is completely integrable. It follows from (55) that (56) admis the representation:

$$\mathcal{X}_{3x} = [\mathcal{X}_1, \mathcal{X}_3]. \qquad (57)$$

Consequently, the determinant $\det(\mu\cdot 1 - \mathcal{X}_3(x,\lambda)) = R(\mu,\lambda)$ does not depend on x and is the integral of the equation.

$$R(\mu,\lambda) = \mu^2 - \wp'(\lambda) - \alpha(c, c_0) = 0 \tag{58}$$

The integral $\alpha(c,c_0)$ is equal to

$$\alpha(c,c_0) = -\frac{u}{2}\left(\frac{\alpha_1-2\alpha_2}{\alpha_1-\alpha_2}\wp(\gamma_1) + \frac{2\alpha_1-\alpha_2}{\alpha_1-\alpha_2}\wp(\gamma_2) - \frac{u^2}{4}\right) + \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1-\alpha_2}\wp'(\gamma_1) - \frac{\alpha_1}{\alpha_1-\alpha_2}\wp'(\gamma_2)\right).$$

The equation (54) depends on c_0, as a parameter.
The set of the stationary solutions of the equation (54) for all c_0 is isomorphic to the space of Tiurin parameters.

The variety of the solution, corresponding to the fixed value of integral $\alpha(c,c_0) = \text{const}$ is isomorphic to three-dimensional Jacoby variety J(R) of the algebraic curve R. This curve is determined by the equation (58) and is two-sheet covering of the initial elliptic curve Γ. The intersection of the varieties, corresponding to α = const, c_0 = const is isomorphic to the so-called "Prim" variety - the odd part of the Jacoby variety.
Consequently, the modular space of the framed holomorphic rank-2 bundles over the elliptic curve is stratified into the two-dimensional Abelian Prim-varieties, corresponding to the coverings of the elliptic curve.

Conclusion. The knoidal waves of the KP equation, which have rank-2 and genus 1, can be represented in terms of theta-function of two variable. They do not coincide with the solutions of KP equation of the genus two and rank 1, which also have the representation in terms of Θ-functions of two variables.

The above-mentioned statements follow directly from the results of the appendix.

Now we shall obtain the exact expression for the operator L_4, which is included into the rank 2 commutative pair $[L_4, L_6] = 0$.

It follows from the results of [11] (§3) that the commutative pair is uniquely determined from the equations (43), (44), where u(x) is an arbitrary function. It is not necessary to solve these equations. If the function c(x) is chosen as the arbitrary functional parameter, then the expression (51) for γ_i, α_i, u permits to obtain all the rank 2 commutative pairs, corresponding to the elliptic curve.
Each function c(x) determines according to (41,51), the logarithmic derivative of Ψ:

$$\Psi'\Psi^{-1} = \chi_1(x) = \begin{pmatrix} 0 & 1 \\ \chi_{21} & \chi_{22} \end{pmatrix}. \tag{59}$$

Here $\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{pmatrix}$ and ψ_i are the eigenfunctions of the operator L_4:

$$L_4\,\psi_i(x,\lambda) = \wp(\lambda)\,\psi_i(x,\lambda). \tag{60}$$

The equation (59) means that $\psi_i'' = \chi_{21}\psi_i + \chi_{22}\psi_i'$.
The formulae for higher derivatives may be obtained from this expression. For example,

$$\psi_i''' = \chi_{21}'\psi_i + \chi_{21}\psi_i' + \chi_{22}'\psi' + \chi_{22}\left(\chi_{21}\psi_i + \chi_{22}\psi_i'\right).$$

These formulae express $\frac{d^n}{dx^n}\psi_i$ in the form of the linear combination of ψ_i and ψ_i', whose coefficients are the polynominals on χ_{21}, χ_{22} and their derivatives.

Consequently, for any operator

$$L_4 = \frac{d^4}{dx^4} + v_2(x)\frac{d^2}{dx^2} + v_1(x)\frac{d}{dx} + v_0(x)$$

we can represent $L_4\psi_i$ in the form of $\ell_1(x,\lambda)\psi_i + \ell_2(x,\lambda)\psi_i'$. The functions ℓ_1, ℓ_2 are meromorphic functions of λ and linearly depend on v_i. The functions v_i can be found from the following conditions on the expansions of ℓ_1 and ℓ_2 in the neigbourhood of $\lambda=0$

$$\ell_1(x,\lambda) = \lambda^{-2} + O(\lambda),$$

$$\ell_2(x,\lambda) = O(\lambda).$$

Finaly, the following expression for L_4 will be obtained.

$$L_4 = L^2 - c_x\left[\wp(c+c_0) - \wp(c+c_1)\right]\frac{d}{dx} - \wp(c+c_0) - \wp(c+c_1),$$
$$L = \frac{\partial^2}{\partial x^2} + u(x).$$

VI. The zero curvature equations with algebraical "spectral" parameter.

It was shown, that for KP equation the construction of the genus 1 and rank 2 "finitegap" solutions leads to the integrable system. This system has the "zero curvature" representation with operators, which algebraically depend on the auxiliary "spectral parameter" - the point of elliptic curve Γ.

The general representation of such type,

$$u_t - v_x + [u,v] = 0, \tag{62}$$

means that the following equations are compatible

$$\left(\frac{\partial}{\partial x} - u(x,t,P)\right)\Psi(x,t,P) = 0, \tag{63}$$

$$\left(\frac{\partial}{\partial t} - v(x,t,P)\right)\Psi(x,t,P) = 0, \tag{64}$$

where P is the point of the algebraic genus g curve Γ with fixed points $P_1, \ldots P_m$.

Let the matrix functions u(x,t,P) and v(x,t,P) be determined, as in § 4 by their singularities at the points P_s and the values $u_0 = u(x,t,P_0)$, $v_0 = v(x,t,P_0)$.

The singularities u and v at the points P_s are the matrix functions:

$$u_s(x,t,k) = \sum_{i=1}^{n_s} u_{si}(x,t)k^i \ ; \ v_s(x,t,k) = \sum_{i=1}^{m_s} v_{si}(x,t)k^i$$

In case of the genus g=o surface the functions u and v are the rational functions. The functions have to satisfy the equation (62) for each point P; but in this case these equations are equivalent to the finite system of the equations. The latter means that the function $w=u_t-v_x+[u,v]$ has no singularities at the points $P_1,\dots,P_m$ and the value $w(x,t,P_0)$ is equal to zero.

If the genus g of surface Γ is greater then $g \geqslant 1$, then the functions u and v have the singularities in the points $\gamma_1,\dots,\gamma_{g\ell}$ except for $P_1 \dots P_m$. For these points there are vectors α_i for which the conditions (28) and the equations (29), (30) are fulfilled.

However, the equation (62) is equivalent to the finite system of the equations, which are associated with points $P_1,\dots,P_m$.

Statement. The systems (63) and (64) are compatible iff

$$u_{0t} - V_{0x} + [u_0, V_0] = 0 \ , \tag{65}$$

$$u_t - v_x + [u,v] = O(1)\Big|_{P=P_s} \ . \tag{66}$$

The latter equations mean that the function $u_t - v_x + [u,v]$ has not any singularities in the points $P_1,\dots,P_m$.

The number of the matrix equations (65), (66) equals to M+N+1, where $M=\sum m_s$, $N=\sum n_s$. The number of the functions determining u and v is equal to M+N+2.

This system is underdeterminate: the gauge transformation

$$u \longrightarrow \partial_x g \cdot g^{-1} + g u g^{-1} \ ,$$

$$v \longrightarrow \partial_t g \cdot g^{-1} + g v g^{-1}$$

with arbitrary invertible matrix g(x,t) transfers the solutions (65), (66) to the solutions of the same equations.

The sketch of the proof. Let us consider the function $w=u_t-v_x+[u,v]$. It follows from the equations (29) that the function w has not the poles of the second degree in the points $\gamma_1,\dots,\gamma_{g\ell}$. In the neighbourhood of the point $\gamma=\gamma_s(x,t)$ we have:

$$u = \frac{u_0}{k-\gamma} + u^1 + u^2\cdot(k-\gamma) + O\left((k-\gamma)^2\right) \ ,$$

$$v = \frac{v_0}{k-\gamma} + v^1 + v^2\cdot(k-\gamma) + O\left((k-\gamma)^2\right) .$$

From the equations (30) we deduce for matrix elements w^{ab}

$$\operatorname{res}_{\gamma_s} w^{ab} = \alpha_{sb} \operatorname{res}_{\gamma_s} w^{al}$$

This means that w has the same type as the functions u and v. Consequently, the function w is uniquely determined by the singularities at the points $P_1 \dots P_m$ and by the value $w(x,t,P_0)$. According to (65), (66) we obtain:

$$W = u_t - v_x + [u,v] = 0 . \tag{67}$$

To complete the proof of the statement it is enough to prove that the equations (29), (30) are compatible.

It follows from (67) that

$$\operatorname{Sp} u^0_t - \operatorname{Sp} v^0_x = 0 \iff \gamma_{xt} = \gamma_{tx} .$$

Let us introduce the vector-row $\beta = (\beta_1, \dots, \beta_l)$ which satisfies the equations

$$\beta_x = -\beta u^1, \tag{68}$$

$$\beta_t = -\beta v^1. \tag{69}$$

The compatibility of this system is equivalent to the compatibility of the equations (30) and $\alpha_i = \beta_i \beta_l^{-1}$. The compatibility of (68), (69) means, that

$$\beta \left(u^1_t - v^1_x + [u^1, v^1] \right) = 0 . \tag{70}$$

The zero degree term of the Loran expansion for W in the neighbourhood of $\gamma = \gamma_s(x,t)$ is equal to

$$\left(u^1_t - v^1_x + [u^1, v^1]\right) + [u^2, v^0] + [u^0, v^2] = 0 \tag{71}$$

Consequently, the (70) is equivalent to

$$\beta \left([v^0, u^2] + [v^2, u^0] \right) = 0 . \tag{72}$$

This relation does not contain the derivatives on x and t .

We shall use the following trick, It is easy to construct the Baker-Akhiezer function $\Psi_1(x,t,P)$, such that:

$$\tilde{u}(x_0,t_0,P) = u(x_0,t_0,P); \quad \tilde{u} = \Psi_{1x} \Psi_1^{-1} ;$$

$$\tilde{v}(x_0,t_0,P)=v(x_0,t_0,P)\ ;\ \tilde{v}=\Psi_{1t}\Psi_1^{-1}$$

For this function the equations (68), (69) are fulfilled for its Tiurin parameters $\tilde{\beta}$. Consequently we have that

$$\tilde{\beta}\left([\tilde{v}^0,\tilde{u}^2]+[\tilde{v}^2,\tilde{u}^0]\right)=0$$

This relation coincides with (72) at $x=x_0$ and $t=t_0$.

Appendix

Algebraic ensembles of the commutative flows.

The λ-representation for KdV equation and for its higher analogues was first found in the work [22]. This is the representation of the whole family of these equations in the form

$$\left[\frac{\partial}{\partial t_i}-u_i(\vec{t},\lambda),\frac{\partial}{\partial t_j}-u_j(\vec{t},\lambda)\right]=0,$$

where $u_i(t_1,\dots,\lambda)$ are the polynominals of λ ($t=t_1$, $x=t_2$)
In the general case of functions u_i, rational or algebraic on λ the in-variant definition of the algebraic ensemble of the operators may be done as follows.
This definition is analogous to the condition (2) in the theory of KdV-type equations.

Let there be the set of the operators L_i

$$L_i=\frac{\partial}{\partial t_i}-u_i(\vec{t},P),$$

where $u_i(\vec{t},P)$ are the meromorphic functions of P on the Riemannian surface Γ of the genus g, which have the same properties as the functions from the §IV. For $g=0$ the functions u_i are the usual rational functions on the Riemannian sphere with the fixed poles, undependent on t.

Definition 1. The family of the operators L_i will be called the "commutative ensemble", if for any i,j the operators L_i, L_j commute:

$$\frac{\partial u_i}{\partial t_j}-\frac{\partial u_j}{\partial t_i}+[u_i,u_j]=0 \tag{73}$$

Definition 2. The commutative ensemble is called algebraic if there exists the matrix function $w(\vec{t},P)$, which algebraically depends on P and such that:

$$\left[\frac{\partial}{\partial t_i}-u_i(\vec{t},P),w(\vec{t},P)\right]=0. \tag{74}$$

The basic example of the algebraic ensemble - are the stationarity conditions of the whole ensemble, with respect to one of the variables

$$\frac{\partial u_j}{\partial t_i} = 0 \qquad , \quad j = 1, \ldots$$

In this case $u_1 = W$. In general, the assumption, that W is connected with $(u_1 \ldots\ldots u_j, \ldots)$, is not necessary a priori. However, it may be shown than this assumption is true.

The linear operators $L_i = \frac{\partial}{\partial t_i} - u_i$, which enter the algebraic ensemble (if they have some Hermitian properties), are "finite-gap" in the sense of the spectral theory of the operators. [22]. Because of this, these operators and corresponding solutions of the nonlinear equations are called the "finitegap".

Any equation (73) with the indices i, j play the role of the "higher KdV" with respect to one of them. A priori all these equations are the partial differential equations. However, the algebraic ("finitegap") conditions (74) lead to the reduce these equations to the set of commuting ordinary differential equations referring to each variable.

Statement. If the operators L_i commute with W, then each of them commutes with the others $[L_i, L_j] = 0$, i.e. the equations (73) follow from the equations (74). If the number and degrees of the poles of W are fixed, then the dimension of the space of the corresponding matrices is finite. The equations (74) determine the commutative deformations of this space. All equations (74) have the common integrals.

Consider the solution $\Psi(\vec{t}, P)$ of the equations:

$$\left(\frac{\partial}{\partial t_i} - u_i(\vec{t}, P)\right) \Psi(\vec{t}, P) = 0 \tag{75}$$

such that $\Psi(o, P) = 1$.

The equality:

$$W(\vec{t}, P)\, \Psi(\vec{t}, P) = \Psi(\vec{t}, P)\, W(o, P) \tag{76}$$

follows from (74).

Hence, the characteristic polynominal

$$R(\mu, \lambda) = \det\left(\mu \cdot 1 - W(\vec{t}, P)\right) = 0 \tag{77}$$

does not depend on $\vec{t}$. Its coefficients are the integrals of (74).

Definition 3. The algebraic ensemble will be called "complete", if the flows determined by (74), cover all the level manifolds of the integrals (77).

In general position, the eigenvalues of $W(o,P)$ are different for almost all points P. Hence, the algebraic curve R which is determined by (77), is l-fold cover of the initial curve Γ. Let us consider for each point γ of R the corresponding eigenvector of $W(o,P)$. If the first coordinate of this vector $h_1(\gamma) = 1$, then the other coordinates are meromorphic functions on R. The vector-functions

$$\psi(\vec{t}, \gamma) = \sum_{i=1}^{l} h_i(\gamma)\, \Psi_i(\vec{t}, P),$$

where Ψ_i are the i-th columns of the matrix Ψ(t,P), possess the following analytical properties:

1°. $\psi(\vec{t},\gamma)$ is meromorphic on R outside the points P_i^j, $j=1,\dots,\ell$, which are the prototypes of the points P_i, $i=1,\dots,m$.

The poles of ψ do not depend on t, their number is equal to g+l-1, where g is the genus of R.

2°. The eigenvalues $u_i(\vec{t},P)$ at $P=P_i$ do not depend on $\vec{t}$, because characteristic polynominal w does not depend on $\vec{t}$. Consequently, in the neighbourhood of P_i^j the coordinates of ψ have the form:

$$\exp\left(\sum_a \lambda_a t_a k\right)\left(\sum_{s=0}^{\infty}\xi_s(\vec{t})\,k^{-s}\right),$$

where λ_a are the constants and $k^{-1}=k^{-1}(\gamma)$ is the local parameter near the P_i^j.

Thats why, $\psi(\vec{t},\gamma)$ is the scalar Baker-Akhiezer function and is uniquely determined by the divisor of the poles $\gamma_1,\dots,\gamma_{g+\ell-1}$. According to the general rule this function may be represented in terms of θ-functions. The function ψ determines the matrix by means of equality:

$$w(\vec{t},P)\,\psi(\vec{t},\gamma)=\mu(\gamma)\,\psi(t,\gamma),$$

where $\gamma=(P,\mu(\gamma))$ is the prototype of P on the curve R.

If we identify the matrices w and $A w A^{-1}$, where A is the constant diagonal matrix, then the factor-manifold of the levels of the integrals is isomorphic to the torus - the Jacobian variety of the surface R. The equations (74) determine the straight line on this torus ([18], III, §3).

In the theory of the KdV-type equations the higher analogus are the complete algebraic ensemble. The following operators, used in [23],[24] for the theory of the chiral field, are another example of the operators' ensemble. These operators have the form:

$$L_i=\frac{\partial}{\partial t_i}-\frac{A_i(\vec{t})}{\lambda-a_i}, \qquad (78)$$

$i=1,2$, $t_1=t-x$ $t_2=t+x$.

The examples of the algebraic ensembles with arbitrary numbers of the operators, which have the form (78), were considered by Garnier [25] . The initial point of his investigations was the Shlezinger theory of the deformations of the ordinary differential equation, which preserve the monodromy group of the singular points. The formal substitution $a_i \longrightarrow t_i$ into (78) leads to Shlezinger equations.

Garnier considered the equation (74) of the special form:

$$\left[\frac{\partial}{\partial t_i}-\frac{A_i}{\lambda-a_i},\ \sum_{i=1}^{n}\frac{A_i}{\lambda-a_i}\right]=0 \qquad (79)$$

where $W = \sum_{i=1}^{n} \frac{A_i}{\lambda - a_i}$.

The ensemble (79) is not complete. The number n of the operator is less than genus g of the curve R, which is determined by the equations:

$$Q(\mu,\lambda) = \det\left(\mu\cdot 1 - \sum_{i=1}^{n} \frac{A_i}{\lambda - a_i}\right) = 0.$$

The equations (79) were used [25] for the construction of new examples of integrable dinamical systems connected with Riemanian surfaces.

$$\xi_i'' = \xi_i\left(\sum_{i=1}^{n} \xi_i\eta_i + a_i\right),$$

$$\eta_i'' = \eta_i\left(\sum_{i=1}^{n} \xi_i\eta_i + a_i\right).$$

This system was discovered in the work [25].

On the different invariant hyperplanes $\xi_i = b_i\eta_i$ this system will reduces with the Newman [26] system of the oscillators, restricted on the sphere $\sum \xi_i^2 = 1$ and with the system of unhormonic oscillators [27].

REFERENCES

[1] Дубровин Б.А., Матвеев В.Б., Новиков С.П., Нелинейные уравнения типа Кортевега-де Фриза, конечнозонные линейные операторы и абелевы многообразия, УМН 31:1 (1976), 55-136.

[2] Кричевер И.М., Методы алгебраической геометрии в теории нелинейных уравнений, УМН 32:6 (1977), 183-208.

[3] Захаров В.Е., Манаков С.В., Новиков С.П., Питаевский Л.П., Теория солитонов. Метод обратной задачи, Москва "Наука",1980.

[4] Ince E.L., Ordinary differential equations, London, Longmans Green, 1927.

[5] Дрюма В.С., Об аналитическом решении двумерного уравнения Кортевега-де Фриза, Письма в ЖЭТФ 19:12 (1973), 219-225.

[6] Захаров В.Е., Шабат А.Б., Схема интегрирования нелинейных уравнений математической физики методом обратной задачи теории рассеяния, Функц. анализ 8:3 (1974), 43-53.

[7] Кричевер И.М., Алгебро-геометрическое построение уравнений Захарова-Шабата и их периодических решений, ДАН 227:2 (1976), 291-294.

[8] Кричевер И.М., Новиков С.П., Голоморфные расслоения над римановыми поверхностями и уравнение Кадомцева-Петвиашвили.I, Функц. анализ 12:4 (1978), 41-52.

[9] Кричевер И.М., Новиков С.П., Голоморфные расслоения и нелинейные уравнения. Конечнозонные решения ранга 2, ДАН 247:1 (1979), 33-36.

[10] Кричевер И.М., Интегрирование нелинейных уравнений методами алгебраической геометрии, Функц. анализ 11:1 (1977),15-32.

II Кричевер И.М., Коммутативные кольца линейных обыкновенных дифференциальных операторов, Функц. анализ I2:3 (I978), 20-3I.

I2 Drinfeld V.G., Krichever I.M., Manin J.I., Novikov S.P., Algebro-geometric methods in the modern mathematic physics, Soviet Science Revieurs, Phys. Reviews 1978, Over. Pub. Ass., Amsterdam, 1980.

I3 Тюрин А.Н., Классификация векторных расслоений над алгебраическими кривыми, Изв. АН СССР, сер. матем. 29 (I965),658-680.

I4 Koppelman W., Singular intergral equations, boundary value problem and Rieman-Roch theorem, J. Math. and Mech. 10, N 2, (1961), 247-277.

I5 Родин Ю.Л., Краевая задача Римана для дифференциалов на римановых поверхностях, Ученые записки Пермского университета, вып.2 (I960), 83-85.

I6 Baker H.F., Note of foregoing paper "Commutative ordinary differential equations, Proc. Royal Soc. Lond., 118 (1928), 570-577.

I7 Манаков С.В., Метод обратной задачи рассеяния и двумерные эволюционные уравнения, УМН 3I:5 (I976), 245-246.

I8 Дубровин Б.А., Кричевер И.М., Новиков С.П., Уравнение Шредингера в периодическом магнитном поле и римановы поверхности, ДАН 229:I (I976), I5-I8.

I9 Чередник И.В., Об условиях вещественности в "конечнозонном интегрировании", ДАН 252:5 (I980), II04-II07.

20 McKean G., Trubovitz E., Hill's operator and Hyperelliptic function theory in the presense of infinitely many branch points, Comm. Pure Appl. Math., 29, N 2, (1976), 143-226.

2I Бейтмен Г., Эрдейи А., Высшие трансцендентные функции. Эллиптические и автоморфные функции. Функции Ламе и Матье, Москва "Наука", I967.

22 Новиков С.П., Периодическая задача Кортевега-де Фриза, Функц. анализ 8:3 (I974), 54-66.

23 Захаров В.Е., Михайлов А.В., Релятивистски-инвариантные двумерные модели теории поля, ЖЭТФ 74:6 (I978), I953-I972.

24 Pohlmeyer K., Intergrable Hamiltonian systems and Interaction through Quadratic Constraints, Com. Math. Phys., 46 (1976), 207-235.

25 Garnier R., Sur une de systemes differentiels abeliens deduits de la theorie des equations lineaires, Rend. Circ. Matem. Palermo, 43 (1919), 155-191.

26 Moser J., Variable aspects of integrable systems, preprint Kurant Ins., 1978.

27 Glaser V., Grosser H., Martin A. Bounds on the number of eigenvalus of the Schrödinger operator, Comm. Math. Phys. 59 (1978), 197-212.