THE LAPLACE METHOD, ALGEBRAIC CURVES, AND NONLINEAR EQUATIONS

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Initially, the present paper has been stimulated by the problem of the construction of the integrable cases of the equation

$$-\varphi'' + e\,(Ex + u\,(x))\,\varphi = \varepsilon\varphi, \tag{1}$$

where u(x) is a periodic function. Equation (1) describes the behavior of an electron in the physically important case of a periodic lattice, to which one has applied a constant exterior electric field E.

The nonstationary Schrödinger equation

$$i\frac{\partial}{\partial t}\varphi + \frac{\partial^2\varphi}{\partial x^2} + e\,(Ex + u\,(x, t))\,\varphi = 0, \tag{2}$$

where u(x, t) is a periodic function with respect to x and t with periods T_1 and T_2, respectively, has been investigated in [1]. If the topological charge, i.e., the flux through the elementary cell

$$\Phi = \int_0^{T_1}\int_0^{T_2} (E + E_0\,(x, t))\,dx\,dt = ET_1T_2$$

($E_0(x, t)$ is a periodic electric field with zero mean, $E_0 = -\partial_x u(x, t)$), satisfies the integer-value condition $e\Phi = 2\pi N$, then one can introduce the "electric Bloch solutions." These solutions of (2) are proper for the group of translations

$$T_1^*\varphi\,(x, t) = \varphi\,(x + T_1, t)\exp\,(-ieEt),$$
$$T_2^*\,\varphi\,(x, t) = \varphi\,(x, t + T_2).$$

As shown in [1], even in the simplest examples, the analytic properties of the "electric Bloch solutions" are extremely interesting and so far they were not investigated at all. We note that if u does not depend on t, then, in general, the family of Bloch solutions is two-dimensional, unlike the case E = 0.

Equation (1) has been examined also in connection with the problem of the integration of the cylindrical Korteweg–de Vries (KdV) equation. For a sufficiently fast decreasing potential u, the direct and the inverse spectral transforms have been constructed in [2-4] (see also [5, 6]). A series of exact solutions of the cylindrical KdV equation (and the corresponding integrable cases (1)), decreasing at infinity, have been constructed in [2-6, 37] with the aid of the Darboux–Bäcklund transformation (see [36]). In all these investigations, the starting point has been the Airy functions, i.e., the solutions of the equation (1) for u = 0. These functions can be constructed with the aid of the well-known Laplace method [7].

In the present paper we give an algebrogeometric variant of a generalization of the Laplace method, using to a large extent the idea of "finite-zone integration," an outline of which can be found in [1, 8-15]. This generalization allows us to construct the integrable cases of Eq. (1) with $\varepsilon = 0$. The corresponding potentials u(x) tend, as $|x| \to \infty$, to periodic or quasiperiodic finite-zone potentials u_0 of the Sturm–Liouville operator [8, 9]. Moreover, the solutions of (1) are expressed in the form of quadratures of differentials on a Riemann surface Γ of the Bloch functions corresponding to the finite-zone Sturm–Liouville operator. These differentials, called in the sequel differentials of Laplace type, are determined

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uniquely by their specific analytic properties on Γ.

It turns out that the multiparameter generalization of the differentials of Laplace type allow us to construct for the equations of type Kadomtsev—Petviashvili (KP) new classes of solutions, depending on the functional parameters. Solutions of another type of the KP equations, also depending on the functional parameters, have been constructed in [12, 16, 17] with the aid of the methods of the theory of multidimensional holomorphic bundles over algebraic curves. These are the so-called finite-zone solutions of rank $l > 1$.

To the extent to which the finite-zone solutions of rank 1 are generalizations of the multisoliton solutions of the KP equation, the solutions, given by the construction of the present paper, are generalizations of the solutions, depending on the functional parameters, constructed in [18]. In a sufficiently general situation, the obtained solutions tend to the finite-zone solutions of rank 1 as $|x, y| \to \infty$. It should be noted that the "phase" of the finite-zone solution depends on the exit direction at infinity.

As called to the author's attention by V. E. Zakharov, such a behavior of the solutions indicates that they must describe the behavior of the "dislocations" in a periodic lattice. I would like to use this opportunity to express my gratitude to him and to S. P. Novikov for their interest in this paper and for their help with their advice.

The distribution of the material in the paper is the following. In the first section we introduce the differentials of Laplace type and we carry out the construction of the solutions of the KP type equations. In the second section we give explicit formulas in terms of theta functions for differentials of Laplace type and we find the asymptotic behavior of the constructed solutions. A proper algebrogeometric Laplace method for the equation (1) is given in Sec. 3. In Sec. 4 we give a brief outline of the possibility of the application of the developed method to difference and differential-difference systems.

1. Differentials of Laplace Type and the Kadomtsev—Petviashvili Equation

Let Γ be a nonsingular algebraic curve of genus g with a distinguished point P_0, in the neighborhood of which one has fixed a local parameter $k^{-1}(P)$, $k^{-1}(P_0) = 0$.

By differentials of Laplace type we shall mean differentials $\Lambda(\mathbf{x}, P)$

1) which are meromorphic on Γ outside the distinguished point P_0;

2) such that in the neighborhood of P_0 the differential

$$\omega_\infty = \Lambda(\mathbf{x}, P) \exp\left(-\sum_{n=1}^{M} x_n k^n(P)\right) \tag{1.1}$$

is meromorphic.

Here $\mathbf{x} = (x_1, \ldots, x_M)$ are arbitrary parameters.

Let $\psi(\mathbf{x}, P)$ be the Baker—Akhiezer function [19, 20], i.e., a function having outside P_0 g poles $\gamma_1, \ldots, \gamma_g$ and representable in the neighborhood of P_0 in the form

$$\psi(\mathbf{x}, P) = \exp\left(\sum_{n=1}^{M} x_n k^n(P)\right)\left(1 + \sum_{s=1}^{\infty} \xi_s(x) k^{-s}\right). \tag{1.2}$$

Then, for any differential Λ of Laplace type, the differential $\omega = \Lambda/\psi$ is meromorphic on Γ. From here, the dimension of the linear space of the differentials of Laplace type with given poles can be easily found with the aid of the Riemann—Roch [21] theorem. In the general case, this dimension is connected with the degree of the divisor of the poles by the relation $\dim \mathscr{L}(D) = \deg D + g - 1$ (the degree of the divisor is the number of points occurring in it with their multiplicities; the point P_0 occurs in D with a multiplicity equal to the order of the pole w_∞ and P_0).

We consider on Γ a collection of contours (not necessarily closed) σ_i, $i = 1, \ldots, N + g$ and a collection of functions on them $G_i(t)$, $t \in \sigma_i$. Then

LEMMA 1.1. For an arbitrary collection of points $\gamma_1, \ldots, \gamma_N$ of general position, there exists a unique differential $\Lambda(\mathbf{x}, P)$ of Laplace type with poles at the points γ_i, having in the neighborhood of P_0 the form

$$\Lambda(\mathbf{x}, P) = \exp\left(\sum x_n k^n\right)\left(1 + \sum_{s=1}^{\infty} \chi_s(\mathbf{x}) k^{-s}\right) dk \tag{1.3}$$

and normalized by the conditions

$$\int_{\sigma_i} G_i(t)\,\Lambda(\mathbf{x}, t) = 0, \qquad i = 1, \ldots, N + g. \tag{1.4}$$

The proof of the lemma follows from the simple comparison of the number of linear equations (1.4) and the dimension of the space of the differentials of Laplace type with the divisor of poles $(\gamma_1 + \ldots + \gamma_{N+q} + 2P_0)$ (the differential dk has at P_0 a pole of order two).

THEOREM 1.1. There exist unique operators

$$L_n = \frac{\partial^n}{\partial x^n} + \sum_{i=0}^{n-2} u_i(\mathbf{x}) \frac{\partial^i}{\partial x^i} \tag{1.5}$$

such that

$$L_n \Lambda(\mathbf{x}, P) = \frac{\partial}{\partial x_n} \Lambda(\mathbf{x}, P), \tag{1.6}$$

where $x = x_1$.

The proof of this theorem is practically identical with the proof of the analogous statements for the Baker–Akhiezer functions [19, 20]. Indeed, as mentioned in these papers, for any formal series (1.4) there exists a unique operator L_n such that

$$\left(\frac{\partial}{\partial x_n} - L_n\right)\Lambda = \exp\left(\sum x_n k^n\right) O(k^{-1})\, dk. \tag{1.7}$$

The differential $\tilde{\Lambda} = \left(\frac{\partial}{\partial x_n} - L_n\right)\Lambda$ satisfies all the requirements which define Λ, except one of them. At the point P_0, by virtue of (1.7), the differential $\tilde{\Lambda}$ has a pole of order at most one. From the uniqueness of Λ, there follows that $\tilde{\Lambda} = 0$.

The coefficients of operator L_n are differential polynomials of $\chi_i(\mathbf{x})$. In particular, $u_{n-2} = -n \frac{\partial}{\partial x} \chi_1$.

COROLLARY. The constructed operators satisfy the equations

$$\left[\frac{\partial}{\partial x_n} - L_n, \frac{\partial}{\partial x_m} - L_m\right] = 0. \tag{1.8}$$

Example. We denote $x_2 = y$, $x_3 = t$. The operators L_2 and L_3 have the form

$$L_2 = \frac{\partial^2}{\partial x^2} + u(x, y, t); \quad L_3 = \frac{\partial^3}{\partial x^3} + \frac{3}{2} u \frac{\partial}{\partial x} + w(x, y, t).$$

From (1.8) there follows that $u(x, y, t) = -2 \frac{\partial}{\partial x} \chi_1(x, y, t)$ is a solution of the KP equation

$$\frac{3}{4} u_{yy} = \frac{\partial}{\partial x}\left(u_t + \frac{1}{4}(6uu_x - u_{xxx})\right). \tag{1.9}$$

(Representation (1.8) for the KP equation has been obtained in [18, 22].) We show that in the particular case the presented construction contains the solutions of KP, depending on the functional parameter, which have been constructed in [18].

Let Γ be a rational curve, i.e., a complex plane. The differential $\Lambda(x, y, t, k)$, having a unique pole at the point $\varkappa$ and the form (1.3) at infinity, is equal to

$$\Lambda(x, y, t, k) = \frac{k + a}{k - \varkappa} e^{kx + k^2 y + k^3 t}.$$

The coefficient $a(x, y, t)$ is obtained from condition (1.4). If one denotes by $\varphi(x, y, t)$, $f(x, y, t)$

$$\varphi = \int_\sigma G(k)\, e^{kx+k^2y+k^3t}; \qquad f = \int_\sigma \frac{G(k)}{k-\varkappa}\, e^{kx+k^2y+k^3t},$$

then $\frac{\partial}{\partial x}(f(x, y, t)\, e^{-\varkappa x}) = \varphi(x, y, t)\, e^{-\varkappa x}$. Since

$$\varphi(x, y, t) + (a-\varkappa) f(x, y, t) = 0,$$

the expression for the solution of the KP

$$u(x, y, t) = -2\frac{\partial}{\partial x} a = 2\frac{\partial}{\partial x}\frac{\varphi(x, y, t)}{f(x, y, t)}$$

coincides, except for notations, with the formula in [18].

We assume that on the curve Γ one has an antiholomorphic involution $\tau\colon \Gamma \to \Gamma$, leaving the point P_0 fixed. In addition, assume that the action of this involution on the local parameter is such that $\tau^*(k) = \bar{k}$.

LEMMA 1.2. If the collection of the poles $\gamma_1, \ldots, \gamma_N$ is invariant relative to τ and if the data σ_i, G_i are invariant relative to τ (i.e., $\tau(\sigma_i) = \sigma_j$ and $\bar{G}_j(\tau(t)) = G_i(t)$), then the operators L_n, constructed by virtue of Theorem 1.1 from the differential $\Lambda(x, P)$ corresponding to this collection of data, are real (for real x_n).

We consider the differential $\bar{\Lambda}(x, \tau(P))$. It satisfies all the requirements which define $\Lambda(x, P)$. From the uniqueness of the latter it follows that $\bar{\Lambda}(x, \tau(P)) = \Lambda(x, P)$ and the assertion of the lemma is proved.

Remark. For the majority of equations, even within the framework of the traditional finite-zone integration, the problem of the selection of the real nonsingular solutions is nontrivial. In recent years, significant progress in this direction has been achieved by Dubrovin (see [15] for the history of this question and for bibliography). Within the framework of the presented construction of a test for reality, a similar reality test for the sine-Gordon equation and for the nonlinear Schrödinger equation, given in [22] for the case of the so-called KP-II equation, has not been obtained. Equation KP-II differs from KP-I (1.9) by the substitution $y \to iy$.

Later in this section and in the following one, fundamental attention will be given to differentials of Laplace type which are holomorphic outside P_0 and have in the neighborhood of P_0 the form (1.3). Formally, in the corresponding class of solutions one also has usual finite-zone solutions of rank 1. If one assumes that $G_i(k)$ is a delta function, then conditions (1.4) yield

$$\Lambda(x, \tilde{\gamma}_i) = 0. \tag{1.10}$$

(Here $i = 1, \ldots, g$.) There exists a unique differential of the second kind ω on Γ, having a unique pole of order two at P_0, $\omega \sim dk(1 + O(k^{-1}))$ and such that $\omega(\tilde{\gamma}_i) = 0$. In addition to $\tilde{\gamma}_i$, the differential ω has also g zeros $\gamma_1, \ldots, \gamma_g$. We consider the Baker–Akhiezer function having poles at the points $\gamma_1, \ldots, \gamma_g$. Then the Laplace type differential, normalized by the conditions (1.10), is equal to

$$\Lambda(x, P) = \psi(x, P)\, \omega(P), \tag{1.11}$$

since both sides of the equality have identical analytic properties. The differential ω does not depend on x and, therefore, $\psi(x, P)$ satisfies the same equation (1.6) as $\Lambda(x, P)$.

Assume that the antiholomorphic involution τ has $g + 1$ fixed ovals $a_1, \ldots, a_{g+1}$ on Γ. (In algebraic geometry, such curves are called M-curves. As a fundamental example of such curves, one can consider in the sequel hyperelliptic curves.) We shall assume that $P_0 \in a_{g+1}$.

THEOREM 1.2. Assume that the contours σ_i coincide with the cycles a_i and assume that the functions G_i are real and positive. Then the solutions of the equations (1.8), given by

Theorem 1.1 in the case of differentials $\Lambda(x, P)$, holomorphic outside P_0, are real and nonsingular.

Proof. The reality of the constructed solutions has been proved above. The absence of the singularities is equivalent to the fact that none of the 2g zeros of the differential $\Lambda(x, P)$ is at the point P_0. Since the ovals a_i are fixed relative to τ, while $\Lambda(x, P) = \overline{\Lambda}(x, \tau(P))$, it follows that the differential Λ is real on these cycles. From the normalization condition (1.4) and the positivity of G_i there follows that Λ has on every cycle at least two zeros. Since there are 2g zeros in all, it follows that they are isolated from P_0, and the theorem is proved.

2. Explicit Formulas and the Asymptotics of the Constructed Solutions

The construction of explicit formulas for the differentials of Laplace type is completely similar to the construction of formulas for the general Baker—Akhiezer function [20] (a similar formula for Bloch functions of finite-zone Sturm—Liouville operators has been obtained for the first time by Its [29]).

We fix on the curve Γ a canonical basis of cycles a_i, b_j with intersection matrix $a_i\circ a_j = b_i\circ b_j = 0,\ a_i \circ b_j = \delta_{ij}$. In a standard manner one derives: a basis of the holomorphic differentials ω_i, normalized by the conditions $\oint_{a_i} \omega_j = \delta_{ij}$; the matrix of the b-periods of these differentials $B_{ik} = \oint_{b_k} \omega_i$, and the corresponding Riemann theta function

$$\theta(u) = \sum_{m\in Z^g} \exp(2\pi i \langle m, u\rangle + \pi i \langle Bm, m\rangle) \tag{2.1}$$

$$u = (u_1, \ldots, u_g), \quad m = (m_1, \ldots, m_g), \quad \langle m, u\rangle = m_1u_1 + \ldots + m_gu_g, \quad \langle Bm, m\rangle = \sum B_{ij}m_im_j.$$

(For more details on the theta function see [10, 13, 15].) The vectors δ_{ik} and B_{ik} form a lattice in C^g. A factor with respect to this lattice is called the Jacobian manifold $J(\Gamma)$ of the curve Γ. The Abel mapping $A: \Gamma \to J(\Gamma)$ is defined by the formula

$$A_k(P) = \int_Q^P \omega_k. \tag{2.2}$$

We denote by $\Omega^{(n)}$ the normalized Abelian second-order differentials, having a unique pole at P_0 of the form dk^{n+1}. The normalization of the differential means that

$$\oint_{a_i} \Omega^{(n)} = 0. \tag{2.3}$$

The general (nonnormalized) Abelian differential with a second-order pole at P_0 can be represented with the aid of the so-called prime-forms [30] (see the appendix of [15]) in the form

$$\widetilde{\Omega}^{(1)} = \frac{\theta(A(P)-\zeta)\,\theta(A(P)+\zeta)}{\theta[\nu](A(P)-A(P_0))\,\theta[\nu](A(P_0)-A(P))}\Big(\sum_j \omega_j(P)\,\theta_j[\nu](0)\Big). \tag{2.4}$$

Here $\theta[\nu](z)$ is a theta function with characteristic $\nu = (\alpha, \beta)$; $\alpha, \beta \in R^g$

$$\theta[\alpha, \beta](z) = \exp(\pi i \langle B\alpha, \alpha\rangle + 2\pi i \langle \alpha, z+\beta\rangle)\theta(z+\beta+B\alpha). \tag{2.5}$$

The vector ζ in (2.4) is arbitrary and the characteristic $[\nu]$ has to be of odd semiperiod, for example, $\alpha = (1/2, 0, \ldots, 0)$, $\beta = (1/2, 0, \ldots, 0)$.

LEMMA 2.1. A differential of Laplace type, holomorphic outside P_0 and having in the neighborhood of P_0 the form (1.3), can be represented in the following form:

$$\Lambda = r(\mathbf{x}) \exp(S(\mathbf{x}, P)) \frac{\theta(A(P) + \langle \mathbf{U}, \mathbf{x}\rangle - \zeta)\,\theta(A(P)+\zeta)}{\theta[\nu](A(P) - A(P_0))\,\theta[\nu](A(P_0) - A(P))} \Big(\sum_j \omega_j(P)\,\theta_j[\nu](0)\Big); \tag{2.6}$$

$$\langle \mathbf{U}, \mathbf{x}\rangle = \sum_n x_n U^{(n)},$$

$$U^{(n)} = (U_1^{(n)}, \ldots, U_g^{(n)}), \qquad U_k^{(n)} = \frac{1}{2\pi i} \oint_{b_k} \Omega^{(n)},$$

where $S(\mathbf{x}, P) = \sum x_n \int_Q^P \Omega^{(n)}$. The normalizing factor $r(\mathbf{x})$ is equal to the value at P_0 of the pre-exponential factor in (2.6). The vector ζ is arbitrary.

The proof of the lemma, just as the proof of the corresponding assertions for the Baker–Akhiezer functions [20], consists in the straightforward verification of the fact that, by virtue of the translation properties of the theta functions, the value of (2.6) does not change when one goes around any cycle on Γ, i.e. (2.6) defines correctly the differential on Γ. In addition, from the definition of $\Omega^{(n)}$ there follows that the differential (2.6) has the required analytic properties.

We select an arbitrary collection of vectors ζ_i, $i = 1, \ldots, g+1$, and we denote by Λ_i the differentials, given by formula (2.6), in which one has set $\zeta = \zeta_i$. Any differential Λ of the form (2.6) can be represented uniquely in the form

$$\Lambda(\mathbf{x}, P) = \sum_{i=1}^{g+1} \alpha_i(\mathbf{x})\,\Lambda_i(\mathbf{x}, P). \tag{2.7}$$

We define the constants α_i from the system of linear equations

$$0 = \sum_i \alpha_i(\mathbf{x}) \int_{\sigma_j} G_j(t)\,\Lambda_i(\mathbf{x}, t), \quad j = 1, \ldots, g,$$
$$\sum_i \alpha_i(\mathbf{x}) = 1. \tag{2.8}$$

The corresponding differential Λ will satisfy the conditions (1.4).

For the expansion of Λ_i in the neighborhood of P_0 one has to make use of the relation $A(P) = A(P_0) - U^{(1)}k^{-1} + O(k^{-2})$.

<u>COROLLARY OF THEOREM 1.1.</u> The function

$$u(x, y, t) = 2\frac{\partial}{\partial x} \sum \alpha_i(x, y, t) \frac{\partial}{\partial x} \ln\theta(U^{(1)}x + U^{(2)}y + U^{(3)}t - \zeta_i) \tag{2.9}$$

is a solution of the KP equation.

We find the asymptotic behavior of the obtained solutions of the KP equation under the assumptions of Theorem 1.2 (ensuring the reality and the nonsingularity of these solutions).

Let r, φ, θ_1 be the spherical coordinates of the three-dimensional vector x, y, t. We denote by $\tilde{S}(\varphi, \theta_1, P) = \int_Q^P d\tilde{S}(\varphi, \theta_1, P)$ an Abelian integral on Γ such that

$$d\tilde{S} = \cos\theta_1(\cos\varphi\,\Omega^{(1)} + \sin\varphi\,\Omega^{(2)}) + \sin\theta_1\,\Omega^{(3)}. \tag{2.10}$$

The differential $d\tilde{S}$ has two zeros on each of the a-cycles. One of them corresponds to the maximum of $\tilde{S}$ on this cycle, while the second one corresponds to the minimum. Let $Z(\varphi, \theta_1)$ be a vector such that $\theta(A(P) - Z(\varphi, \theta_1))$ vanishes at the points $\tilde{\gamma}_i$ of the maxima on the cycles a_i. The vector $Z(\varphi, \theta_1)$ is equal to

$$Z(\varphi, \theta_1) = \sum_i A(\tilde{\gamma}_i) + \mathscr{K}.$$

THEOREM 2.1. Under the assumptions of Theorem 1.2, the solutions of the KP equation, given by formulas (2.8), (2.9), for $r \to \infty$ have the form

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\ln\theta(U^{(1)}x + U^{(2)}y + U^{(3)}t + Z(\varphi, \theta_1)) + O(e^{-\mathrm{const}\, r}). \tag{2.11}$$

Proof. In order to prove the theorem it is necessary to switch from representation (2.7) for the differential Λ to formula (2.6). The parameter ζ in formula (2.5) must be determined from conditions (1.4).

For the computation of integrals (1.4) we make use of considerations from the saddle-point method. Under the assumptions of Theorem 1.2, the vectors $U^{(n)}$ are real. The theta function is periodic relative to the shifts of their arguments by the vector (0, ..., 0, 1, 0, ..., 0). It follows that the factor of the exponent in (2.6) is a quasiperiodic function of x, y, t. For the vanishing of the integrals (1.4) as $r \to \infty$, it is necessary that the zeros of the differential Λ should tend exponentially to the points of maximum of $\tilde{S}(\varphi, \theta_1, P)$ on a_i. Consequently, $\zeta \to Z(\varphi, \theta_1)$ and, expanding (2.6) in the neighborhood of P_0, we obtain the expression (2.11).

This formula has the form of a usual finite-zone solution, which is consistent with the remark made in the first section regarding the fact that conditions (1.10), instead of (1.4), lead to finite-zone solutions of rank 1.

At the conclusion of this section we consider as an example the case of an elliptic curve Γ with periods 2ω, $2\omega'$. In this case, formula (2.6) can be represented as

$$\Lambda(x, y, t, z) = \exp\left(\zeta(z)x + \wp(z)y - \frac{1}{2}\wp'(z)t\right)\frac{\sigma(z+x-a)\sigma(z+a)}{\sigma^2(z)\sigma(x-a)\sigma(a)}, \quad z \in \Gamma. \tag{2.12}$$

Here σ, ζ, $\wp$ are the Weierstrass functions (all the necessary information regarding these can be found in [31]).

We fix a couple of complex numbers a_i and we denote by Λ_i the differentials (2.12) in which $a = a_i$.

A special case of the preceding theorems is the following assertion.

The function

$$u(x, y, t) = 2\frac{\partial}{\partial x}\frac{\left(\oint G(z)\Lambda_1\right)\zeta(x-a_1) - \left(\oint G(z)\Lambda_2\right)\zeta(x-a_2)}{\left(\oint G(z)(\Lambda_1 - \Lambda_2)\right)}, \quad \Lambda_i = \Lambda_i(x, y, t, z) \tag{2.13}$$

is a solution of the KP equation. Here the integrals are taken from $\omega' - \omega$ to $\omega' + \omega$.

If the period ω is real and ω' is pure imaginary, then for the real positive function G(z) this formula gives a real and nonsingular solution (the assumptions made are equivalent to the assumptions of Theorem 1.2). For $r \to \infty$, where r, φ, θ are the spherical coordinates of x, y, t, we have x, y, t, $u(x, y, t) \to 2\wp(x + Z(\varphi, \theta))$, where $Z(\varphi, \theta)$ is the maximum on the cycle $[\omega' - \omega, \omega' + \omega]$ of the function

$$\cos\theta(\cos\varphi(\zeta(z)2\omega - 2\eta z) + \wp(z)\sin\varphi) - \frac{1}{2}\sin\theta\wp'(z),$$

and is a solution of the equation

$$-\cos\theta(\cos\varphi(\wp(z)2\omega + 2\eta) - \wp'(z)\sin\varphi) - \frac{1}{2}\sin\theta\wp''(z) = 0.$$

3. Algebraic-Geometric Laplace Method

In the previous sections, we have basically restricted ourselves to the consideration of differentials of Laplace type, holomorphic outside P_0. In this section, we shall consider some other differentials.

A special case of Lemma 1.1 is the following assertion. For any collection of points $\gamma_i(\varepsilon, E)$ of general position, there exists a unique differential $\Lambda^*(x, \varepsilon, E, P)$, exact outside

P_0, with poles of order two at the points $\gamma_s(\varepsilon, E)$. In the neighborhood of P_0 this differential has the form

$$\Lambda^* = \exp\left(k\left(x - \frac{\varepsilon}{E}\right) - \frac{k^3}{3E}\right) dk\left(1 + \sum_{s=1}^{\infty} \chi_s(x, \varepsilon, E) k^{-s}\right). \tag{3.1}$$

For comparison with Lemma 1.1 we note that the divisor of the poles of Λ^* has degree $2g + 2$. The condition of the exactness of Λ^* means that $\oint_{\sigma_i} \Lambda^* = 0$, where σ_i is the collection of a- and b-cycles on Γ and cycles surrounding the points γ_s.

The differential Λ^* depends on the functional parameters $\gamma_s(\varepsilon, E)$, which will be determined later (see (3.17)). For the time, we formulate a theorem, valid for any choice of the parameters.

Let $\psi(x, \varepsilon, E, P)$ be the Baker–Akhiezer function having poles at the points $\gamma_s(\varepsilon, E)$ and representable in the neighborhood of P_0 in the form

$$\psi = \exp\left(k\left(x - \frac{\varepsilon}{E}\right) - \frac{k^3}{3E}\right)\left(1 + \sum_{s=1}^{\infty} \xi_s(x, \varepsilon, E) k^{-s}\right). \tag{3.2}$$

We denote by $c(x, \varepsilon, E)$ and $u(x, \varepsilon, E)$ functions such that

$$\frac{c'}{c} = \frac{1}{2}(\xi_1 - \chi_1), \tag{3.3}$$

$$u = 2\chi_1' + \chi_1(\xi_1 - \chi_1) + \frac{1}{2}(\xi_1' - \chi_1') + \frac{1}{4}(\xi_1 - \chi_1)^2. \tag{3.4}$$

THEOREM 3.1. The differential $\Lambda^*(x, \varepsilon, E, P)$ satisfies the equation

$$-(c\Lambda^*)'' + (Ex - \varepsilon + u)(c\Lambda^*) = Ec\, d\psi. \tag{3.5}$$

The proof of the theorem is standard. The differential $\tilde{\Lambda}$, equal to the difference between the right- and left-hand sides of equality (3.5), is exact outside P_0 and has possible poles of order two at the points γ_s. From (3.1)-(3.4) there follows that in the neighborhood of P_0 it has the form

$$\tilde{\Lambda} = \exp\left(k\left(x - \frac{\varepsilon}{E}\right) - \frac{k^3}{3E}\right) dk\,(O(k^{-1})).$$

From the uniqueness of Λ^* we have $\tilde{\Lambda} = 0$, and the equality (3.5) is proved.

In the neighborhood of P_0 we consider the sectors in which $\operatorname{Re} k^3/3E > 0$. Assume that they are numbered as in Fig. 1.

If σ_i, $i = 1, 2$ are contours emanating from P_0 in sector III and going to P_0 in the sectors I and II, respectively, then for $P \to P_0$ we have on these contours $\psi(x, \varepsilon, E, P) \to 0$. From here, integrating (3.5), we obtain

COROLLARY. The functions $Y_i(x, \varepsilon, E)$,

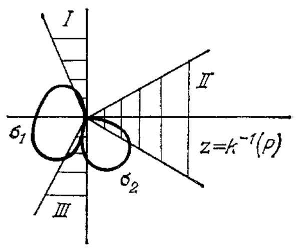

Fig. 1

$$Y_i = \frac{1}{\pi} \int_{\sigma_i} (c\Lambda^*(x, \varepsilon, E, P)), \quad c = c(x, \varepsilon, E),$$

satisfy the equation

$$-Y'' + (Ex + u(x, \varepsilon, E))\, Y = \varepsilon Y. \tag{3.6}$$

Now, under the assumptions of Theorem 1.2, we find the asymptotic behaviors of the potentials $u(x, \varepsilon, E)$ and of the corresponding solutions $Y_i(x, \varepsilon, E)$.

LEMMA 3.1. Any differential $\Lambda(x, \varepsilon, E, P)$, having form (3.1) in the neighborhood of P_0 and poles of order two at the points $\gamma_s(\varepsilon, E)$, can be represented uniquely in the form

$$\Lambda = r(x, E, \varepsilon) \cdot \exp(S) \frac{\left(\sum_j \omega_j(P)\, \theta_j[\nu](0)\right)}{\theta[\nu](A(P) - A(P_0))\, \theta[\nu](A(P_0) - A(P))} \; \frac{\prod_{i=1}^{4} \theta(A(P) + \zeta_i)}{\theta^2(A(P) + \zeta_0(\varepsilon, E))}, \tag{3.7}$$

where

$$S = S(x, \varepsilon, E, P) = \left(x - \frac{\varepsilon}{E}\right) \int_Q^P \Omega^{(1)} - \frac{1}{3E} \int_Q^P \Omega^{(3)}, \tag{3.8}$$

$$\sum_{i=1}^{4} \zeta_i = 2\zeta_0 + \left(x - \frac{\varepsilon}{E}\right) U^{(1)} - \frac{1}{3E} U^{(3)}. \tag{3.9}$$

The vector $-\zeta_0$, within the accuracy of a shift by the vector of the Riemann constants, is equal to the Abel mapping of the points $\gamma_s(\varepsilon, E)$

$$\zeta_0(\varepsilon, E) = -\sum_s A(\gamma_s(\varepsilon, E)) + \mathcal{K}. \tag{3.10}$$

The proof of the lemma reduces to a straightforward verification of the correctness on Γ of the differential defined by the formula (3.10).

For $|x| \to \infty$, the absence of the residues of Λ^* at the points $\gamma_s(\varepsilon, E)$ yields

$$\zeta_1 = \zeta_0 + O\left(\frac{1}{|x|}\right). \tag{3.11}$$

The vanishing of the integrals of Λ^* along the a-cycles means that the g zeros of Λ^* tend exponentially to the zeros of $\Omega^{(1)}$, corresponding to the maxima of $\int\Omega^{(1)}$ on these cycles. Similarly, the vanishing of the integrals of Λ^* along the b-cycles means that Λ^* tends to zero also at the other zeros of $\Omega^{(1)}$. From here it follows that with exponential accuracy we have

$$\Lambda^* = r(x, \varepsilon, E) \exp(S)\, \Omega^{(1)}(P) \frac{\theta(A(P) + \zeta_1)\, \theta(A(P) + \tilde{\zeta})}{\theta^2(A(P) + \zeta_0)}, \tag{3.12}$$

where

$$\tilde{\zeta} = 2\zeta_0 - \zeta_1 + \left(x - \frac{\varepsilon}{E}\right) U^{(1)} - \frac{1}{3E} U^{(3)}. \tag{3.13}$$

Expanding (3.12) in the neighborhood of P_0 (as well as the analogous formula for

$$\psi = r_1(x, \varepsilon, E) \exp(S) \frac{\theta\left(A(P) + \zeta_0 + \left(x - \frac{\varepsilon}{F}\right) U^{(1)} - \frac{1}{3E} U^{(3)}\right)}{\theta(A(P) + \zeta_0)}\Bigg),$$

we obtain

$$\chi_1 = \xi_1 + O\left(\frac{1}{|x|}\right), \tag{3.14}$$

$$\chi_2 = \xi_2 + O\left(\frac{1}{|x|}\right) \tag{3.15}$$

and

$$u(x, \varepsilon, E) = 2\frac{\partial^2}{\partial x^2}\ln\theta(A(P_0)+\tilde{\zeta}). \tag{3.16}$$

Now we define the relation $\gamma_s(\varepsilon, E)$ so that the vector ζ_0 in (3.10) should satisfy the condition

$$A(P_0)+\zeta_0-\frac{\varepsilon}{E}U^{(1)}-\frac{1}{3E}U^{(3)}=\zeta=\text{const.} \tag{3.17}$$

Summing up, we obtain the following statement.

THEOREM 3.2. Assume that the Abel mapping of the divisor $\gamma_s(\varepsilon, E)$ satisfies relation (3.17). Then the potential $u(x, \varepsilon, E)$, corresponding by Theorem 3.1 to these data, for $|x| \to \infty$ behaves as

$$u(x, \varepsilon, E) = 2\frac{\partial^2}{\partial x^2}\ln\theta(xU^{(1)}+\xi). \tag{3.18}$$

The computation of the asymptotics of $Y_i(x, \varepsilon, E)$ is carried out similarly to the computation of the asymptotics of the Airy function (see, for example, the appendix in [32]). For $x \to +\infty$ the exponent has extremes at the points $\pm z_0 = \pm(Ex)^{-1/2}$, $z = k^{-1}(P)$. We deform the contours σ_i so that σ_1 should intersect the axis at the point $-z_0$ and the contour σ_2 should coincide in the neighborhood of z_0 with the real axis. The difference with the computations of the asymptotics of the Airy functions consists only in the presence of a preexponential factor in the expression (3.12).

After simple computations we have

$$Y_1=\frac{1}{2\sqrt{\pi}\,y^{1/4}}e^{-2/3y^{3/2}}\left(1-y^{-1/2}\left(\frac{\partial}{\partial z}\ln\theta(Uz+\zeta)\Big|_{z=0}^{z=y}\right)+O(y^{-1})\right), \tag{3.19}$$

$$Y_2=\frac{1}{2\sqrt{\pi}\,y^{1/4}}e^{2/3y^{3/2}}\left(1+y^{-1/2}\left(\frac{\partial}{\partial z}\ln\theta(Uz+\zeta)\Big|_{z=0}^{z=y}\right)+O(y^{-1})\right), \tag{3.20}$$

where $y = E^{1/3}x$.

Remark. One can see that expansions (3.19), (3.20) coincide with the formal asymptotic series which can be obtained for Eq. (3.7) following the general scheme [33] of the construction of the WKB expansions. The same assertion refers also to the following formulas. Regarding this comparison one has to note that

$$v(y)=\frac{\partial}{\partial z}\ln\theta(Uz+\zeta)\Big|_{z=0}^{z=y}\approx\int_0^y u(x, \varepsilon, E)\,dx.$$

For $x \to -\infty$ the extremals of the exponent are attained at the points $\pm z_0 = \pm i(Ex)^{-1/2}$. Performing a standard deformation of the contours σ_i so that they should pass through $\pm z_0$ and should intersect the imaginary axis in the direction of the "steepest descent" (under the angle $\pm\pi/4$), we obtain

$$Y_1=\frac{1}{\sqrt{\pi}\,|y|^{1/4}}\left[\sin\left(\frac{2}{3}|y|^{3/2}+\frac{\pi}{4}\right)\left(1+O(|y|^{-1})\right)+|y|^{-1/2}\cos\left(\frac{2}{3}|y|^{3/2}+\frac{\pi}{4}\right)v(y)\left(1+O(|y|^{-1})\right)\right], \tag{3.21}$$

$$Y_2=\frac{1}{\sqrt{\pi}\,|y|^{1/4}}\left[\cos\left(\frac{2}{3}|y|^{3/2}+\frac{\pi}{4}\right)\left(1+O(|y|^{-1})\right)-|y|^{-1/2}\sin\left(\frac{2}{3}|y|^{3/2}+\frac{\pi}{4}\right)v(y)\left(1+O(|y|^{-1})\right)\right]. \tag{3.22}$$

Remark. Both the Laplace method and its algebrogeometric variant, presented above, allow us to construct the solutions of the corresponding linear equations only for one value of the "spectral parameter" ε. Nevertheless, it is possible, making use of the above obtained results, to construct in analogy with the fast-descending case the direct and the inverse spectral transformation for the equation

$$-\varphi''+(Ex+u_0(x))\varphi=\varepsilon\varphi, \tag{3.23}$$

where $u_0(x)$ is a finite-zone potential. The complete presentation of these results and the analysis of other possibilities in the algebrogeometric Laplace method will be published in the second part of this investigation.

4. Differential-Difference Systems

As it is known, the methods of algebraic geometry allow to construct the periodic and quasiperiodic solutions not only for partial differential equations but also for certain differential-difference systems. As an example we consider the construction of the solutions of the two-dimensionalized Toda chain

$$\varphi_{n\xi\eta}=e^{\varphi_n-\varphi_{n-1}}-e^{\varphi_{n+1}-\varphi_n}. \tag{4.1}$$

Here $\xi = x + t$, $\eta = x - t$ are the conic variables. These equations, just as the general equations of the principal chiral field, possess a remarkable property: the method of the inverse problem allows us to reduce to the linear Riemann problem the construction of the general solutions of the periodic, $\varphi_n = \varphi_{n+N}$, two-dimensionalized Toda chain [14, 23]. (The construction of the solutions for these equations, depending on the required number of functional parameters, has been presented also in [24]. The solution has been represented in the form of series, whose convergence has been proved with the methods of the infinite-dimensional Lie algebras. The relationship between the functional parameters of [24] and the initial data has been obtained in [25].) A large class of periodic and quasiperiodic solutions of (4.1), expressed in terms of the Riemann theta function, have been obtained by the author (see the appendix to [13]).

Below, making use of an analogue of the constructions of Sec. 1, we shall construct solutions of equation (4.1) which tend asymptotically to the finite-zone solutions.

Equation (4.1) has been obtained in [26] with the aid of the Zakharov–Shabat two-dimensionalization of the Lax pair

$$L\psi_n=c_n\psi_{n+1}+v_n\psi_n+c_{n-1}\psi_{n-1}, \tag{4.2}$$

$$A\psi_n=\frac{c_n}{2}\,\psi_{n+1}-\frac{c_{n-1}}{2}\,\psi_{n-1}, \tag{4.3}$$

for the equations of the Toda chain $\ddot{x}_n=e^{x_{n+1}-x_n}-e^{x_n-x_{n-1}}$. Here $c_n^2=e^{x_{n+1}-x_n}$, $v_n=\dot{x}_n$.

Remark. Due to the absence of a historical survey of discrete systems in the author's paper [14], we shall consider this question in more detail.

The first sequential construction of the algebrogeometric Bloch–Floquet spectral theory in $\mathscr{L}_2(Z)$ of the Schrödinger difference operator (4.2) has been started by Novikov [8, Chap. III, Sec. 1] and by Tanaka–Date [27]. Making use of the trace formula for the function $\chi_n = \psi_{n+1}/\psi_n$, one has obtained the formulas for v_n. In [8] one has also investigated the symmetric case $v_n \equiv 0$. However, in [8] this theory has been carried out to the end only in the elliptic case. We note that the operator L in the Lax pair for the discrete KdV equation has the form (4.2), where $v_n = 0$ [34].

In [27] the expressions for v_n have been written in the following form (in [8], this formula is not correct; it has been corrected in [9]):

$$v_n=\frac{d}{dt}\ln\frac{\theta((n-1)U+Vt+W)}{\theta(nU+Vt+W)}+\text{const}. \tag{4.4}$$

In the case of the Toda chain, by virtue of the condition $\dot{x}_n = v_n$, the formula (4.4) determines $x_n(t)$, except for the collection of numbers $x_n(0)$, $-\infty < n < \infty$. In [27], the question of the KdV difference equation has not been considered.

These investigations have been concluded in the author's paper [35], where one has obtained explicit expressions for x_n and solutions of the KdV difference equation. The idea in [35] consists in the use of the explicit expressions for ψ_n in terms of theta functions, in contrast to [8, 27], where, as already mentioned, one has made use of the trace formula for χ_n, similar to the continuous case. In a more recent paper [36] there are explicit "local trace identities" $c_n = c_n(\gamma_1, \ldots, \gamma_g)$, whose existence have been inefficiently proved in [8].

Thus, assume that on the curve Γ we have two distinguished points $P^{\pm}$ with local parameters $k_{\pm}^{-1}$ in their neighborhoods.

LEMMA 4.1. Let $\gamma_1, \ldots, \gamma_N$ be an arbitrary collection of points of general position. There exist unique differentials $\Lambda_n(\xi, \eta, P)$ that

1°) are mermorphic on Γ outside $P^{\pm}$ and have poles at the points $\gamma_1, \ldots, \gamma_N$;

2°) in the neighborhoods of $P^{\pm}$ have the form

$$\Lambda_n^{\pm} = \pm dk^{\pm 1} k^{\pm(n-1)} \left(\sum_{s=0}^{\infty} \chi_s^{\pm}(\xi, \eta, n) k^{-s}\right) e^{k(x \pm t)}; \tag{4.5}$$

3°) satisfy the normalization conditions

$$\int_{\sigma_i} G_i(P) \Lambda_n(\xi, \eta, P) = 0, \quad P \in \sigma_i, \quad i = 1, \ldots, N+g, \tag{4.6}$$

$$\chi_0^{+}(\xi, \eta, n) \equiv 1. \tag{4.7}$$

THEOREM 4.1. The functions $\varphi_n = \ln \chi_0^{-}(\xi, \eta, n)$ satisfy the equations of the two-dimensionalized Toda chain (4.1).

The proof of the theorem is entirely similar to the proof of the corresponding assertions of Sec. 1. From the uniqueness of Λ_n and from the comparison of the principal parts of the expansions at $P^{\pm}$ of the right- and left-hand sides of the following equalities there follows that these equalities take place indeed:

$$\partial_\xi \Lambda_n = \Lambda_{n+1} + \varphi_{n\xi} \Lambda_n, \tag{4.8}$$

$$\partial_\eta \Lambda_n = e^{\varphi_n - \varphi_{n-1}} \Lambda_{n-1}. \tag{4.9}$$

The consistency condition of (4.8) and (4.9) is equivalent to Eq. (4.1).

Clearly, practically all the results of the preceding sections can be carried over without fundamental modifications also to the discrete case. We bring without proof only parts of them.

LEMMA 4.2. The differentials $\tilde{\Lambda}_n(\xi, \eta, P)$, holomorphic outside $P^{\pm}$ and having in the neighborhoods of $P^{\pm}$ the form (4.5), can be uniquely represented in the form

$$\tilde{\Lambda}_n = r_n(\xi, \eta) \exp(S(n, \xi, \eta, P)) \left(\sum_i{}' \omega_i(P) \theta_i[\nu](0)\right) \frac{\theta(A(P) + Un + V^+\xi + V^-\eta + \zeta)\, \theta(A(P) - A(P^+) - A(P^-) - \zeta)}{\theta[\nu](A(P) - A(P^+))\, \theta[\nu](A(P^-) - A(P))}, \tag{4.10}$$

$$S(n, \xi, \eta, P) = \int_Q^P (n\Omega_3 + \xi\Omega_+ + \eta\Omega_-),$$

where $\Omega_{\pm}$ are normalized differentials of the second kind with second-order poles at $P^{\pm}$; Ω_3 is a normalized differential of the third kind with simple poles at $P^{\pm}$ and residues ± 1. The vectors $2\pi iU$, $2\pi V^{\pm}$ are the vectors of the b-periods of the differentials Ω_3, $\Omega_{\pm}$, respectively. The normalizing function $r_n(\xi, \eta)$ is determined from the condition (4.7) and is equal to

$$r_n^{-1}(\xi, \eta) = \theta(A(P^+) + Un + V^+\xi + V^-\eta + \zeta)\, e^{d + nI_0 + I_+\xi + I_-\eta} \cdot d,\ I_0,\ I_{\pm} - \text{const}. \tag{4.11}$$

The vector ζ, which is arbitrary in formula (4.10), must be determined from the normalization conditions (4.6). The corresponding nonlinear equations can be reduced to linear ones if one denotes by Λ_{nj} the differentials given by the formula (4.10), where ζ has been taken equal to $\zeta_j - j = 1, \ldots, g + 1$.

We determine $\alpha_j^{(n)} = \alpha_j^{(n)}(\xi, \eta)$ from the system of linear equations

$$\sum_j \alpha_j^{(n)} \Big(\int_{\sigma_i} G_i \tilde{\Lambda}_{nj} \Big) = 0, \tag{4.12}$$

$$\sum_j \alpha_j^{(n)} = 1. \tag{4.13}$$

Then from the previous theorem we obtain

COROLLARY. The functions

$$\varphi_n = \ln \sum_j \alpha_j^{(n)} \frac{\theta(A(P^-) + Un + V^+\xi + V^-\eta + \zeta_j)}{\theta(A(P^+) + Un + V^+\xi + V^-\eta + \zeta_j)} + d + I_0 n + I_+\xi + I_-\eta \tag{4.14}$$

are solutions of Eqs. (4.1).

We assume that on Γ there exists an antiholomorphic involution τ with fixed ovals $a_1, \ldots, a_{g+1}$. Assume that the cycles σ_i, occurring in the normalization conditions (4.6), coincide with a_i and $P^\pm \in a_{g+1}$.

THEOREM 4.2. Under the assumptions made, the solutions (4.14) of Eqs. (4.1) are real and nonsingular if the functions G_i are real and positive. For large n, ξ, η, these solutions tend to the finite-zone solutions of (4.1):

$$\varphi_n \to \frac{\theta(A(P^-) + Un + V^+\xi + V^-\eta + z(n, \xi, \eta))}{\theta(A(P^+) + Un + V^+\xi + V^-\eta + z(n, \xi, \eta))} + d + I_0 n + I_+\xi + I_-\eta.$$

Here the phase $z(n, \xi, \eta)$ corresponds to the Abel transformation of the collection of zeros of the differential dS corresponding to the maxima of $S(n, \xi, \eta, P)$ on the cycles a_i. (It is clear from the definition that this phase depends only on the direction of the three-dimensional vector (n, ξ, η) and does not depend on its length.)

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