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METHOD OF AVERAGING FOR TWO-DIMENSIONAL "INTEGRABLE" EQUATIONS

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The Whitham method of averaging (or, as it is also called, the nonlinear WKB method) is a generalization to the case of partial differential equations of the classical Bogolyubov–Krylov method of averaging. This method is applicable to nonlinear equations which have a set of exact solutions of the form $u(Ux + Wt + \zeta|I)$. Here $u(z_1,\ldots,z_g|I)$ is a function with unit periods with respect to the z_i; $U = (U_1,\ldots,U_g)$, $W = (W_1,\ldots,W_g)$ are vectors which, like u itself, depend on the parameters $I = (I_1,\ldots,I_N)$, $U = U(I)$, $W = W(I)$. These solutions serve as a basis for the construction of asymptotic solutions, whose leading terms have the form

$$u(\varepsilon^{-1}S(X, T) + \zeta(X, T)|I(X, T)), \tag{1}$$

where the I_k depend on the "slow" variables $X = \varepsilon x$ and $T = \varepsilon t$, where ε is a small parameter, and the vector-valued function S(X, T) is defined by the equations

$$\partial_X S = U(I(X, T)) = U(X, T); \quad \partial_T S = W(I(X, T)) = W(X, T). \tag{2}$$

The equations which describe the "slow" modulation of the parameters $I_k(X, T)$ are called Whitham equations. They can be obtained by requiring that the following terms of the asymptotic series have a uniform bound of lower order than that of the leading term. (For details, see [1, 2], where a larger bibliography concerning this problem can be found.)

If the parameters I_k are integrals of the original equations with local densities, i.e., $I_k = \int P_k(u, u', \ldots)\,dx$, $\partial_t P_k = \partial_x Q_k$, where P_k and $Q_k(u, u',\ldots)$ are differential polynomials in u, then it is possible to obtain a closed system of equations for the I_k (see [3]) by averaging the last equality with respect to the "fast" variables x and t:

$$\partial_T I_k = \partial_X J_k, \quad J_k = \int Q_k(u, u', \ldots)\,dx. \tag{3}$$

We must note that Eq. (3) is very often postulated as a first principle without further analysis, and without a precise statement of the connection of the averaged system with the problem of constructing the solutions of the original equation.

The Hamiltonian theory of the averaged equations (3) has been constructed in [4], where a classification of nonsingular general Hamiltonian systems of "hydrodynamic" type was also obtained: $\partial_T I_k = v_k^i \partial_X I_i$ [here $v_k^i(I)$ depends on I and does not depend on the derivatives]. These results served as a starting point for [5], in which a scheme is proposed for the construction of solutions in general position for "diagonalizable" Hamiltonian systems of hydrodynamic type, i.e., systems for which there exist Riemann invariants – variables $r_i(I)$ in terms of which the matrix v_k^i becomes diagonal.

The presence of single-phase (g = 1) periodic solutions is characteristic of many nonlinear equations, and the existence of multiphase periodic solutions is the exception. Those equations to which we apply the inverse-problem method constitute the largest class of such equations. In particular, these are equations which admit a Lax representation $\dot{L} = [A, L]$, where L and A are differential operators in x whose coefficients depend on x and t. The Korteweg–de Vries equation, the nonlinear Schrödinger equation, and the sine-Gordon equation are among them. Using the methods of algebraic geometry, S. P. Novikov, B. A. Dubrovin, V. B. Matveev, and A. R. Its have constructed multiphase periodic solutions, called finite-zone solutions, of these and a number of other evolution equations with one spatial dimension. These results are summarized in [6, 7]. Some later portions of them were obtained in [8, 9].

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The main purpose of the present article is the generalization of the Whitham method to the case of "integrable" equations with two spatial dimensions, for which an analog of the Lax representation has been proposed in [10] in the form

$$[\partial_y - L, \partial_t - A] = 0, \tag{4}$$

where L and A are the differential operators

$$L = \sum_{i=0}^{n} u_i(x, y, t)\,\partial_x^i, \quad A = \sum_{j=0}^{m} v_j(x, y, t)\,\partial_x^j \tag{5}$$

with scalar or $\ell \times \ell$ matrix coefficients. In what follows we assume that the leading coefficients of L and A are constant diagonal matrices $u_n^{\alpha\beta} = u_n^{\alpha}\delta_{\alpha\beta}$, $v_m^{\alpha\beta} = v_m^{\alpha}\delta_{\alpha\beta}$ with distinct elements on the diagonal. In this case, using a diagonal matrix g(x), it is possible, by means of the adjoints $L' = gLg^{-1}$ and $A' = gAg^{-1}$, to achieve the result that $v_{m-1}^{\alpha\alpha} = 0$.

A general scheme for constructing finite-zone solutions of such equations was proposed in [11] (see also [12]; further steps in the development of the theory of finite-zone integration are described in the summaries [13-17]). These solutions are given explicitly in terms of the Riemann theta-function. Diagonal matrices $a = a(I)$, $b = b(I)$, $c = c(I)$, and Φ such that

$$L = g\hat{L}g^{-1}, \quad A = g\hat{A}g^{-1} \tag{6}$$

are found for the respective operators L and A, where $g = \exp(ax + by + ct + \Phi)$, and the coefficients $\hat{u}_i$ and $\hat{v}_j$ of the operators $\hat{L}$ and $\hat{A}$ have the form

$$\hat{u}_i = \hat{u}_i(Ux + Vy + Wt + \zeta \mid I), \quad \hat{v}_j = \hat{v}_j(Ux + Vy + Wt + \zeta \mid I). \tag{7}$$

Here $\hat{u}_i(z_1, \ldots, z_{2g} \mid I)$ and $\hat{v}_j(z_1, \ldots, z_{2g} \mid I)$ are functions with unit periods in the z_ℓ which depend analytically on the parameters $I = (I_1, \ldots, I_N)$. The vectors $U = U(I)$, $V = V(I)$, and $W = W(I)$ are real, and like a, b, and c, depend on I. The matrix Φ and the real vector ζ in (7) and (8) are arbitrary.

The proposed method for constructing the Whitham equations is based only on the internal self-consistency of the choice of the leading term of the asymptotic series in such a way that for the respective operators we have that

$$L_0 = G\hat{L}_0G^{-1}, \quad A_0 = G\hat{A}_0G^{-1}, \tag{8}$$

where $G = \exp(\varepsilon^{-1}S_0(X, Y, T) + \Phi(X, Y, T))$, and the coefficients of $\hat{L}_0$ and $\hat{A}_0$ have the form

$$\begin{gathered}\hat{u}_i(\varepsilon^{-1}S(X, Y, T) + \zeta(X, Y, T) \mid I(X, Y, T)),\\ \hat{v}_j(\varepsilon^{-1}S(X, Y, T) + \zeta(X, Y, T) \mid I(X, Y, T)).\end{gathered} \tag{9}$$

The vector-valued function S(X, Y, T) and the diagonal matrix $S_0(X, Y, T)$ must satisfy the conditions

$$\begin{gathered}\partial_X S = U(X, Y, T), \quad \partial_Y S = V(X, Y, T), \quad \partial_T S = W(X, Y, T),\\ \partial_X S_0 = a(X, Y, T), \quad \partial_Y S_0 = b(X, Y, T), \quad \partial_T S_0 = c(X, Y, T),\end{gathered} \tag{10}$$

which are analogous to (2). Here $Y = \varepsilon y$ (just as for X and T) is a "slow" variable.

The Whitham equations obtained in Sec. 2 are necessary conditions for the existence of an asymptotic solution of Eqs. (4) with leading term of the form (8), (9), such that the remaining terms of the asymptotic series admit a uniform bound of lower degree than that of the leading term. In Sec. 1 we present material which is necessary for the subsequent development concerning the construction of finite-zone solutions of equations which admit the commutator representation (4).

In Sec. 3 we propose a scheme for constructing solutions of the Whitham equations for the case of two spatial dimensions. In the special case of equations with one spatial dimension, it gives a more effective formulation of the construction in [5]. In addition, this section contains solutions obtained in [18] which describe shock waves in the Korteweg–de Vries equation.

We must note explicitly here that in the simplest case of "zero-zone" solutions, the proposed construction allows us to obtain solutions of the Whitham equations which in this particular case are none other than the quasiclassical limit of the original equations. For systems with one spatial dimension, our construction becomes a scheme for constructing solutions of quasiclassical limit equations of Lax type, which were proposed in a number of examples by V. A. Geogdzhaev, which, in turn, were developments of the results of V. E. Zakharov, who proved the integrability of these equations for the first time.

As an example, we construct solutions of the well-known Khokhlov–Zabolotskii equation in the nonlinear theory of sound beams,

$$\frac{3}{4}\sigma^2 u_{yy}+\left(u_t-\frac{3}{2}uu_x\right)_x=0. \tag{11}$$

Equation (11) is the Whitham equation (38) for "zero-zone" solutions of the Kadomtsev–Petviashvili equation. According to the construction in Sec. 3, it is possible to obtain its solution by giving an arbitrary contour $\mathscr{L}$ in the complex k-plane and a differential $dh(\tau)$ on it.

We define the function $\mathscr{F}$ by

$$\mathscr{F}(k,k_1,k_2)=\oint_{\mathscr{L}}\frac{dh(\tau)}{k-\xi(\tau)},\quad \tau\in\mathscr{L},$$

where $\xi(\tau)$ is determined by means of the equation

$$\xi^4+2u\xi^2+\frac{4}{3}w\xi-(k_1+k_2)^4-2u(k_1+k_2)^2+\frac{4}{3}w(k_1+k_2)=\tau^4,$$

$$u=k_1k_2-(k_1+k_2)^2,$$
$$w=3k_1k_2(k_1+k_2).$$

Here k_1 and k_2 are arbitrary parameters. As a consequence of Theorem 2, if these parameters are determined by the system of equations

$$\mathscr{F}(k_j,k_1,k_2)=x+2\frac{i}{\sigma}k_jy+\left(3k_j^2+\frac{3}{2}u\right)t,\quad j=1,2,$$

which determines k_1 and k_2 implicitly as functions of x, y, and t, then the function

$$u(x,\ y,\ t)=k_1k_2-(k_1+k_2)^2$$

will satisfy (11).

A detailed discussion of the construction of solutions of other equations which are quasiclassical limits of integrable equations in two spatial dimensions and an anlysis of the physical applications of these solutions will be the subjects of a separate article.

If the periodic problem for the original equation is integrable, then the Whitham equations obtained in this article are sufficient for the construction of the entire asymptotic series. The integrability of this problem for the Kadomtsev–Petviashvili-2 equation was proved in a recent article by the present author; this allows us to prove that, in the case of this equation, the Whitham equations are not only necessary but also sufficient. Unfortunately, the limitations of a single article do not permit a complete exposition of this question. A separate article will be devoted to it.

1. NECESSARY MATERIAL FROM THE THEORY OF FINITE-ZONE INTEGRATION

The initial object in the construction of the finite-zone solutions of Eqs. (4) is a nonsingular algebraic curve Γ of genus g with distinguished points $P_1,\ldots,P_\ell$, in a neighborhood of which are fixed the local parameters $k_\alpha^{-1}(P)$, $k_\alpha^{-1}(P_\alpha)=0$, $\alpha=1,\ldots,l$. In addition, we fix a set of polynomials $Q_\alpha(k)$ of degree n, $R_\alpha(k)$ of degree m, and $\sigma_{i\alpha}(k)$ of arbitrary degree, $i=1,\ldots,2g$.

For any set of points $\gamma_1,\ldots,\gamma_{g+l-1}$ in general position there exists a unique function $\psi_\alpha(x,y,t,\zeta,P)$, $P\in\Gamma$, $\zeta=(\zeta_1,\ldots,\zeta_{2g})$, which:

1°) is meromorphic outside of the points P_β and has poles at the points γ_j;

2°) is representable in the form

$$\psi_\alpha=\exp\left(k_\beta x+Q_\beta(k_\beta)y+R_\beta(k_\beta)t+\sum_i\sigma_{i\beta}(k_\beta)\zeta_i\right)\left(\sum_{s=0}^{\infty}\xi_s(x,y,t,\zeta)k_\beta^{-s}\right) \tag{12}$$

in a neighborhood of P_β, where $k_\beta=k_\beta(P)$, and $\xi_0^{\alpha\beta}=e^{\Phi_\alpha}\delta_{\alpha\beta}\Phi_\alpha$ are arbitrary constants. (Functions of a similar type are called Baker–Akhiezer functions.)

We denote the column vector with components ψ_α, $\alpha=1,\ldots,\ell$ by $\psi(x,y,t,\zeta,P)$. As is shown in [12], there exist unique operators L and A of the form (5) with $\ell\times\ell$ matrix coefficients (which depend on the ζ_i as well as on the parameters) such that

$$(\partial_y - L)\psi(x, y, t, \zeta, P) = 0, \quad (\partial_t - A)\psi(x, y, t, \zeta, P) = 0. \tag{13}$$

Since (13) is satisfied identically in P, the operators L and A satisfy (4) for all ζ.

Let the function $\psi_0(\zeta, P)$ be determined by the equality $\psi_0 = \sum_\alpha \exp(\Phi'_\alpha - \Phi_\alpha)\psi_\alpha(0, 0, 0, \zeta, P)$. Then the functions $\tilde{\psi}_\alpha(x, y, t, P) = \psi_\alpha(x, y, t, \zeta, P)\psi_0^{-1}(\zeta, P)$ are the Baker–Akhiezer functions corresponding to the values of the parameters Φ'_α and $\zeta_i = 0$ and to the set of poles $\gamma'_1, \ldots, \gamma'_{g+l-1}$, which coincide with the zeros of ψ_0. Since the vector-valued function $\tilde{\psi}$ with components $\tilde{\psi}_\alpha$ satisfies the same equalities (13) as ψ, it follows that the variation of the parameters ζ and Φ is equivalent to the variation of the set of poles γ_s. Ordinarily the γ_s are chosen as the independent parameters which determine the finite-zone operators L and A, putting $\Phi_\alpha = 0$ and $\zeta_i = 0$ (see [12]). If we fix any set $\gamma_1^0, \ldots, \gamma_{g+l-1}^0$ on Γ, then it is possible as independent parameters to take $\Phi_1, \ldots, \Phi_{\ell-1}$ (in what follows, we will always suppose that $\Phi_\ell = 0$) and ζ_i, which are real. We will confine ourselves to this parametrization.

The operator equation (4) is a system of nonlinear equations for the coefficients u_i and v_j of the operators L and A. It turns out that if $n \leqslant m$, then this system reduces to a pencil of systems only for the coefficients of A, parametrized by the constants $h_{\alpha i}$, $i = 0, \ldots, n$; $\alpha = 1, \ldots, \ell$ (see the details in [12]). In order to express the coefficients of L in terms of the v_j, it suffices to use the fact, which follows from (4), that the operator [L, A] must have degree m − 1, and the diagonal elements of the leading coefficient must be equal to zero. If we equate the coefficients of [L, A] to zero for ∂_x^{m-1+i}, i = n, n − 1, ..., 1 in turn, we find $\partial_x u_i^{\alpha\alpha}$ and $u_{i-1}^{\alpha\beta}$, $\alpha \neq \beta$ (the $h_{\alpha i}$ are constants of integration). As is shown in [12], the matrix elements u_i are differential polynomials in $v_j^{\alpha\beta}$ and $h_{\alpha i_1}$, $j \leqslant i$, $i_1 \leqslant i$.

If we put $R_\alpha = v_m^\alpha k^m$ and $Q_\alpha = \sum_{i=0}^n h_{\alpha i}k^i$, then the above construction gives solutions of the reduced system corresponding to the set of constants $h_{\alpha i}$. Thus the polynomials Q_α parametrize the nonlinear equations, and, in general, the remaining parameters parametrize the solutions of the corresponding equation.

In what follows, we will describe a choice of the polynomials $\sigma_{i\alpha}$ such that under changes of the local parameter k' = k'(k), the corresponding polynomials satisfy the condition $\sigma'_{i\alpha}(k') - \sigma_{i\alpha}(k) = O(k^{-1})$. In this case, it follows from the definition of the Baker–Akhiezer function that, under changes of the local parameter such that $k'_\alpha = k_\alpha + O(k_\alpha^{-m})$, two local parameters related to each other in the above way will be called equivalent, and the set of equivalence classes, called m-germs of local parameters, will be denoted by $[k_\alpha^{-1}]_m$.

Thus the manifold of solutions corresponding to curves of genus g is parametrized by the data

$$(\Gamma, P_1, [k_1^{-1}]_m, \ldots, P_l, [k_l^{-1}]_m) \tag{14}$$

and the quantities $\Phi_1, \ldots, \Phi_{l-1}$, and ζ_i.

The complex dimension of the space of moduli of curves of genus g is equal to 3g − 3. Therefore the dimension of the manifold of data (14), which we will henceforth denote by M_g, is equal to $N = 3g - 3 + \ell(m + 2)$. It is possible to introduce a complex-analytic structure on M_g. Let $I = (I_1, \ldots, I_N)$ be an arbitrary local system of coordinates on M_g. The dependence of all quantities on I_k in subsequent expressions is complex-analytic.

In order to establish the statement made in the introduction concerning the form of the coefficients of L and A, it suffices to reduce the expression for the Baker–Akhiezer function in terms of the Riemann theta-function to a form which is slightly different from the standard one [12].

On Γ, we fix a canonical basis of cycles a_i, b_j with intersection matrices $a_i \circ a_j = b_i \circ b_j = 0$, $a_i \circ b_j = \delta_{ij}$. We define in the standard way (see [12] or [20]) a basis for the normalized holomorphic differentials ω_k, the vectors $B_k = (B_{ki})$ of their b-periods and the corresponding Riemann theta-function — an entire function g of the complex variables which is transformed under shifts of the arguments by the unit basis vectors e_k in C^g and by the vectors B_k in the following way:

$$\theta(\tau + e_k) = \theta(\tau), \quad \theta(\tau + B_k) = e^{-\pi i B_{kk} - 2\pi i \tau_k}\theta(\tau). \tag{15}$$

Let q be an arbitrary point of Γ. A correspondence under which the vector A(P) with coordinates $A_k = \int_q^P \omega_k$ is associated with a point P is called an Abel transformation. For any set g of points $\tilde{\gamma}_s$ in general position the function $\theta(A(P) + Z)$, where

$$Z = -A(\tilde{\gamma}_1) - \ldots - A(\tilde{\gamma}_g) + K \tag{16}$$

(K is the vector of Riemann constants), has precisely g zeros, which coincide with the $\tilde{\gamma}_s$.

Let $\gamma_1^0, \ldots, \gamma_{g+l-1}^0$ be some fixed set of points on Γ. According to the Riemann–Roch theorem there exists a unique function h_α which has poles at the points γ_s^0 and satisfies the normalization condition $h_\alpha(P_\beta) = \delta_{\alpha\beta}$.

We define the function $\varphi_\alpha(z, P)$, $z = (z_1, \ldots, z_{2g})$ by means of the expression

$$\varphi_\alpha = h_\alpha(P) \exp\Big(2\pi i \sum_{k=1}^{g} (A_k(P) - A_k(P_\alpha)) z_{k+g}\Big) \frac{\theta\big(A(P) + Z_\alpha + \sum_{k=1}^{g} (z_k l_k + z_{k+g} B_k)\big)\, \theta(A(P_\alpha) + Z_\alpha)}{\theta(A(P) + Z_\alpha)\, \theta\big(A(P_\alpha) + Z_\alpha + \sum_{k=1}^{g} (z_k l_k + z_{k+g} B_k)\big)}, \tag{17}$$

where

$$Z_\alpha = K + \sum_{\beta \neq \alpha} A(P_\beta) - Z_0, \quad Z_0 = \sum_s A(\gamma_s^0). \tag{18}$$

From (16) it follows that φ_α has unit periods in all of the variables z_i.

We define the differentials dp, dE, and $d\Omega$ as meromorphic differentials on Γ with singularities at the P_α of the form dk_α, $dQ_\alpha(k_\alpha)$, and $dR_\alpha(k_\alpha)$, respectively, uniquely normalized by the requirement that their periods in every cycle are imaginary. Let U be the real vector with coordinates

$$U_k = \frac{1}{2\pi i} \oint_{a_k} dp, \quad U_{k+g} = -\frac{1}{2\pi i} \oint_{b_k} dp, \quad k = 1, \ldots, g. \tag{19}$$

The 2g-dimensional vectors V and W are defined analogously with respect to dE and $d\Omega$.

Cutting Γ along the cycles a_i and b_j, we can choose a unique branch of the integrals p(P), E(P), and Ω(P) of the corresponding differentials. In a neighborhood of P_α they have the form

$$p = k_\alpha - a_\alpha + O(k_\alpha^{-1}), \quad E = Q_\alpha(k_\alpha) - b_\alpha + O(k_\alpha^{-1}), \quad \Omega = R_\alpha(k_\alpha) - c_\alpha + O(k_\alpha^{-1}); \tag{20}$$

it is possible to define p, E, and Ω uniquely if we require that $a_j = b_j = c_l = 0$.

We let $d\sigma_j$ and $d\sigma_{j+g}$ denote arbitrary differentials with singularities at the P_α and which have identical nonzero periods $\pm 2\pi i$, j = 1,...,g, in the cycles a_j and b_j, respectively. Their primitives will be denoted by $\sigma_j(P)$, j = 1,...,2g.

LEMMA 1. The Baker–Akhiezer vector-valued function having poles in the separate set γ_s^0 is representable in the form

$$\psi = e^{ax+by+ct+\Phi} \varphi(Ux + Vy + Wt + \zeta, P)\, e^{px+Ey+\Omega t+\sum_{i=1}^{2g} \sigma_i \zeta_i}, \tag{21}$$

where φ is a vector with coordinates determined by (17); a, b, c, and Φ are diagonal matrices with elements a_α, b_α, c_α, and Φ_α on the diagonals (by virtue of the assumptions we have made, $a_l = b_l = c_l = \Phi_l = 0$); $p = p(P)$, $E = E(P)$, $\Omega = \Omega(P)$, and $\sigma_i = \sigma_i(P)$.

The proof of the lemma consists in a direct verification of the fact that all of the coordinates of the vector on the right-hand side of (21) are correctly determined by the functions P with the required analytic properties.

As follows from the proof of (13) (see [12]), the coefficients of the operators $\hat{L}$ and $\hat{A}$ related to L and A by (6) are differential polynomials in the matrices $\tilde{\xi}_s^{\alpha\beta}$, whose elements are coefficients in the expansion of φ_α in a neighborhood of P_β. Since $\varphi(z, P)$ is periodic in the z_j, the relations (7) are proved.

Following (21), we define the concept of the dual Baker–Akhiezer vector function. For any set $\gamma_1, \ldots, \gamma_{g+l-1}$ in general position, there exists an Abel differential $\hat{\omega}$ of the second kind with second-order poles at the P_β which vanishes at all of the points γ and which is unique to within proportionality. The set of points $\gamma_1^+, \ldots, \gamma_{g+l-1}^+$ consisting of the remaining zeros of $\hat{\omega}$ is called the dual. From this definition it follows that the vectors Z and Z^+ which correspond to the divisors $\{\gamma_s\}$ and $\{\gamma_s^+\}$ under an Abel transformation are related by

$$Z + Z^+ = \mathcal{K} + 2\sum_\alpha A(P_\alpha). \tag{22}$$

The dual Baker—Akhiezer vector function is defined to be the row vector with coordinates $\psi_\alpha^+(x, y, t, \zeta, P)$, which are meromorphic outside of the P_β and have poles at γ_s^+; in a neighborhood of P_β they are representable in the form

$$\psi_\alpha^+ = e^{-k_\beta x - Q_\beta(k_\beta)y - R_\beta(k_\beta)t - \sum_i \sigma_{i\beta}(k_\beta)\zeta_i} \left(\sum_{s=0}^\infty \xi_s^{+\alpha\beta}(x, y, t, \zeta)\, k_\beta^{-s}\right), \tag{23}$$

where $\xi_0^{+\alpha\beta} = e^{-\Phi_\alpha}\delta_{\alpha\beta}$.

The dual Baker—Akhiezer vector function has the form

$$\psi^+ = e^{-px - Ey - \Omega t - \sum_{i=1}^{2g} \sigma_i \zeta_i} \varphi^+(-Ux - Vy - Wt - \zeta, P)\, e^{-ax - by - ct - \Phi}, \tag{24}$$

where the components $\varphi_\alpha^+(z, P)$ are the row vectors of $\varphi^+(z, P)$ given by (17), in which the vectors Z_α must be replaced by $Z_\alpha^+ = Z_\alpha + Z_0 - Z_0^+$. In addition, it is necessary to replace h_α by h_α^+, which has poles at the γ_s^{0+} and is such that $h_\alpha^+(P_\beta) = \delta_{\alpha\beta}$.

In [21] it is shown that ψ^+ satisfies the equations

$$\psi^+(x, y, t, \zeta, P)(\partial_y - L) = 0, \quad \psi^+(x, y, t, \zeta, P)(\partial_t - A) = 0, \tag{25}$$

where the operators L and A are the same as in (13).

In these expressions, as in what follows, the right action of any operator $D = \sum_{i=0}^k w_i \partial_x^i$ on the row vector f^+ is equal to the action of the formal adjoint operator, i.e.,

$$f^+ D = \sum_{i=0}^k (-\partial_x)^i (f^+ w_i). \tag{26}$$

In concluding this section, for any differential operator D with respect to x of order k we give the definition of the operators $D^{(j)}$, $j = 0, \ldots, k$, of order $k - j$ "associated" with it. They are uniquely defined by the requirement that, for any row vector f_1^+ and column vector f_2,

$$(f_1^+ D) f_2 = \sum_{j=0}^k \partial_x^j (f_1^+ (D^{(j)} f_2)). \tag{27}$$

Hence it follows immediately that $D^{(0)} = D$,

$$D^{(1)} = -\sum_{i=1}^k i w_i \partial_x^{i-1}, \quad D^{(2)} = \sum_{i=2}^k \frac{i(i-1)}{2} w_i \partial_x^{i-2} \tag{28}$$

and so forth. It is convenient to define $D^{(j)}$ be the equal to zero for $j > k$.

2. THE WHITHAM EQUATIONS

As stated above, we consider (4) (for $n \leqslant m$) as a system of nonlinear equations for the coefficients of A, parametrized by a set of constants. In addition, the coefficients of L can be expressed in terms of the coefficients of A and the $h_{\alpha i}$, which we write in the convential form $L = \mathcal{L}(A, h_{\alpha i})$. Once more we note that if we fix these expressions, then for any operator with the same leading coefficient as A, the operator $L' = \mathcal{L}(A', h_{\alpha i})$ has the same property that [L', A'] is an operator of order $m - 1$ with zero diagonal elements in the leading coefficient.

We consider the problem of the construction of asymptotic solutions in a more general setting than in the introduction. Let K(A) be a differential operator of order $m - 1$ with zero diagonal elements in the leading coefficient. Its coefficients are differential polynomials in the coefficients of the operator A. The only requirement that these polynomials must satisfy is that if A has the form (6), (7), then K(A) must also have this form (in the case of scalar operators, this condition is automatically satisfied).

We consider the problem of constructing asymptotic solutions

$$\tilde{A} = A_0 + \varepsilon A_1 + \ldots; \quad \tilde{L} = \mathcal{L}(\tilde{A}, h_{\alpha i}) = L_0 + \varepsilon L_1 + \ldots, \tag{29}$$

(where A_i are differential operators of order m − 1 with zero diagonal elements in the leading coefficients) for the equations

$$\partial_t L - \partial_y A + [L, A] - \varepsilon K(A) = 0, \qquad (30)$$

which for k ≠ 0 are weak perturbations of the original equation (4).

We let $\Delta^{m-1} = \Delta^{m-1}(a, b, c, U, V, W)$ denote the space of differential (with respect to x) operators of order m − 1 which have the form $D = g\hat{D}g^{-1}$, where $g = \exp(ax + by + ct)$, and the coefficients of $\hat{D}$ are quasiperiodic functions with vector periods U, V, and W in the corresponding variables, i.e., $\hat{w}_i = \hat{w}_i(Ux + Vy + Wt)$, where $\hat{w}_i(z_1,\ldots,z_{2g})$ is a function with unit period in z_i. For any operator $D \subset \Delta^{m-1}$, we define the operator D^Σ to be $\exp(\varepsilon^{-1}S_0)\hat{D}^S \exp(\varepsilon^{-1}S_0)$, where the coefficients of the operator $\hat{D}^S$ are $\hat{w}_i(\varepsilon^{-1}S(X, Y, T))$. Here $S_0(X, Y, T)$ is a diagonal matrix, S is a vector, and $\Sigma = (S_0, S)$. In this notation, the operators L_0 and A_0 given by (8) and (9) are $L_0 = L^\Sigma$ and $A_0 = A^\Sigma$.

Suppose that S_0 and S satisfy the conditions (10). Then the operators $\tilde{L}$ and $\tilde{A}$, the principal parts of which are L_0 and A_0, satisfy (30) to within O(ε). In order to write the equations which define L_1 and A_1 we must introduce the following definition.

Suppose that the quantities I, ζ, and Φ which parametrize the finite-zone operators L and A depend on some parameter τ. Then the operators $\hat{\partial}_\tau L$ and $\hat{\partial}_\tau A$ obtained by differentiating (6) and (7), in which it has been assumed formally that the vectors U, V, and W and the matrices a, b, and c are constants, are called the "truncated derivatives" of the operators L(τ) and A(τ) along τ. From this definition it follows that

$$\partial_\tau A = \hat{\partial}_\tau A + [\partial_\tau a \cdot x + y\partial_\tau b + t\partial_\tau c, A] + \sum_{i=1}^{2g} (x\partial_\tau U + y\partial_\tau V + t\partial_\tau W)\frac{\partial A}{\partial \zeta_i} \qquad (31)$$

(the same equality holds for $\hat{\partial}_\tau L$).

If the parameters L and A depend on X, Y, and T, then we define F = F(L, A):

$$F = \hat{\partial}_T L - \hat{\partial}_Y A + \{L, A\}, \qquad (32)$$

$$\{L, A\} = \sum_{i=0}^{n} u_i \sum_{k=0}^{i} k\, C_i^k \partial_x^{k-1}(\hat{\partial}_X v_j)\,\partial_x^{i+j-k} - \sum_{j=0}^{m} v_j \sum_{k=0}^{j} kC_j^k \partial_x^{k-1}(\hat{\partial}_X u_i)\,\partial_x^{i+j-k}. \qquad (33)$$

We obtain the operator {L, A} from [L, A] by replacing ∂_x by $\partial_x + \varepsilon\hat{\partial}_X$ in all of the differential expressions in the coefficients of the latter and taking the terms of the first degree in ε. Hence it follows that F has degree m − 1 with zero diagonal elements in the leading coefficient. From the definition of truncated derivatives we have that $F \subset \Delta_0^{m-1}$ (here and in what follows $\Delta_0^{m-1} \subset \Delta^{m-1}$ is the subspace of operators with the above leading coefficients).

Substituting (29) into (30) and setting the coefficient of ε equal to zero, we get that

$$L_{1t} - A_{1y} + [L_0, A_1] + [L_1, A_0] + F^\Sigma - K^\Sigma = O(\varepsilon). \qquad (34)$$

Suppose that ψ and ψ^+ are the Baker–Akhiezer functions corresponding to L and A. Then the functions ψ_0 and ψ_0^+ obtained by replacing Ux + Vy + Wt by $\varepsilon^{-1}S(X, Y, T)$, the diagonal matrix $ax + by + ct$ by $\varepsilon^{-1}S_0(X, Y, T)$, and px + Ey + Ωt by $\varepsilon^{-1}\sum_i \sigma_i S_i(X, Y, T)$ (the S_i are the components of S) in (21) and (24) satisfy (13) and (25), in which L and A have been replaced by L_0 and A_0, to within O(ε).

Using the resulting equality and (34), we get that

$$\partial_t(\psi_0^+(L_1\psi_0)) - \partial_y(\psi_0^+(A_1\psi_0)) + \sum_{j\geqslant 1} \partial_x^j(\psi_0^+((L_0^{(j)}A_1 - A_0^{(j)}L_1)\psi_0)) + O(\varepsilon) = -(\psi_0^+((F^\Sigma - K^\Sigma)\psi_0). \qquad (35)$$

Consequently, if (34) has a uniformly bounded solution, the mean with respect to x, y, and t (in what follows, we will denote it by $\langle\cdot\rangle_0$) of the right-hand side of (35) must be equal to zero. Hence it follows that

$$\langle\psi^+ F\psi\rangle_0 - \langle\psi^+ K\psi\rangle_0 = 0. \qquad (36)$$

This equation must be satified identically in P. From the Riemann–Roch theorem it follows immediately that, among the equations (36), for various P, not more than $N_1 = g + lm - 1$ are independent. Indeed, the left-hand side of (36) is a meromorphic function on Γ which has poles at the poles of ψ and ψ^+ and poles of multiplicity m − 2 at the points P_α. According to the Riemann–Roch theorem, the dimension of the linear space of such functions is N_1.

In order to obtain a complete system which describes the dynamics with respect to X, Y, and T of the parameters I of the finite-zone solutions, we must adjoin to (36) the conditions of compatibility with Eq. (10), which define S_0 and S:

$$\partial_Y U = \partial_X V, \quad \partial_T U = \partial_X W, \quad \partial_T V = \partial_Y W,$$
$$\partial_Y a = \partial_X b, \quad \partial_T a = \partial_X c, \quad \partial_T b = \partial_Y c. \tag{37}$$

We consider the manifold $\hat{M}_g$ of pairs (P, μ), where μ is the set of data (14) and P is a point of Γ in this set. This manifold is naturally stratified over M_g. Let $(\lambda, I_1, \ldots, I_N)$ be a local system of coordinates on $\hat{M}_g$ such that, for fixed I_k, λ(P) parametrizes some region Γ = Γ(I). We will call any such system of coordinates a connection of the stratification $\hat{M}_g \to M_g$ since, for every path I(τ) in $\hat{M}_g$ and any point $P_0 \in \Gamma(I(\tau_0))$, it is possible to define the concept of this path in M_g by defining P(τ) by the condition $\lambda(P(\tau)) = \lambda(P_0)$.

The multivalued functions p, E, and Ω, defined on every curve, are multivalued functions on $\hat{M}_g$, i.e., p = p(λ, I), E = E(λ, I), and Ω = Ω(λ, I).

THEOREM 1. The system of equations (36), (37) is equivalent to the following equation in p(λ, X, Y, T), E(λ, X, Y, T), and Ω(λ, X, Y, T):

$$\frac{\partial p}{\partial\lambda}\left(\frac{\partial E}{\partial Y}-\frac{\partial\Omega}{\partial Y}\right)-\frac{\partial E}{\partial\lambda}\left(\frac{\partial p}{\partial T}-\frac{\partial\Omega}{\partial X}\right)+\frac{\partial\Omega}{\partial\lambda}\left(\frac{\partial p}{\partial Y}-\frac{\partial E}{\partial X}\right)=\frac{\langle\psi^+K\psi\rangle_0}{\langle\psi^+\psi\rangle_0}\frac{\partial p}{\partial\lambda}. \tag{38}$$

Proof. Let P(τ) and I(τ) be an arbitrary curve in $\hat{M}_g$. Then along this curve, p = p(τ), E = E(τ), and Ω = Ω(τ). If ζ and Φ also depend on τ, then the corresponding finite-zone operators also depend on τ: L = L(τ) and A = A(τ).

LEMMA 2. The following relations hold:

$$\partial_\tau\Omega\langle\psi^+\psi\rangle_{xt}+\langle\psi^+\partial_\tau c\psi\rangle_{xt}+\sum_{i=1}^{2g}\left\langle\varphi^+\partial_\tau W_i\frac{\partial\varphi}{\partial\zeta_i}\right\rangle_{xt}=-\partial_\tau p\langle\psi^+(A^{(1)}\psi)\rangle_{xt})-\langle\psi^+(A^{(1)}\partial_\tau a\psi\rangle_{xt})-$$
$$-\sum_{i=1}^{2g}\left\langle\varphi^+\left(\hat{A}^{(1)}\partial_\tau U_i\frac{\partial\varphi}{\partial\zeta_i}\right\rangle_{xt}\right)+\langle\psi^+(\hat{\partial}_\tau A\psi)\rangle_{xt}), \tag{39}$$

$$\left\langle\psi^+\left(\frac{\partial A}{\partial\zeta_i}\psi\right)\right\rangle_{xt}=\left\langle\psi^+\left(\frac{\partial A}{\partial\Phi_\alpha}\psi\right\rangle_{xt}=0. \tag{40}$$

Here φ and φ^+ are the multipliers of the exponentials in (21) and (24); the operator $A^{(1)}$ is defined in terms of A in (27), $\hat{A}^{(1)}=g^{-1}A^{(1)}g$, $g=\exp(ax+by+ct)$, and $\langle\cdot\rangle_{xt}$ is the mean with respect to x and t.

Proof. Let $\psi^+ = \psi^+(x, y, t, P(\tau))$ and let $\psi_1 = \psi(x, y, t, P(\tau_1))$. From (13) and (25) it follows that

$$\partial_t(\psi^+\psi_1)=-\sum_{j=1}^{m}\partial_x^j(\psi^+(A^{(j)}\psi))+(\psi^+((A_1-A)\psi)), \tag{41}$$

where $A_1 = A(\tau_1)$ and $A = A(\tau)$. Differentiating (41) with respect to τ_1 and putting $\tau_1 = \tau$, we get

$$\partial_\tau\Omega(\varphi^+\varphi)+(\varphi^+\partial_\tau c\varphi)+\sum_{i=1}^{2g}\left(\varphi^+\partial_\tau W_i\frac{\partial\varphi}{\partial\zeta_i}\right)=-\partial_\tau p(\varphi^+(\hat{A}^{(1)}\varphi))-(\varphi^+(\hat{A}^{(1)}\partial_\tau a\varphi))-$$
$$-\sum_{i=1}^{2g}\left(\varphi^+\left(\hat{A}^{(1)}\partial_\tau U_i\frac{\partial\varphi}{\partial\zeta_i}\right)\right)+(\psi^+((\partial_\tau A)\psi))+R. \tag{42}$$

Here the remainder term R is a sum of terms of the form

$$R=\sum_s(q_s^0+q_s^1x+q_s^2y+q_s^3t)(q_s^4\partial_x\tilde{w}_s(Ux+Vy+Wt+\zeta)+q_s^5\partial_t\tilde{w}_s(Ux+Vy+Wt+\zeta)), \tag{43}$$

where the q_s^i are constants, and the $\tilde{w}_s$ are periodic in z_i.

The vectors U and W define the rectilinear windings on T^{2g}. We let $T_1(\zeta)$ denote the closure of the winding Ux + Wt + ζ. It is a k-dimensional subtorus in T^{2g}. For any function of the form w(Ux + Wt + ζ), we may consider the mean over the subtorus T_1, which is denoted by $\langle w\rangle_{T_1}$. It coincides with the mean with respect to x and t, i.e., $\langle w\rangle_{T_1} = \langle w\rangle_{xt}$.

We average (42) over $T_1(\zeta + Vy)$ [we note that it is impossible to take the mean with respect to x and t since the part of the coefficients in (43) depends linearly on x and t]. From (43) it follows that $\langle R\rangle_{T_1} = 0$.

We examine the averaged equality (42) for variations under which P and I do not change, and consequently ζ_i and Φ_α change. For these variations, all of the terms except the next to the last are equal to zero. Hence

$$\left\langle \psi^+ \left(\frac{\partial A}{\partial \zeta_i}\psi\right)\right\rangle_{T_1} = \left\langle \psi^+ \left(\frac{\partial A}{\partial \Phi_\alpha}\psi\right)\right\rangle_{T_1} = 0 \tag{44}$$

and since in (44) the average over T_1 coincides with the average with respect to x and t, (40) is fulfilled.

For any constant diagonal matrix r,

$$\langle \psi^+ ([r, A]\psi)_{T_1} = \langle \psi^+ ([r, A]\psi)\rangle_{xt} = \langle \partial_t (\psi^+ r\psi)\rangle_{xt} = 0. \tag{45}$$

Hence it also follows from (31) that

$$\langle \psi^+ (\partial_\tau A\psi)\rangle_{T_1} = \langle \psi^+ (\hat{\partial}_\tau A\psi)\rangle_{xt} \tag{46}$$

and the averaged equality (42) goes into (39).

COROLLARY 1. The differentials dp and dΩ are related by

$$d\Omega \langle \psi^+\psi\rangle_{xt} + dp \langle \psi^+ (A^{(1)}\psi)\rangle_{xt} = 0. \tag{47}$$

Proof. We consider a variation P in Γ with constant I, ζ, and Φ. Then all of the terms except the first two on both sides of (39) are equal to zero. Hence (47) follows. (A similar equality for the case where A is a Sturm–Liouville operator was obtained for the first time in [22].)

From (47) it follows that in the case of general position, when the zeros of dp and dΩ do not intersect, the zeros of the functions $\langle \psi^+\psi\rangle_{xt}$ and $\langle \psi^+ A^{(1)}\psi\rangle_{xt}$, which are meromorphic on Γ, coincide with the zeros of these differentials. Because of the analytic dependence of all of these zeros on I_k, this last statement holds for any curve Γ. We note that it follows from this that the mean in (47) does not depend on y.

Equalities completely analogous to (39) and (40) also hold for L (we omit them for the sake of brevity). From these relations it follows, in particular, that

$$dE \langle \psi^+\psi\rangle_{xy} + dp \langle \psi^+ (L^{(1)}\psi)\rangle_{xy} = 0. \tag{48}$$

Note. If we choose one-half of the zeros of dp as the distinguished set γ_S^0 of poles of the Baker–Akhiezer function, then such a function satisfies the relation $\langle \psi^+\psi\rangle_{xt} = \langle \psi^+\psi\rangle_{xy} = 1$ (as follows from what we have proved).

From (27) it follows that, for the operators $L^{(j)}$ and $A^{(j)}$ associated with L and A satisfying (4), we have that

$$L_t^{(j)} - A_y^{(j)} + \sum_{k=0}^{j} [L^{(k)}, A^{(j-k)}] = 0. \tag{49}$$

Using these relations and the equations which $\psi^+ = \psi^+(x, y, t, P(\tau))$ and $\psi_1 = \psi(x, y, t, P(\tau_1))$ satisfy, we get that

$$\sum_{j=1}^{m} \partial_x^{j-1} [\partial_t (\psi^+ (L^{(j)}\psi_1)) - \partial_y (\psi^+ (A^{(j)}\psi_1)) = \sum_{j=1}^{m} \partial_x^{j-1} [(\psi^+ (L^{(j)} (A_1 - A) \psi_1)) - (\psi^+ (A^{(j)} (L_1 - L) \psi_1)), \tag{50}$$

where $L = L(\tau)$, $L_1 = L(\tau_1)$, $A = A(\tau)$, $A_1 = A(\tau_1)$.

We differentiate (50) with respect to τ_1 and put $\tau_1 = \tau$. We average the resulting equation over the subtorus T_0 corresponding to the winding Ux + Vy + Wt + ζ. The mean of all of the terms in (50) except those corresponding to j = 1 is equal to zero. In a way analogous to the proof of Lemma 2, we can show that the averaged equation (50) reduces to (51).

LEMMA 3. We have the equalities

$$\partial_\tau\Omega \langle \psi^+ (L^{(1)}\psi)\rangle_0 - \partial_\tau E \langle \psi^+ (A^{(1)}\psi)\rangle_0 + \langle \psi^+ L^{(1)}\partial_\tau c\psi\rangle_0 - \langle \psi^+ A^{(1)}\partial_\tau b\psi\rangle_0 + \\ + \sum_{i=1}^{2g} \left\langle \varphi^+ (L^{(1)}\partial_\tau W_i - A^{(1)}\partial_\tau V_i) \frac{\partial\varphi}{\partial\zeta_i}\right\rangle_0 = \langle \psi^+ ((L^{(1)}\hat{\partial}_\tau A - A^{(1)}\hat{\partial}_\tau L)\psi)\rangle_0. \tag{51}$$

It is possible to verify directly that, for any two operators L and A,

$$\{L, A\} = L^{(1)}\hat{\partial}_X A - A^{(1)}\hat{\partial}_X L. \tag{52}$$

Therefore the assertion in Lemma 3 allows us to find the mean $\langle\psi^+(\{L, A\}\psi)\rangle_0$. From (39) we can find an expression for $\langle\psi^+\partial_Y A\psi\rangle_0$ for $\tau = Y$. Putting $\tau = T$ into the analogous equality for L, we find $\langle\psi^+\partial_T L\psi\rangle_0$.

Summing the resulting expression and using (47) and (48), we get that $\frac{\partial p}{\partial\lambda}\frac{\langle\psi^+F\psi\rangle_0}{\langle\psi^+\psi\rangle_0}$ is equal to the left-hand side of (38) plus a sum of terms each of which is zero by virtue of (37). [These additional terms have the form $\langle\psi^+L^{(1)}(\partial_T a - \partial_X c)\psi\rangle_0$, etc.] This proves the theorem.

The description of the construction of solutions of (4) in Sec. 1 also contains as a special case the construction of solutions of the Lax equation $L_t = [A, L]$. We consider the submanifold of data (14), $M_g^0 \subset M_g$, for which the corresponding differential dE is exact, i.e., the function E(P) is single-valued on $\Gamma \subset M_g^0$. In this case, the coefficients of L and A do not depend on y, and (4) becomes a Lax equation. It is possible to use the function E(P) to parametrize neighborhoods of all points of the corresponding curves except a finite number. In addition, $p = p(E, X, T)$ and $\Omega = \Omega(E, X, T)$, and (38) becomes

$$\partial_X\Omega - \partial_T p = \frac{\langle\psi^+K\psi\rangle_0}{\langle\psi^+\psi\rangle_0}\frac{dp}{dE}. \tag{53}$$

For $K \equiv 0$ (53) coincides with $\partial_T p = \partial_X\Omega$, first obtained in the special case of the Korteweg–de Vries equation in [3]. It is necessary to note that this equation was obtained in [3] as a consequence of averaged conservation laws, i.e., a consequence of the equations (3), which were postulated a priori. The derivation of (3) as necessary conditions for the boundedness of the correction term of an asymptotic series was given in [1] in the csae of the Korteweg–de Vries equation.

3. CONSTRUCTION OF SOLUTIONS OF THE AVERAGED EQUATIONS

Let $n_\alpha \geqslant m$ be an integer, and let $\sum_\alpha n_\alpha = g + l(m+1)$. For any curve Γ of genus g with distinguished points P_α in general position and with local parameters k_α^{-1} there exists a function λ(P) which is unique to within an additive constant, which has poles only at P_α (of multiplicity n_α), and such that $P_\alpha\,\lambda^{1/n_\alpha}(P) = k_\alpha + O(k_\alpha^{-m})$ in a neighborhood of P_α. In the case of general position, it is possible to assume that the zeros of dλ are simple, $d\lambda(q_i) = 0$, $i = 0,\ldots, N = 3g - 3 + \ell(m+2)$. It is possible to normalize λ(P) uniquely by putting $\lambda(q_0) = 0$. Then the quantities $\lambda_i = \lambda(q_i)$, $i = 1,\ldots,N$, are local coordinates on M_g. (In the case of Lax equations similar coordinates on M_g^0 are Riemann invariants). The sets $(\lambda(P), \lambda_i)$ constitute local coordinates on $\hat{M}_g$ everywhere except in a neighborhood of q_j. (The connections on $\hat{M}_g$ given in this way will be called canonical.)

On an arbitrary curve in general position Γ_0 we fix a piecewise smooth contour $\mathscr{L}$ (consisting of a finite number of closed or open curves with a finite number of intersections) and sets of points t_ν and $\tilde{P}_\mu$ on and outside of this contour, respectively. Using a canonical connection, it is possible to define a corresponding contour and corresponding points on any curve Γ sufficiently close to Γ_0. (For example, it suffices to put the point $t' \subset \mathscr{L}' \subset \Gamma$ which is determined by the condition $\lambda(t') = \lambda(t)$ into correspondence with the point $t \subset \mathscr{L} \subset \Gamma_0$.

<u>LEMMA 4.</u> For any differential dh on $\mathscr{L}$, which is H-continuous (Hölder continuous) everywhere except at the points t_ν and is such that in a neighborhood of t_ν the differential $(\lambda - \lambda(t_\nu))^{s_\nu}dh$ is bounded, there exists a unique differential dΛ which satisfies the conditions:

1°. dΛ is meromorphic to Γ outside of $\mathscr{L}$, and has a simple pole in q_0 there and poles at P_μ of the form

$$d\Lambda = d\lambda\Big(\sum_{i=1}^{\varkappa_\mu}\tilde{r}_{\mu i}(\lambda - \lambda(\tilde{P}_\mu))^{-i} + O(1)\Big). \tag{54}$$

2°. The limiting values $d\Lambda^\pm$ on $\mathscr{L}$ are H-continuous outside of t_ν and satisfy the "jump relation"

$$d\Lambda^+ - d\Lambda^- = dh. \tag{55}$$

In addition, $(\lambda - \lambda(t_\nu))^{s_\nu}d\Lambda$ is bounded in a neighborhood of t_ν.

3°.

$$\oint(\lambda - \lambda(t_\nu))^i d\Lambda = r_{\nu i},\ i = 1,\ldots,s_\nu, \tag{56}$$

(the integral is taken over a small neighborhood on Γ containing t_μ).

Here $\tilde{r}_{\mu i}$, $i = 1, \ldots, \varkappa_\mu$, and $r_{\nu i}$, $i = 1, \ldots, s_\nu$, are arbitrary sets of numbers.

4°. $d\Lambda$ has imaginary periods with respect to all cycles on Γ.

The assertion in the lemma is a standard one in the theory of boundary value problems. We give a brief sketch of the proof. Let $d\tilde{\Lambda}$ be the differential defined by the Cauchy integral

$$d\tilde{\Lambda} = \frac{d\lambda}{2\pi i} \int_{\mathscr{L}} A(\lambda, t)\, dh(t), \tag{57}$$

where $A(\lambda, q)d\lambda$, the meromorphic analog of the Cauchy kernel, is a function which is meromorphic in q with zeros of multiplicity s_ν at t_ν, and is a differential in λ with poles at t_ν of multiplicity s_ν and simple poles at q and q_0. In a neighborhood of q it has the form $d\lambda\left(\frac{1}{\lambda - q} + O(1)\right)$. It is possible to give such a differential by (2.5) of [20], where one can find further details on boundary value problems on Riemann surfaces. The limiting values $d\tilde{\Lambda}^{\pm}$ on $\mathscr{L}$ satisfy (55). Therefore $d\Lambda = d\tilde{\Lambda} + d\tilde{w}$, where $d\tilde{w}$ is a meromorphic differential with poles of multiplicity s_ν and $\varkappa_\mu$ at t_ν and $\tilde{P}_\mu$, respectively, and a simple pole at q_0. The dimension of the space of such differentials is $g + \sum_\nu s_\nu + \sum_\mu \varkappa_\mu$, and therefore the conditions (54), (56), and 4° allow us to fix $d\tilde{w}$ univalently.

THEOREM 2. Suppose that $\lambda_i = \lambda(q_i)$ depends on X, Y, and T is such a way that, for any $i = 1, \ldots, N$, one of the two conditions

$$\oint \frac{1}{\sqrt{\lambda - \lambda_i}} (d\Lambda + X\,dp + Y\,dE + T\,d\Omega) = 0 \quad \text{or} \quad \lambda_i = \text{const} \tag{58}$$

is fulfilled. Then $p = p(\lambda, X, Y, T)$, $E = E(\lambda, X, Y, T)$, $\Omega = \Omega(\lambda, X, Y, T)$ satisfy the equations

$$\partial_T p = \partial_X \Omega, \quad \partial_Y p = \partial_X E, \quad \partial_Y \Omega = \partial_T E. \tag{59}$$

The integral in (58) is taken over a small contour containing q_i. If q_i does not lie on $\mathscr{L}$, then the first of the conditions (58) means that the differential in the parentheses vanishes at q_i.

Proof. We consider the differential $\partial_X d\hat{S}$, where $d\hat{S} = d\Lambda + X\,dp + Y\,dE + T\,d\Omega$. From the constancy of the "jump" in $d\hat{S}$ on $\mathscr{L}$ it follows that $\partial_X d\hat{S}$ is meromorphic on Γ. From (54) and (56) it follows that this differential is holomorphic at t_ν and $\hat{P}_\mu$. Besides P_α and q_0, the only points where it could have poles are the points q_i, where the connection has singularities. The differential $d\hat{S}$ has no singularities at q_i; therefore for any $j = 0, 1, 2, \ldots$ the first of the equalities in (60) holds. The second is a consequence of it.

$$\oint (\sqrt{\lambda - \lambda_i})^j\, d\hat{S} = 0 \Rightarrow \oint (\lambda - \lambda_i)^{j/2}\, \partial_X\, d\hat{S} - j\,\frac{\lambda_{iX}}{2} \oint (\lambda - \lambda_i)^{j/2-1} d\hat{S} = 0. \tag{60}$$

Hence we have that, for all j, the first of the terms in (60) is equal to zero [for $j \neq 1$, this follows from the first equality, and for $j = 1$ we must use (58)]. Consequently $\partial_X d\hat{S}$ is holomorphic outside of the points P_α and q_0. At the points P_α, it has the same singularity as dp. This means, in particular, that its residues at these points are zero. From the fact that the sum of all of the residues of any meromorphic differential is equal to zero, it follows that its residue at q_0 is also zero. Hence $\partial_X d\hat{S} - dp$ is a holomorphic differential on Γ. By virtue of condition 4° and the normalization conditions on dp, dE, and $d\Omega$, this holomorphic differential must have imaginary periods with respect to all cycles. Consequently it is equal to zero. In a similar way, we can prove that $\partial_Y d\hat{S} = dE$ and $\partial_Y d\hat{S} = d\Omega$. The equalities (59) are a consequence of the fact that the mixed derivatives of $\hat{S}(P) = \int^P d\hat{S}$ are equal.

Remark 1. As follows from the proof of the theorem, the vector of periods of the differential $d\hat{S}$ and the matrix S_0 with diagonal elements which are equal to the coefficients of the zero powers of k_α in the expansions of S(P) in powers of k_α^{-1} in neighborhoods of P_α satisfy the conditions (10).

The quantities $v_i^1 = \frac{dE}{dp}(q_i), v_i^2 = \frac{d\Omega}{dp}(q_i)$ and

$$w_i = \left(\oint \frac{1}{\sqrt{\lambda - \lambda_i}} d\Lambda\right) \left(\oint \frac{dp}{\sqrt{\lambda - \lambda_i}}\right)^{-1} \quad (61)$$

depend on λ_j and determine the functions $v_i^{1,2} = v_i^{1,2}(\lambda_1, \ldots, \lambda_N), w_i = w_i(\lambda_1, \ldots, \lambda_N)$. In this notation, (58) has the form

$$w_i + X + Yv_i^1 + Tv_i^2 = 0 \quad \text{or} \quad \lambda_i = \text{const}. \quad (62)$$

For given X, Y, and T, (62) give a system of N equations in the N unknowns λ_j. Their solutions $\lambda_j(X, Y, T)$ determine particular solutions of the Whitham equations for the unperturbed equations (4), i.e., in the case $K \equiv 0$.

These solutions depend on the choice of canonical connection and on the parameters in the definition of $d\Lambda$, i.e., $\mathcal{L}$, dh, t_ν, $r_{\nu i}$, $\tilde{P}_\mu$, and $\tilde{r}_{\mu i}$. It is possible to extend the method of choosing the canonical connection described in the first paragraph.

Let $\mathfrak{M} \subset M_g$ be a submanifold of M_g (possibly equal to it). We will say that an admissible connection is defined on the stratification $\hat{\mathfrak{M}} \to \mathfrak{M}$, which is a restriction of the stratification $\hat{M}_g$ on $\mathfrak{M}$, if on each curve Γ in the set of data which define a point of $\mathfrak{M}$ there is defined a function λ(P) such that, for any number λ_0 belonging to a small neighborhood of $\lambda(P_\alpha)$, the quantities $k_\alpha^{i}(P)$, i = 1,...,m, where P is determined by the conditions $\lambda(P) = \lambda_0$, are well-defined functions of λ_0, i.e., do not depend on the curve Γ. We note that canonical connections are admissible. The points q_i at which dλ vanishes are singularities of the connection.

<u>THEOREM 2'.</u> Suppose $(\Gamma, P_\alpha, [k_\alpha^{-1}]_m) \subset \mathfrak{M}$ depends on X, Y, and T in such a way that at all singularitis of an admissible connection on Γ one of the conditions (58) is fulfilled. Then the corresponding Abelian integrals p, E, and Ω satisfy (59).

In the special case of the submanifold of data $\mathfrak{M} = M_g^0$, which determine solutions of Lax equations, the connection given by the function E(P), which connection exists on each curve in the data sets in M_g^0, is an admissible connection. Moreover, if all $E_i \neq$ const, Eqs. (62) go into equations proposed in [5]. It must be noted, however, that [5] lacks an effective construction of the functions w_i except in the case where the w_i are produced by averaged polynomial laws of conservation of the original Lax equation. The corresponding solutions are called "averaged n-zone." In our framework, the constructions of such solutions correspond to the case dH = 0, and all of the points $\tilde{P}_\mu$ coincide with the P_α. From this interpretation of averaged n-zone solutions follows their similarity, a property omitted in [5]. More precisely, we consider the Whitham equations corresponding to the Korteweg–de Vries equation.

<u>COROLLARY.</u> Let $d\Omega_n$ be a differential which is holomorphic everywhere except E = ∞ on the hyperelliptic curve Γ_g given by the equation $y^2 = \prod_{j=1}^{2g+1}(E - E_j)$. In a neighborhood of E = ∞, it has a singularity of the form $d\Omega_n = d(E^{n/2}) + O(1)$. Then for $w_i = \frac{d\Omega_n}{dp}(E_i)$, (62) defines solutions of the Whitham equations with similarity index $\gamma = 2/(n-3)$.

The Whitham equations for the Korteweg–de Vries equation have similar solutions of the form $E_i = t^\gamma E_i\left(\frac{x}{t^{1+\gamma}}\right)$ with arbitrary index γ. To construct such solutions, it suffices to take as the contour $\mathcal{L}$ a cut on Γ_g along the entire real axis and to put $dh = \alpha_i d(E^{n/2})$, where $n = 3 + 2/\gamma$ and the constants α_i may be different on different banks of the cut. We note that, if we analyze the corollary in more detail, we can show that the similar solution with index γ = 1/2 used in [18] is an averaged 7-zone.

An important problem is the definition of "equivalent" sets of construction parameters $(\mathcal{L}, dh, \ldots)$, i.e., sets which reduce to the same solution of the Whitham equations. The problem of the effective solution of the Cauchy problem for Whitham equations in the case of equations in one spatial dimension is also closely related to this problem.

<u>Supplement.</u> In [22] a nontrivial generalization of the Lax equations to the case of systems in two spatial dimensions, different from that of [4], was proposed. The greatest interest in such equations centers on the Novikov–Veselov equation [23].

$$u_t = \partial^3 u + \bar{\partial}^3 u + \partial (vu) + \bar{\partial} (\bar{v} u), \; 3\partial u = \bar{\partial} v, \qquad \partial = \partial/\partial z, \; \bar{\partial} = \partial/\partial \bar{z}, \; z = x + iy. \tag{63}$$

It turns out that, although the commutator representation for this equation differs from (4), the Whitham equation has the form (38) in this case. More precisely: the parameters which determine the finite-zone solutions of (63) are the curve Γ, on which there is a holomorphic involution σ with two fixed points P_1 and P_2, and an antiholomorphic involution τ which commutes with σ, $\tau\sigma = \sigma\tau$, $\tau(P_1) = P_2$. In addition, in neighborhoods of P_1 and P_2 are fixed the germs $[k_1]_3$ and $[k_2]_3$ of the local parameters k_1 and k_2 such that $\sigma^* k_i = -k_i$, and $\tau^* k_1 = k_2$. On such curves, we define differentials dp_x, dp_y, and $d\Omega$ which have the forms $dp_x \approx i\,dk$, $dp_y \approx \pm dk$, and $d\Omega \approx i\,dk^3$ in neighborhoods of P_1 and P_2 and are normalized by the condition that their periods with respect to all cycles are real.

THEOREM 3. The Whitham equation for the Novikov–Veselov equation has the form (38), where $dp = dp_x$, $dE = dp_y$, and $d\Omega = d\Omega$.

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ASYMPTOTIC OF THE SPECTRAL FUNCTION OF A POSITIVE ELLIPTIC OPERATOR WITHOUT THE NONTRAP CONDITION

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In this article we investigate the asymptotic with respect to the spectral parameter of the spectral function of a scalar (pseudo) differential operator on a smooth manifold (with or without boundary). A large number of works have been devoted to this problem; we indicate [1-13] without pretention of completeness. Till now, in the general situation a one-term asymptotic formula with a sharp estimate of the remainder for the trace of the spectral function on the diagonal and an order-sharp estimate of the spectral function off the diagonal have been obtained. It is established that these formulas are uniform on compacta that lie outside a small neighborhood of the boundary (and sometimes also up to the boundary; see [8, 12]). In this article, under certain (quite weak) conditions we find the subsequent (with respect to order) term of the asymptotic.

In [5, 6, 11, 13] the complete asymptotic expansion of the spectral function was obtained for the operators in $\mathbf{R}^n$ (in [5, 6], also for the operators in the exterior of a bounded domain) under a series of additional conditions. The nontrap condition (see, e.g., [11, 13]) is the most fundamental of these conditions. In the present article, in place of the nontrap condition we introduce the substantially weaker nonfocality condition (see Theorems 3.2 and 3.3). At the nonfocal points the two leading terms of the asymptotic have the same form as for the problems with the nontrap condition (these terms vanish off the diagonal); we will call these asymptotics the Weyl asymptotics. On the boundaryless manifold the Weyl asymptotic is uniform on compacta that do not contain focal points.

The behavior of the spectral function at the focal points depends on the properties of certain partially isometric operators, generated by a Hamiltonian flow and acting in the spaces L_2 on the unit cotangent spheres (see Sec. 4). Without the nonfocality condition a two-sided asymptotic inequality (Theorem 4.3) is obtained for the trace of the spectral function on the diagonal. A similar formula is established in [14, 15] for the distribution function of eigenvalues. In the "regular" cases the Weyl asymptotic, and in a somewhat more general situation the "quasi-Weyl" asymptotic (the coefficient of the second term is a bounded uniformly continuous function of the spectral parameter), is obtained from these formulas. Moreover, in analogy with works on the distribution function of eigenvalues the notion of a cluster asymptotic is introduced (see Sec. 5).

In this article we use the method of hyperbolic equation. In the main body of the article, for simplicity we consider an operator that acts in the space of half-densities on a boundaryless manifold. All the results are easily carried over to an operator that acts in a function space (then for the formulation of the problem it is necessary to fix a positive smooth density on the manifold). With certain stipulations, the results are carried over to manifolds with boundary (see Sec. 6). In this case the obtained formulas are valid

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