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Elliptic Solutions of Nonlinear Integrable Equations and Related Topics

Dedicated to the memory of J.-L. Verdier

I. M. KRICHEVER
Landau Institute for Theoretical Physics, Academy of Sciences, Kosygina str. 2, 117940 Moscow, Russia

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Abstract. The theory of elliptic solitons for the Kadomtsev–Petviashvili (KP) equation and the dynamics of the corresponding Calogero–Moser system is integrated. It is found that all the elliptic solutions for the KP equation manifest themselves in terms of Riemann theta functions which are associated with algebraic curves admitting a realization in the form of a covering of the initial elliptic curve with some special properties. These curves are given in the paper by explicit formulae. We further give applications of the elliptic Baker–Akhiezer function to generalized elliptic genera of manifolds and to algebraic 2-valued formal groups.

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0. Introduction

Algebraic geometrical methods in the theory of nonlinear integrable systems are now one of the most powerful tools in soliton theory. They provide the construction of periodic and quasi-periodic solutions of the corresponding equations which can be explicitly represented in terms of Riemann theta functions of auxiliary algebraic curves. These solutions are the periodic analog of the multi-soliton solutions of nonlinear equations. The concept of solitons is the most fundamental feature of modern nonlinear physics.

The famous one-soliton solutions of the Korteweg–de Vries (KdV) equation

$$u_t + \tfrac{3}{2}uu_x - \tfrac{1}{4}u_{xxx} = 0 \tag{0.1}$$

have the form

$$u(x,t) = \frac{2\kappa^2}{\cosh^2 \kappa\left(x - \kappa^2 t - \varphi\right)} \tag{0.2}$$

and depend on two parameters. They are particular cases of the generic cnoidal wave solutions which have the form $u(x,t) = v(x - Vt)$. The function $v(x)$ should satisfy the ordinary differential equation

$$-Vv' + \tfrac{3}{2}vv' - \tfrac{1}{4}v''' = 0. \tag{0.3}$$

It has been well known since the end of the last century that all the solutions of Equation (0.3) are elliptic functions,

$$v(x) = 2\wp(x+\varphi) + c, \tag{0.4}$$

where $\wp(x) = \wp(x \mid \omega, \omega')$ is the Weierstrass elliptic function [1].

In present-day theory, the nonlinear differential equations for Abelian functions are considered as the output of the theory rather than its origin.

All nonlinear equations which are considered in soliton theory are equivalent to the compatibility conditions of the auxiliary linear problems. For example, the KdV equation is the compatibility condition

$$\dot{L} = [A, L] \tag{0.5}$$

of the linear equations

$$L\psi = E\psi, \qquad (\partial_t - A)\psi = 0, \tag{0.6}$$

where

$$L = \partial_x^2 - u(x,t), \qquad A = \partial_x^3 - \tfrac{3}{2}u\partial_x - \tfrac{3}{4}u_x. \tag{0.7}$$

Many other physically important nonlinear equations can be represented in the form (0.5), where the linear operators L and A have matrix coefficients and higher orders.

Equation (0.5) means that the corresponding soliton equations are 'isospectral flows' of the auxiliary linear problems.

The effectivization program of the spectral theory of the Schrödinger operator with periodic potentials was developed in the papers by Novikov, Dubrovin, Matveev, and Its (see reviews [2, 3]). It not only provides the construction of the 'finite-gap' solutions of the KdV equation but changes the whole approach to the spectral theory of periodic linear ordinary differential operators.

From the modern point of view, the appearance of Riemann surfaces in the spectral theory of the linear periodic ordinary differential operator L is absolutely obvious. The Bloch solutions of the equation

$$L\psi_i = E\psi_i, \tag{0.8}$$

which are by definition the eigenfunctions of the monodromy operator

$$\psi_i(x+T, E) = w_i\psi(x, E), \tag{0.9}$$

become a single-valued function $\psi(x, P)$ on the corresponding Riemann surface, $P = (E, w_i) \in \Gamma$. The analytical properties of this Bloch function on the Riemann surface are so specific that they uniquely define $\psi(x,t)$. In the case when the Riemann surface of the Bloch function has a finite genus, the inverse problem of reconstructing the operator L from algebro-geometrical data is solved in terms of the Riemann theta functions.

The corresponding 'finite-gap' or algebro-geometrical solutions of the KdV equation have the form [4]

$$u(x,t) = 2\partial_x^2 \ln\theta(\mathbf{U}x + \mathbf{W}t + \Phi \mid \mathbf{B}) + \text{const}, \tag{0.10}$$

where

$$\theta(z_1, \dots, z_g \mid \mathbf{B}) = \sum_{m \in \mathbb{Z}^g} \exp\left(2\pi i(m,z) + \pi i(Bm,m)\right) \tag{0.11}$$

is the Riemann theta function of the hyperelliptic curve Γ

$$y^2 = \prod_{i=1}^{2g+1} (E - E_i) = R(E). \tag{0.12}$$

This means that the matrix B_{ij} is the matrix of b-periods of normalized holomorphic differentials on Γ. The vectors $2\pi\mathbf{U}$, $2\pi\mathbf{V}$ are the vectors of the b-periods of normalized second-kind differentials with poles of 2nd and 3rd orders at infinity (i.e. at the point P_0, corresponding to $E = \infty$). In the general case, the function $u(x,t)$ which is given by (0.10) is quasi-periodic. The branching points E_i, when $u(x,t)$ is periodic in x, are simple eigenvalues for periodic and anti-periodic problems for the Schrödinger equation with the potential $u(x,t)$. The segments $\left[E_{2i}, E_{2i+1}\right]$, $i = 1, \ldots, g$, are forbidden gaps for this operator.

For the general Lax-type equation (sine–Gordon equation, nonlinear Schrödinger equation, and so on), algebro-geometrical solutions have a structure. They are expressed in terms of theta functions as differential polynomials. The argument of theta functions contains the linear-dependence $\mathbf{U}x + \mathbf{V}t + \Phi$ with respect to the variables x, t. The matrices of the theta functions, the vectors $\mathbf{U}$, $\mathbf{V}$ as well, are determined by relevant algebraic curves (in the case of the KdV equation, they are hyperelliptic; for the Boussinesque equation, they are three-fold coverings of the complex plane, and so on).

There is more 'freedom' for the parameters of the algebro-geometrical solutions of two-dimensional (2+1) integrable solution equations [5, 6] . This means, for example, that the formula

$$u(x,y,t) = 2\partial_x^2 \ \ln\ \theta(\mathbf{U}x + \mathbf{V}y + \mathbf{W}t + \Phi \mid \mathbf{B}) + \ \text{const}, \tag{0.13}$$

defines the solutions of the Kadomtsev–Petviashvili (KP) equation, if matrix B is the matrix of b-periods of holomorphic differentials on the *arbitrary* algebraic curve ([6]).

Unfortunately, the arbitrariness of the curve Γ doesn't mean that matrix B is an arbitrary matrix with a positive imaginary part. The problem of selecting Jacobian matrices (i.e. the matrices of b-period algebraic curves) is the famous Riemann–Schottky problem. (The solution of this problem, with the help of the KP theory, was recently obtained in [8].) That's why formulae of the form (0.13) for higher values of g (when B is not arbitrary) should be considered together with the expressions of U, V, B through the free parameters = { curves Γ with punctures }. These expressions are quadratures, but nevertheless they are too complicated for many physical applications.

The generic algebro-geometrical solutions contain in limiting and in particular cases, solutions which can be expressed through more elementary objects than generic theta functions. From the early days of the 'finite-gap theory', it has been well known that multi-soliton solutions of the KdV equation are nothing more than algebro-geometrical solutions corresponding to the rational algebraic curves with simplicity singularities – the double points. In this case, the theta functions degenerate into the determinants of some matrices and such solutions are expressed in terms of elementary functions.

Of course, it's impossible to express the algebro-geometrical solutions corresponding to the smooth algebraic curves of the genus $g > 0$ through elementary functions, but there are no a-priori obstructions for their expressions (in particular cases) in terms of elliptic functions. The elliptic functions are the simplest after elementary functions. Moreover, there are remarkable classical results concerning the reduction of hyperelliptic integrals to the elliptic ones, and the reduction of theta functions. All of them can be and have been applied to the effectivization of some special cases of generic algebro-geometrical solutions (see the review [9]).

Now we are going to explain the other point of view which makes the theory of elliptic solutions especially interesting. At first sight, it has nothing in common with the classical and new results of reduction theory, but the reader will later see how all these approaches to the theory of elliptic solutions will be married to each other.

In the pioneering work of Airault, McKean, and Moser [10], it was found that rational and elliptic solutions of the KdV equation are connected to the completely integrable finite-dimensional Hamiltonian system – the Moser–Calogero system.

The Hamiltonian of this system of one-dimensional particles in the general elliptic case, has the form

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - 2\sum_{i\neq j} \wp(x_i - x_j). \tag{0.14}$$

In the degenerate case (when both the periods of the elliptic function tend to infinity), the $\wp$-function becomes x^{-2} and the Hamiltonian (0.14) transforms into

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - 2\sum_{i\neq j} \frac{1}{(x_i - x_j)^2}. \tag{0.15}$$

We shall call the systems (0.14) and (0.15) elliptic and rational Moser–Calogero systems, respectively.

In both these cases, the motion equation of Moser–Calogero systems have the Lax-type representation (0.5), where L and A are $(n \times n)$ matrices with their elements being the functions of the variables p_i, x_i [11, 12]. It follows from (0.5) that the values $H_k = \operatorname{tr} L^k$ are the integrals of motion $(H_2 = H)$. These integrals are in the involution and, hence, the systems (0.14), (0.15) are completely integrable Hamiltonian systems. The integrals H_k generate the commuting flows on the phase space.

Any locally meromorphic solutions of the KdV equation, i.e. solutions of the form

$$u(x,t) = \sum_{k=-N}^{\infty} \alpha_k(t)\left(x - x_0(t)\right)^k \tag{0.16}$$

may have a singularity only of the following type

$$u(x,t) = 2\left(x - x_0(t)\right)^{-2} + O(1). \tag{0.17}$$

That's why any rational (in respect to the variable x) solution of the KdV equation tending to zero at infinity has the form

$$u(x,t) = 2\sum_{i=1}^{N} \frac{1}{\left(x - x_i(t)\right)^2}. \tag{0.18}$$

In the elliptic case, $u(x,t)$ should be

$$u(x,t) = 2\sum_{i=1}^{N} \wp\left(x - x_i(t)\right) + \text{const.} \tag{0.19}$$

For $N = 3$, such solutions were obtained for the first time in [40].

As was found in [10], the dynamics of the poles of the rational or elliptic solutions of the KdV equation coincide with the dynamics generated by the third integral H_3 of Moser–Calogero systems restricted to the stationary points of the initial Hamiltonian $H_2 = H$. In the rational case

$$\nabla H = 0 \Longleftrightarrow \sum_{j=1}^{N} \frac{1}{\left(x_i - x_j(t)\right)^3} = 0. \tag{0.20}$$

The stationary points of the elliptic Moser–Calogero system are defined by the equations

$$\sum_{j} \wp'\left(x_i - x_j\right) = 0, \quad i = 1, \ldots, N. \tag{0.21}$$

A complete investigation of the so-called 'locus' described by Equations (0.21) was conducted in [10]. It has been proved that it is not empty iff the number of particles and, hence, the number poles of the rational solutions of the KdV equation, is equal to $N = d(d+1)/2$, d are integers. All the rational solutions of the KdV equation are obtained with the help of higher KdV flows from the function $d(d+1)x^{-2}$.

In the elliptic case, part of these results can be easily generalized. It has been well known since the beginning of this century that the Lamé potential $d(d+1)\wp(x)$ finite-gap, and the corresponding Schrödinger operator has d gaps in its spectrum. Therefore, in the elliptic case, again the 'locus' (0.21) is nonempty when N equals $d(d+1)/2$. It contains the sets $\{x_i\}$, which are the poles of the elliptic solutions of the KdV equation obtained with the help of the higher KdV flows from the Lamé potentials.

But unlike the rational case, the question about the existence of other elliptic solutions of the KdV equation had remained open until the appearance of Verdier paper [13]. We shall return to this question a bit later.

1. Elliptic Solutions of the KP Equation

The restricted isomorphism between the elliptic (rational) Moser–Calogero system and the elliptic (rational) solutions of the KdV equation becomes complete in the case of the 'two-dimensional KdV equation' which is a KP equation.

$$\tfrac{3}{4} u_{yy} = \frac{\partial}{\partial x}\left(u_t + \tfrac{1}{4}\left(6uu_x - u_{xxx}\right)\right). \tag{1.1}$$

The elliptic solutions of this equation have the form:

$$u(x,y,t) = 2\sum_{i=1}^{N} \wp\left(x - x_i(y,t)\right) + \text{const.} \tag{1.2}$$

The dynamics of the poles $x_j(y,t)$ with respect to the variables y and t, coincide with the Hamiltonian flows corresponding to the Moser–Calogero system integrals H and H_3, respectively. This isomorphism [14, 15] can be used in two directions. If one knows the solutions of the Moser–Calogero system equations, the corresponding solutions of the KP equation can be written in the form (1.2) or

$$u(x,y,t) = 2\sum_{i=1}^{N} \left(x - x_i(y,t)\right)^{-2},$$

(here N is arbitrary) in the case of the rational Moser–Calogero system. (The solutions of motion equations of the rational Moser–Calogero system were found in [16].) On the other hand, if there exists exact formulae for the corresponding solutions of the KP equation, their poles give solutions of the Moser–Calogero system. The second direction was realized in [14], where the 'algebro-geometrical' construction of the rational solutions of two-dimensional integrable equations was proposed. It turns out that such solutions within the framework of the general algebro-geometrical scheme, correspond to the singular algebraic curves which are rational curves with 'cusps'.

In the elliptic case, the isomorphism between the Moser–Calogero system and the corresponding solutions of the KP equation, remained the isomorphism between the unsolved problems, until the paper [17] appeared (except for the obvious case $N = 1$ and the case $N = 2$, which was considered in [18]).

In the two-dimensional case, the generalization of the Lax-pair representation has the form

$$\left[\frac{\partial}{\partial y} - L, \frac{\partial}{\partial t} - M\right] = 0. \tag{1.3}$$

The operators L and M for the KP equation are

$$L = \partial_x^2 - u(x,y,t), \qquad M = \partial_x^3 - \tfrac{3}{2} u\partial_x + w(x,y,t). \tag{1.4}$$

As was shown in [17], Hamiltonian systems of the form (0.14) are connected with the existence of special solutions of auxiliary linear equations with elliptic coefficients.

THEOREM 1 [17]. *The equation*

$$\left(\partial_t - \partial_x^2 + 2\sum_{i=1}^{n} \wp\left(x - x_j(t)\right)\right)\psi = 0, \tag{1.5}$$

has a solution ψ *of the form*

$$\psi = \sum_{i=1}^{n} a_i(t,k,\alpha)\Phi\left(x - x_i, \alpha\right)\, \mathrm{e}^{kx+k^2t}, \tag{1.6}$$

where

$$\Phi(x,\alpha) = \frac{\sigma(\alpha - x)}{\sigma(\alpha)\,\sigma(x)}\, \mathrm{e}^{\zeta(\alpha)x}, \tag{1.7}$$

if and only if $x_i(t)$ satisfies the motion equations of the Moser–Calogero system (0.14)

$$\ddot{x}_i = 4\sum_{k\neq i} \wp'(x_i - x_k), \quad i = 1, \ldots, n. \tag{1.8}$$

Here

$$\sigma(z) = \sigma(z \mid \omega, \omega'), \qquad \zeta(z) = \zeta(z \mid \omega, \omega')$$

are σ- and ζ-Weierstrass functions.

The choice of the form (1.6) for the solution ψ is connected with a well-known fact that $\Phi(x, \alpha)$ is the solution of the Lamé equation

$$\left(\partial_x^2 - 2\wp(x)\right)\Phi(x, \alpha) = \wp(\alpha)\Phi(x, \alpha). \tag{1.9}$$

The function $\Phi(x, \alpha)$ is two-periodic in relation to the variable α

$$\Phi(x, 2\omega_l + \alpha) = \Phi(x, \alpha), \tag{1.10}$$

and has the following monodromy properties in respect to the variable x:

$$\Phi(x + 2\omega_l, \alpha) = \Phi(x, \alpha)\ \exp\left(\zeta(\alpha)\omega_l - \eta_l\alpha\right). \tag{1.11}$$

Let us give a brief sketch of the proof of this theorem. The function ψ of the form (1.6) has simple poles at the points $x = x_j$. Expanding it around these points, one can obtain from the singular terms of the left-hand side of (1.9), the equations

$$\begin{aligned} a_i\dot{x}_i + 2ka_i + 2\sum_{j\neq i} a_j\Phi\left(x_i - x_j, \alpha\right) &= 0, \\ \dot{a}_i - \wp(\alpha)a_i + a_i\sum_{k\neq i} 2\wp(x_i - x_k) + 2\sum_{j\neq i} a_i\Phi'\left(x_i - x_j, \alpha\right) &= 0. \end{aligned} \tag{1.12}$$

(They are equivalent to the equality of the coefficients to zero before $(x - x_i)^{-2}$ and $(x - x_i)^{-1}$, respectively.) Equations (1.12) have the form

$$\left(L(\alpha) + 2k\right)\mathbf{a} = 0, \qquad (\partial_t + T)\mathbf{a} = 0, \tag{1.13}$$

where $\mathbf{a}$ is a vector with coordinates $\mathbf{a} = (a_1, \ldots, a_n)$, and L and T are the corresponding matrices

$$\begin{aligned} L_{ij} &= \dot{x}_i\delta_{ij} + 2\left(1 - \delta_{ij}\right)\Phi\left(x_i - x_j, \alpha\right), \\ T_{ij} &= \delta_{ij}\left(-\wp(\alpha) + 2\sum_{k\neq i}\wp(x_i - x_k)\right) + 2\left(1 - \delta_{ij}\right)\Phi'\left(x_i - x_j, \alpha\right). \end{aligned} \tag{1.14}$$

The compatibility conditions of the linear Equations (1.13)

$$[L, \partial_t + T] = 0 \Longleftrightarrow L = [L, T] \tag{1.15}$$

are equivalent to the motion equations of the Moser–Calogero system. The last statement can be checked directly, but it can be obtained in the following way, as well. As has been proved in [12], the compatibility conditions (1.15) are equivalent to Equations (1.8) iff the function Φ satisfies the functional equations

$$\frac{\Phi'(x)\Phi(y) - \Phi(x)\Phi'(y)}{\Phi(x+y)} = \wp(y) - \wp(x), \tag{1.16}$$

$$\Phi(x)\Phi(-x) = \wp(\alpha) - \wp(x) \tag{1.17}$$

In [12], particular solutions of these functional equation were found corresponding to the values of $\alpha = \omega_l$, $l = 1, 2, 3$. The existence of the Lax-type representation (1.15) containing 'spectral parameter' α is very essential because it makes it possible to apply the algebro-geometrical methods.

Remark. Equations (1.16), (1.17) for $\Phi(x, \alpha)$, which is given by formula (1.7), are equivalent to the addition formula for σ-functions. They will be very important in the next section where we will consider the applications of the simplest Baker–Akhiezer function Φ for the theory of the multiplicative genera of quasi-complex manifolds with rigidity properties. There, we shall propose the generalization of (1.16) for the generic Baker–Akhiezer functions in Section 2.

It follows from (1.15) that the parameters α and k in (1.6) should satisfy the equation (see [17]).

$$R(k, \alpha) = \sum_{i=0}^{n} r_i(\alpha) k^i = \det\big(2k + L(t, \alpha)\big) = 0. \tag{1.18}$$

The coefficients $r_i(\alpha)$ are elliptic functions with their poles at $\alpha = 0$. Hence, they can be represented as linear combinations of the $\wp$-function and its derivatives. The coefficients of such representations are integrals of the Moser–Calogero system.

EXAMPLE. Let $n = 2$, then

$$R(k, \alpha) = 4k^2 + 2k\,(\dot{x}_1 + \dot{x}_2) + \dot{x}_1\dot{x}_2 + 4\wp\,(x_1 - x_2) - 4\wp(\alpha). \tag{1.19}$$

Equation (1.18) defines the algebraic curve Γ_n, which is the n-fold covering of the elliptic curve of the spectral parameter α. In [17], it was proved that these covering have a very specific property (in [13], such coverings were called tangent coverings). In the generic case, the curve Γ_n has n-sheeted over the neighbourhood of the origin $\alpha = 0$. In this neighbourhood, the function $R(k, \alpha)$ can be represented in the form

$$R(k, \alpha) = \big(k - (n-1)^{-1}\alpha + b_n(\alpha)\big) \prod_{l=1}^{n-1} \big(k + \alpha^{-1} + b_l(\alpha)\big) \tag{1.20}$$

Therefore, the function $k(\alpha)$ has simple poles on each sheet at the points P_j over the origin, $\alpha(P_j) = 0$. Its expansions in the neighbourhood of P_α are given by the factors of the right-hand side of (1.20). From (1.20), it follows that one sheet of the covering is distinguished. We shall call it the 'upper sheet'.

The function $\psi(x,t,\alpha,k)$ of the form (1.6) satisfying Equation (1.5), is the multivalued function of the variable α because the parameter k should be the root of Equation (1.18). As is usual in 'finite-gap' theory, it becomes the single-valued function $\psi(x,t,P)$ on the algebraic curve Γ_n, $P=(k,\alpha)\in\Gamma_n$. (From (1.20), it follows that, in general, the genus of Γ_n equals n.)

THEOREM 2 [17]. *The eigenfunction $\psi(x,t,P)$ of the nonstationary Schrödinger equation (1.5) normalized by the condition $\psi(0,0,P)\equiv 1$, is defined on the n-sheeted covering Γ_n of the initial elliptic curve. The function $\psi(x,t,P)$ is meromorphic on Γ_n except for one point P_n. In general, it has $n=g$ (the genus of Γ_n) poles $\gamma_1,\dots,\gamma_n$, which do not depend on x, t. At the neighbourhood of P_n, it has the form*

$$\psi(x,t,P)=\left(1+\sum_{s=1}^{\infty}\zeta_s(x,t)\alpha^s\right)\exp\left(\lambda(\alpha)x+\lambda^2 t\right), \tag{1.21}$$

where $\lambda(\alpha)=n\alpha^{-1}+b_n(0)$.

Therefore, the function $\psi(x,t,P)$ is a Baker–Akheizer function [5,6]. As has been proved in [6], any function $\psi(x,t,P)$ with the analytical properties which were formulated above, on an arbitrary smooth algebraic curve is the solution of the nonstationary Schrödinger operator with the potential

$$u(x,t)=2\partial_x^2\ln\theta(\mathbf{U}x+\mathbf{V}t+\varphi)+\text{const.} \tag{1.22}$$

Here θ is the Riemann theta function of the corresponding curve; vectors $2\pi\mathbf{U}$, $2\pi\mathbf{V}$ are the vectors of b-periods of normalized Abelian differentials of the second kind with poles of orders 2 and 3 at fixed point P_n, respectively.

COROLLARY. *The coordinates $x_i(t)$ of the Moser–Calogero system are solutions of the equation*

$$\theta(\mathbf{U}x+\mathbf{V}t+\varphi)=0, \tag{1.23}$$

where θ is the theta function corresponding to the curve Γ_n defined by Equation (1.18).

The statement of the Corollary means that Equation (1.23), considered for the fixed value of t as the equation on x, has n roots in the fundamental domain of the elliptic curve with the periods $2\omega_1$, $2\omega_2$ coinciding with $x_i(t)$.

The proof of the Corollary follows from a simple comparison of (1.22) and the formula of the potential $u(x,t)$ in (1.5). The poles of u coincide with zeros of θ on the one hand and with $x_i(t)$ on the other. From this, it also follows that, in the case of the curves Γ_n, the equality

$$\theta(\mathbf{U}x+\mathbf{V}t+\varphi)=\text{const}\prod_{i=1}^{N}\sigma\left(x-x_i(t)\right) \tag{1.24}$$

is fulfilled.

As was shown above, all the algebraic curves Γ_n corresponding to the elliptic Moser–Calogero systems have the property that their characteristic polynomial $R(k,\alpha)$ can be represented in the form (1.20). Now we are going to prove the inverse statement that any algebraic curve Γ, which is defined with the help of the equation

$$R(k,\alpha) = \sum_{i=0}^{n} r_i(\alpha)k^i = 0, \tag{1.25}$$

with the elliptic coefficients $r_i(\alpha)$ having the poles at $\alpha = 0$, and such that the function $R(k,\alpha)$ has the decomposition (1.20), corresponds to the elliptic integrable potentials. Hence, all such functions $R(k,\alpha)$ are characteristic polynomials of some matrix L of the form (1.14).

Let us consider, on the curve Γ, the functions

$$\varphi_i(P) = \exp\left(k(P)\omega_i - \zeta(\alpha)\omega_i + \eta_i\alpha\right), \quad i = 1, 2, \tag{1.26}$$

(here $P = (k,\alpha) \in \Gamma$ is the point of the algebraic curve defined by the equation $R(k,\alpha) = 0$). From the monodromy properties of the theta function, it follows that the functions φ_i are correctly defined on Γ. From (1.20), it follows that φ_i are holomorphic everywhere except for the only point P_n over $\alpha = 0$, which corresponds to the first factor in the decomposition (1.20). In the neighbourhood of this point, the functions φ_i have the form

$$\varphi_i(P) = \exp\left(n\alpha^{-1} + b_n(0)\right)\omega_i\left(1 + O(\alpha)\right). \tag{1.27}$$

The Baker–Akheizer function $\psi(x,t,P)$ is uniquely defined on Γ by its analytical properties which were formulated in Theorem 2. Hence, comparison of the analytical properties of the left- and right-hand sides proves the following equality

$$\psi(x + 2\omega_i, t, P) = \psi(x,t,P)\varphi_i(P), \quad i = 1, 2. \tag{1.28}$$

It implies that the potential (1.22) of the corresponding nonstationary Schrödinger operator has two periods, $2\omega_1$ and $2\omega_2$, and, hence, is an elliptic function. As was stated above, all elliptic solutions of the KP equation have the form (1.2) and the dynamics of their poles in relation to the variable y coincides with the dynamics of the Moser–Calogero system. Therefore, from the previous results (and after changing the notion $t \to y$) it follows that all the solutions of the KP equation are given by formula (0.13), where the theta function corresponds to the n-sheeted covering Γ_n, which is defined by Equation (1.18) (or, equivalently, by (1.25), where $R(k,\alpha)$ has the decomposition (1.20) near $\alpha = 0$).

The KdV equation is a particular case of the KP equation. Hence, all elliptic solutions of the KdV equation are given by formula (0.10), where the theta function corresponds to a 'tangent covering' Γ_n, which at the same time, should be the hyperelliptic curve (i.e. there should exists the function $E(P)$ on Γ_n with the only pole of the second order at the point P_n over $\alpha = 0$ on the 'upper sheet').

The problem of the elliptic solutions of the KdV equation had once again attracted the attention of many specialists after the works of Verdier and Treibich [13, 19], where an unexpected result was obtained. The new elliptic solutions

$$u(x,t) = 2\sum_{j=1}^{4} \wp\left(x - x_j(t)\right) \tag{1.29}$$

(which are not isospectral deformations of the Lamé potentials) were constructed.

The papers of this special issue of *Acta Applicandae Mathematicae* contain the latest results in the theory of elliptic solutions of the KdV equation. Recent progress in the theory is based on the unification of the classical results concerning the reduction of the hyperelliptic integrals and theta functions, with the theory of the Moser–Calogero system.

2. The Generalized Elliptic Genera

The simpliest elliptic Baker–Akhiezer function $\Phi(x, \alpha)$ is the 'cornerstone' of the theory of elliptic solutions. It turns out that $\Phi(x, \alpha)$ also links the solution theory with topological quantum field theory and with the theory of the multiplicative genera of manifolds.

Since the middle of the 70's, it has become clear that a number of quantum anomalies have a topological nature. In all such theories, these anomalies are connected with different versions of the Atiyah–Singer theorem.

The 'topological' anomalies in the strings models were considered in [20–22] where, in particular, the partition function of the supersymmetric nonlinear string σ-model was found. This model is the string analogue of the model of spin-particle M. The partition function of the spin-particle model coincides with the topological invariants of the manifold M which is the index of the Dirac operator (coinciding with the so-called $\hat{A}$-genus). After slight modifications of the models, the corresponding partition functions will coincide with other topological invariants such as signature, Euler characteristic, and so on.

An interpretation of the partition function of the nonlinear sypersymmetric string σ-model as the 'index of the Dirac-type operator on the loop space' of target manifold M was proposed in [23]. The 'index' of such operators was defined as the 'index-character' of the natural S^1 action on the loop space LM. This means the following. The loop space is infinite-dimensional. Hence, the kernel and co-kernel of 'elliptic operator D' on it (even if D is well defined) could be infinite-dimensional. Suppose, that the operator D 'commutes' with the natural S^1 action on LM. Then the $\ker(D)$ and $\operatorname{coker}(D)$ are invariant under the corresponding S^1 action. They can be expanded into the sums

$$\ker D = \sum_{k=-\infty}^{\infty} V_k, \qquad \operatorname{coker} D = \sum_{k=-\infty}^{\infty} W_k,$$

where V_k, W_k are the invariant subspaces, such that the element $q = e^{i\varphi} \in S^1$ acts on them as the multiplication on q^k. The index character is defined in the following way:

$$\operatorname{ind} D(q) = \sum_{k=-\infty}^{\infty} (\dim V_k - \dim W_k)\, q^k. \tag{2.1}$$

Let us present here some heuristic arguments concerning the topological invariants, which can be obtained in this way.

A-priori, the index characters should be the invariants of the loop space LM. But, according to the Atiyah–Singer equivariant theorem, the index characters of the elliptic operator on the finite-dimensional manifold with S^1 action, can be expressed in terms of invariants of fixed-point manifolds. The fixed points of natural S^1 action on the loop space LM correspond to a trivial loop. Hence, this fixed-point submanifold coincides with $M \subset LM$ and if the finite-dimensional arguments can be applicable to this case, one can expect that the index characters of the Dirac-type operators on the loop space of manifold M provide topological invariants of that manifold. Moreover, if the index of the elliptic operator on LM is a really topological invariant, then as was shown in [24], the corresponding invariant has a 'rigidity' property, which means the following in the situation under consideration.

Let us assume that there exists S^1 action on M. Then each of the subspaces V_k, W_k can be represented as the sums

$$V_k = \sum_l V_{k,l}, \qquad W_k = \sum_l W_{k,l}, \tag{2.2}$$

where $V_{k,l}$, $W_{k,l}$ are eigenspaces of an $S^1 \times S^1$ action on LM.

The 'rigidity property' means that

$$b_l(q) = \sum_k \left(\dim V_{k,l} - \dim W_{k,l}\right) q^k = 0, \qquad l \neq 0. \tag{2.3}$$

(The rigidity of all the classical multiplicative genera was proved in [25].)

The above-formulated heuristic arguments of [23] were made rigorous in [24]. (A general reference to the mathematical works within the framework of this theory is in [26]. There one can find Ochanin's definition of the elliptic genus and the related definition of the elliptic cohomology.)

The indices of natural elliptic operators (Dirac, signature, etc.) are examples of the so-called multiplicative genera. The multiplicative genus, by definition, is a ring homomorphism $h\colon \Omega_*^{\mathrm{U,SO},\dots} \longrightarrow \Lambda$ of the bordism of quasi-complex (oriented, etc.) manifolds. According to Hirzebruch, any multiplicative genus of quasi-complex manifolds is defined with the help of the formal series

$$h(x) = x + \sum_{i=1}^{\infty} \lambda_i x^i. \tag{2.4}$$

The value of the corresponding genus for any manifold is given by the formula

$$h\left(M^{2n}\right) = \left\langle \prod_{i=1}^{n} \frac{x_i}{h(x_i)}, \left[M^{2n}\right] \right\rangle. \tag{2.5}$$

Here, x_i are Wu generators. Their symmetric polynomials are Chern classes of the tangent bundle of M

$$\prod_{i=1}^{n} (1 + x_i) = 1 + \sum_{i=1}^{n} c_i(M). \tag{2.6}$$

The brackets means the evaluation of the corresponding cohomology class on the fundamental class of the manifold. The theory of multiplicative genera is closely related to the theory of formal groups. The series $F(u,v)$ in two variables u, v is called a formal group if

$$\text{(a)} \qquad F(u,0) = F(0,u) = u, \tag{2.7}$$

$$\text{(b)} \qquad F(u,F(v,w)) = F(F(u,v),w). \tag{2.8}$$

From condition (2.7), it follows that F has the form

$$F(u,v) = u + v + \sum_{\substack{i+j>1\\ i\geqslant 1,\ j\geqslant 1}} \alpha_{ij} u^i v^j. \tag{2.9}$$

If the ring of the coefficients of formal group $\alpha_{ij} \in \Lambda$ is a Q-module, then it has the logarithm $g_F(u)$, i.e.

$$F(u,v) = g_F^{-1}\left(g_F(u) + g_F(v)\right). \tag{2.10}$$

As was shown in [27], the formal group of the 'geometrical cobordism'

$$\begin{aligned} f(u,v) &= g^{-1}(g(u)+g(v)), \qquad g(u) = \sum_{n=0}^{\infty} \frac{[CP^n]}{n+1} u^{n+1}, \\ f(u,v) &= u + v + \sum \alpha^0_{ij} u^i v^j, \end{aligned} \tag{2.11}$$

is universal, i.e. for any formal group F there exists the ring homomorphism

$$h_F\colon\ \Omega^U_* \longrightarrow \Lambda,$$

from the bordism ring of the quasi-complex manifolds to the ring of the coefficients of F, such that the coefficients of F are equal to

$$\alpha_{ij} = h_F\left(\alpha^0_{ij}\right). \tag{2.12}$$

Consequently, there exists a one-to-one correspondence between the multiplicative genera and the formal groups. Direct connections were found by Novikov [28], who proved that the generating series $h(x)$ coincides with the 'exponent' of the formal group

$$\begin{aligned} &h(x) = g_h^{-1}(x), \qquad g_h^{-1}\left(g_h(u)\right) = u, \\ &g_h(u) = \sum_{n=0}^{\infty} \frac{h([CP^n])}{u^{n+1}}. \end{aligned} \tag{2.13}$$

The classical multiplicative genera correspond to the following formal groups

$$u\sqrt{1+\frac{v^2}{4}} + v\sqrt{1+\frac{u^2}{4}} - A \quad \text{– genus},$$

$$\frac{u+v}{1+uv} \quad \text{– signature,}$$

$$\frac{u+v+2uv}{1-uv} \quad \text{– Euler characteristic.} \tag{2.14}$$

The index of the signature-like operator on the loop space is the so-called, elliptic genus which was introduced by Ochanin [29]. The logarithm of the corresponding formal group equals

$$g_{\mathrm{ell}}(x) = \sum \frac{\varphi([CP^n])}{n+1} x^{n+1} = \int_0^x \frac{\mathrm{d}t}{R(t)^{1/2}}, \tag{2.15}$$

where $R(t) = 1 - 2\delta t^2 + \varepsilon t^4$. The formal group by itself is the Euler group

$$F_{\mathrm{ell}} = \frac{u\sqrt{R(v)} + v\sqrt{R(u)}}{1 - \varepsilon u^2 v^2}. \tag{2.16}$$

Let Γ be the elliptic curve with the periods $2\omega_1$, $2\omega_2$. The elliptic Baker–Akhiezer function $\Phi(x, \alpha) = \Phi(x, \alpha \mid \omega, \omega')$ has the expansion

$$\Phi(x, \alpha) = \frac{1}{x} + \mathrm{O}(1)$$

in the neighbourhood of $x = 0$. Hence, it can be used for the definition of the multiplicative genus. Let us define the complex-valued genus $\hat{\varphi} = \hat{\varphi}(\alpha, k_0 \mid \omega_1, \omega_2)$ [30]

$$\hat{\varphi}\colon \Omega_*^U \longrightarrow C \tag{2.17}$$

with the help of formula (2.5), where (depending on the parameters $(\alpha, k_0 \mid \omega_1, \omega_2)$) $1/h(x)$ equals

$$\Phi(x, \alpha, k_0 \mid \omega_1, \omega_2) = \Phi(x, \alpha \mid \omega_1, \omega_2)\, \mathrm{e}^{-k_0 x}. \tag{2.18}$$

Using the functional equations (1.16), (1.17), it is easy to show that the function $\Phi_l(x) = \Phi(x, \alpha = \omega_l \mid \omega_1, \omega_2)$ generates the elliptic genus. The functions $\Phi_l(x)$ are odd, i.e. $\Phi_l(x) = -\Phi_l(-x)$. Therefore,

$$\Phi_s^2(x) = \wp(x) - e_s, \qquad e_s = \wp(\omega_s), \tag{2.19}$$

and

$$2\Phi_s'(x)\Phi_s(x) = \wp'(x) = 2\sqrt{\prod_{s=1}^{3} \left(\wp(x) - e_s\right)}. \tag{2.20}$$

The corresponding formal group is equal to

$$f_s(u, v) = \frac{1}{\Phi_s\left(g_s(u) + g_s(v)\right)}, \qquad \Phi_s\left(g_s(u)\right) = \frac{1}{u}. \tag{2.21}$$

From (2.20), it follows that

$$\Phi_s'\left(g_s(u)\right) = \frac{\sqrt{R_s(u)}}{u^2}, \tag{2.22}$$

where the coefficients of the polynomial $R_s = 1 - 2\delta_s u^2 + \varepsilon_s u^4$ are equal to

$$2\delta_s = \sum_{i\neq s} (e_s - e_i), \qquad \varepsilon_s = \prod_{i\neq s} (e_s - e_i). \tag{2.23}$$

From (1.16) and (2.22), it follows that the formal group (2.21) coincide with the Euler formal group . Let us consider the functions

$$\hat{\Phi}_{nm}\left(x \mid \omega_1, \omega_2\right) = \hat{\Phi}\left(x, \alpha_{nm}, k_{nm} \mid \omega_1, \omega_2\right), \tag{2.24}$$

where

$$\alpha_{nm} = \frac{2n}{N}\omega_1 + \frac{2m}{N}\omega_2, \qquad n, m = 0, \dots, N-1, \tag{2.25}$$

and

$$k_{nm} = -\frac{2n}{N}\eta_1 - \frac{2m}{N}\eta_2 + \zeta\left(\alpha_{nm}\right). \tag{2.26}$$

They have the properties

$$\hat{\Phi}_{nm}\left(x + 2\omega_1 \mid \omega_1, \omega_2\right) = \hat{\Phi}_{nm}\left(x \mid \omega_1, \omega_2\right) e^{2\pi i n/N}, \tag{2.27}$$

$$\hat{\Phi}_{nm}\left(x + 2\omega_2 \mid \omega_1, \omega_2\right) = \hat{\Phi}_{nm}\left(x \mid \omega_1, \omega_2\right) e^{2\pi i m/N}. \tag{2.28}$$

The function Φ_{nm} generate 'the elliptic genera of the level N' which were introduced in [31]. All the classical multiplicative genera correspond to the degenerate cases Φ_{sing} of the function Φ, corresponding to the degeneration of an elliptic curve into a rational singular curve with double points. The function Φ_{sing}:

$$\Phi_{\text{sing}}(x, k \mid \eta) = \left(-k + \eta \coth(\eta x)\right) e^{kx}, \tag{2.29}$$

depends on the parameters k, η and satisfies the Schrödinger equation with a one-soliton potential (0.2). For different values of the parameters k, k_0, η of the generating function

$$\hat{\Phi}_{\text{sing}}(x, k, k_0 \mid \eta) = \left(-k + \eta \coth(\eta x)\right) e^{(k-k_0)x}, \tag{2.30}$$

all the classical genera can be obtained:

(1) $k = k_0$, $T_{a,b}$ genus [32, 33]
$$a = \eta - k_0, \qquad b = \eta + k_0, \tag{2.31}$$
which coincides with the Todd genus, signature, Euler characteristics for $a = 1$ and $b = 0, 1, -1$, respectively.

(2) $k = \eta$, $k_0 = ((N-2)/N)\eta$ – A_N genus [32, 34], $A_{N-2} = A$ genus.

In [33] the equivariant analog of any multiplicative genus h: $U_* = \Omega_*^U \longrightarrow Q$ was defined:

$$h^G\colon\ U_*^G \longrightarrow K(BG) \otimes Q, \tag{2.32}$$

where U_*^G is the ring of the bordism of manifolds with the action of compact Lie group G. For any G manifolds M, the projection

$$p: M_G = (M \times EG)/G \longrightarrow BG \tag{2.33}$$

on the universal classifying space BG defines the cobordism class

$$\chi_0^G([X, G]) = p_!(1) \in U^*(BG), \tag{2.34}$$

where

$$p_!: U^*(M_G) \longrightarrow U^*(BG) \tag{2.35}$$

is a Gisin homomorphism (direct image).

According to the Dold theorem [35], each homomorphism h has a unique extension as the homomorphism of the functors

$$\tilde{h}: U^*(\cdot) \longrightarrow K(\cdot) \otimes Q. \tag{2.36}$$

The equivariant genus is, by definition, the composition

$$h^G = \tilde{h} \cdot \chi_0^G: U_*^G \longrightarrow U^*(BG) \longrightarrow K^*(BG) \otimes Q. \tag{2.37}$$

The rigidity property of h^G for some special classes of manifolds means that its values belong to the ring of constants

$$h^G([X, G]) \in Q \subset K(BG) \otimes Q \tag{2.38}$$

in the case of connected compact Lie groups.

The universal classifying space in the case $G = S^1$ is CP^∞ and the ring $U^*(CP^\infty)$ is the ring of formal series in relation to the degree 2 variable u with the coefficients from the ring U^*:

$$U^*(CP^\infty) = U^*[[u]]. \tag{2.39}$$

The expression of the cobordism class $\chi_0^G([X, S^1])$ in terms of the fixed point's submanifolds are called Conner–Floyd expressions. In the case of the S^1 action with isolated fixed points, they have the form

$$\chi_0^{S^1}([X, S^1]) = \sum_s \prod_{i=1}^n \frac{1}{[u]_{j_{si}}} \tag{2.40}$$

(see [28, 36, 37], for the generic action they were found in [38] and some formulas of which slightly corrected in [39]). Here $[u]_j$ is the jth power in the formal group of the 'geometrical cobordisms' (2.11)

$$[u]_j = g^{-1}\left(jg(\omega)\right);$$

the integers j_{si}, $i = 1, \ldots, n = \dim_{\mathbb{C}} X$, are 'the exponents' of the S^1 action in the fiber of a tangent bundle over the fixed point x_s.

Each term in the right-hand side of (2.40) is singular. The equality (2.40) means, in particular, that the sum of them is regular. This generates a lot of relations for the local exponents.

The rigidity property of some genus means that, after the application of this genus to the coefficients of the series (2.40), the only constant term will be obtained.

The basic cobordism class $u \in U^*(CP^\infty)$ is transformed under the homomorphism $\tilde{h}$ into $\tilde{h}(u) = g_h^{-1}(\ln \eta)$, where η, η^{-1} are generators of the ring $K(CP^\infty)$. Let us introduce the formal variable $x = \ln \eta$. Then, from (2.40), it follows that the equivariant genus $\varphi^{S^1}([X, S^1])$ corresponding to the generalized elliptic genus (2.7) for the S^1 manifold X with the isolated fixed points has the form

$$\varphi^{S^1}([X, S^1]) = \varphi_X(x) = \sum_s \prod_{i=1}^n \hat{\Phi}\left(j_{si} \cdot x, \alpha, k_0 \mid \omega_1, \omega_2\right). \tag{2.41}$$

By definition, the left-hand side is regular at the origin $x = 0$. The main idea of the proof [33, 34] of the rigidity property is the investigation of global analytical properties of the function φ. It turns out that the function $\varphi_X(x)$ is regular at all points of the lattice $x_{nm} = 2n\omega_1 + 2m\omega_2$ in the case of SU manifolds (i.e. manifolds with zero first Chern class, $c_1(X) = 0$). Indeed, from the monodromy properties (1.22), it follows that

$$\varphi_X\left(x + 2\omega_l\right) = \sum_s \mathrm{e}^{r_s Q_l} \prod_{i=1}^n \hat{\Phi}\left(j_{si} \cdot x\right), \tag{2.42}$$

where

$$r_s = \sum_{i=1}^n j_{si}, \qquad Q_l = 2\left(\zeta(\alpha)\omega_l - \eta_l\alpha - k_0\omega_l\right). \tag{2.43}$$

If the values r_s do not depend on s, i.e. $r_s = N$, then

$$\varphi_X(x + 2\omega_l) = \varphi_X(x)\mathrm{e}^{NQ_l} \tag{2.44}$$

and, hence, φ_X is regular at all the points of the lattice x_{nm} (because it is regular at $x = 0$). It turns out that the values r_s do not depend on s in the case of S^1 actions on the SU manifolds [30]. (In [31], the number $N = r_s$ was called a type of action.)

In [30], it was proved that $\varphi_X(x)$ has no poles at the points x such that $j_{si}x \equiv 0$ (mod $2\omega_1, 2\omega_2$) (it was proved for an arbitrary S^1 action and not only for actions with isolated fixed points). Consequently,

THEOREM 2.2 [30]. *For any SU manifold X the equivariant genus*

$$\varphi_X(x) = \varphi^{S^1}([X, S^1]) \equiv \varphi([X]) \subset K(CP^\infty) \otimes Q$$

is a constant. If the action of S^1 *on X has the type* $N \neq 0$, *then*

$$\varphi_X(x) \equiv \varphi([X]) = 0.$$

The 'addition formula'-type relations (1.16), (1.17) for the elliptic Baker–Akhiezer function $\Phi(x, \alpha \mid \omega_1, \omega_2)$ play an extremely important role in the proof of this theorem.

Finally, at the end of this paper we would like to mention another application of this remarkable functional equation and its solutions. It turned out that they lead to universal algebraic two-valued formal groups.

The definition of the n-valued formal group is the natural generalization of the ordinary one-valued formal group.

The algebraic function $z = z(x, y)$, which is defined by the equation

$$z^n + \sum_{i=0}^{n-1} r_i(x, y) z^i = 0 \tag{2.45}$$

where $r_i(x, y)$ are formal series with respect to the variables x, y, is called an n-valued commutative formal group if the following relations hold:

(1) $z(x, y) = z(y, x)$,
(2) $z(x, z(y, t)) = z(z(x, y), t)$,
(3) $r_i(x, 0) = (-1)^i \binom{n}{i} x^i$ (i.e. for $y = 0$, Equation (2.45) has the form $(z - x)^n = 0$) are fulfilled [41].

The substitution of one n-valued function into another m-valued function defines n, m-valued functions. In this sense, relation (2) means that its right- and left-hand sides coincide with each other as n^1-valued functions.

In [42], it was proved that functional equation (1.16) corresponds to the 'algebraic' two-valued formal group. (The multi-valued formal group is called algebraic if the defining relation (2.45) is polynomial in all the variables.) It turns out that a three-parametrical set of solutions of the functional equation (1.16), which are given by the formula (1.7), generates the universal two-valued formal group which is quadratic in all variables [42]. This formal group has the form [42]

$$(x + y + z - a_2 xyz)^2 = 4 + a_3 xyz(xy + xz + yz + a_1 xyz).$$

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