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Nuclear Physics B 505 (1997) 387–414

Solitons in high-energy QCD

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Received 15 April 1997; accepted 10 July 1997

Abstract

We study the asymptotic solutions of the Schrödinger equation for the color-singlet reggeon compound states in multi-color QCD. We show that in the leading order of asymptotic expansion, quasiclassical reggeon trajectories have the form of soliton waves propagating on the 2-dimensional plane of transverse coordinates. Applying the methods of the finite-gap theory we construct their explicit form in terms of Riemann θ-functions and examine their properties. © 1997 Elsevier Science B.V.

PACS: 12.38.-t; 12.40.Nn; 11.25.Hf; 75.10.Jm

Keywords: QCD pomeron; Regge theory; Exactly solvable model

1. Introduction

It is widely believed that the Regge asymptotics of hadronic scattering amplitudes in high-energy QCD should be described by an effective Regge theory. In this effective theory reggeized gluons, or reggeons, play the role of new collective degrees of freedom. Reggeons form color-singlet compound states, QCD pomerons and odderons, which propagate in the t-channel between scattering hadrons and contribute to a power rise with energy of the physical cross sections [1]. In the Bartels–Kwiecinski–Praszalowicz approach (BKP) [2], the color-singlet reggeon compound states are built from a conserved number $N = 2, 3, \ldots$ of reggeons. For a given number of reggeons, N, the

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PII S0550-3213(97)00487-2

wave function of these states χ_N depends only on the transverse reggeon coordinates $b_{\perp,j} = (x_j, y_j)$ $(j = 1, \dots, N)$ and it satisfies the BKP equation

$$\mathcal{H}_N |\chi_N\rangle = \varepsilon_N |\chi_N\rangle, \tag{1.1}$$

where $\mathcal{H}_N$ is an effective QCD Hamiltonian acting on the transverse coordinates of N interacting reggeons. QCD pomerons and odderons appear as the states in the spectrum of $\mathcal{H}_N$ with the maximal energy ε_N.

The BKP equation has a number of remarkable properties in multi-color limit of QCD, $N_c \to \infty$ and $\alpha_s N_c$ = fixed. Firstly, introducing holomorphic and antiholomorphic coordinates on the 2-dimensional plane of transverse reggeon coordinates, $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$, respectively, one can find that the wave function χ_N splits into a product of holomorphic and antiholomorphic parts [3],

$$\begin{aligned}&\chi_N(x_1, y_1, \dots, x_N, y_N; x_0, y_0)\\ &\quad = \varphi_N(z_1 - z_0, \dots, z_N - z_0) \cdot \overline{\varphi}_N(\bar{z}_1 - \bar{z}_0, \dots, \bar{z}_N - \bar{z}_0),\end{aligned}$$

where (x_0, y_0) is the coordinate of the center-of-mass of the N reggeon compound state. The effective Hamiltonian $\mathcal{H}_N$ is invariant under SL_2 transformations

$$z_j \to \frac{az_j + b}{cz_j + d}, \qquad \bar{z}_j \to \frac{\bar{a}\bar{z}_j + \bar{b}}{\bar{c}\bar{z}_j + \bar{d}}, \tag{1.2}$$

with $ad - bc = \bar{a}\bar{d} - \bar{b}\bar{c} = 1$ and $j = 0, 1, \dots, N$, while the wave function is transformed as

$$\chi_N \to (cz_0 + d)^{2h} (\bar{c}\bar{z}_0 + \bar{d})^{2\bar{h}} \chi_N . \tag{1.3}$$

The conformal weights take the values $h = \frac{1+m}{2} - i\nu$ and $\bar{h} = \frac{1-m}{2} - i\nu$ with $m = \mathbb{Z}$, $\nu = \mathbb{R}$ corresponding to the principal value representation of the $SL(2,\mathbb{C})$ group. Secondly, the QCD effective Hamiltonian $\mathcal{H}_N$ is closely related to the Hamiltonian of the XXX Heisenberg magnet of non-compact $SL(2,\mathbb{C})$ spin $s = 0$ and, as a consequence, the system of N interacting reggeons possesses a large enough family of conserved charges q_k in order for the Schrödinger equation (1.1) to be completely integrable [4,5]. This implies that instead of solving the BKP equation one can define the holomorphic wave function φ_N as a common eigenfunction of the family of $N - 1$ mutually commuting operators $[q_k, q_n] = 0$,

$$q_k = \sum_{1 \leqslant j_1 < j_2 < \dots < j_k \leqslant N} z_{j_1 j_2} z_{j_2 j_3} \dots z_{j_k j_1} p_{j_1} p_{j_2} \dots p_{j_k}, \qquad k = 2, \dots, N \tag{1.4}$$

with $z_{ab} = z_a - z_b$ and $p_j = -i\frac{\partial}{\partial z_j}$. There are also additional constraints on φ_N,

$$(L_3 + h)\, \varphi_N(z_1, \dots, z_N) = 0, \qquad L_+ \varphi_N(z_1, \dots, z_N) = 0, \tag{1.5}$$

which follow from the transformation properties of the wave function (1.3). Here, the notation was introduced for the SL_2 generators

$$L_3 = i\sum_{k=1}^{N} z_k p_k\,, \qquad L_+ = i\sum_{k=1}^{N} z_k^2 p_k\,, \qquad L_- = -i\sum_{k=1}^{N} p_k$$

with $p_k = -i\partial_k$. The conserved charges q_k defined in (1.4) have the meaning of higher Casimir operators of the $SL(2,\mathbb{C})$ group whose eigenvalues determine the quantum numbers of the N reggeon compound states. In particular,

$$q_2 = -\sum_{j>k} z_{jk}^2 p_j p_k = -L_3 L_3 - \frac{1}{2}\left(L_- L_+ + L_+ L_-\right)$$

is a quadratic Casimir operator and its eigenvalue corresponding to the wave function φ_N is given by $q_2 = -h(h-1)$ provided that conditions (1.5) are fulfilled.

To find the holomorphic reggeon wave function φ_N one has to diagonalize simultaneously the remaining $N-2$ commuting Hamiltonians $q_3,\dots,q_N$ on the space of functions satisfying (1.5). This leads to a system of N coupled partial differential equations on the wave function φ_N. The same system can be also interpreted [6] as a set of Schrödinger equations, in which the conformal weight h plays the role of the effective Planck constant, $\hbar = 1/h$. This suggests us to consider the quasiclassical limit of large h and develop the WKB expansion for the holomorphic reggeon wave function

$$\varphi_N(z_1,\dots,z_N) = \exp(iS_0 + iS_1 + \dots)\,, \tag{1.6}$$

where $S_0 = \mathcal{O}(h)$, $S_1 = \mathcal{O}(h^0),\dots$ at large h and $S_k = S_k(z_1,\dots,z_N) = \mathcal{O}(h^{1-k})$. In complete analogy with quantum mechanics, the leading term, S_0, defines the action function for the system of N reggeons. For a given set of quantum numbers q_k, it satisfies the system of Hamilton-Jacobi equations

$$\sum_{k=1}^{N} z_k \frac{\partial S_0}{\partial z_k} = ih\,, \qquad \sum_{k=1}^{N} z_k^2 \frac{\partial S_0}{\partial z_k} = 0\,, \qquad \widehat{q}_\alpha\left(z, \frac{\partial S_0}{\partial z}\right) = q_\alpha\,, \quad \alpha = 3,\dots,N\,, \tag{1.7}$$

where $z = (z_1,\dots,z_N)$ and $\widehat{q}_\alpha(z,p)$ stands for the symbol of the operator (1.4).

For the $N=2$ reggeon state, the BFKL pomeron, the Hamilton-Jacobi equations (1.7) can be easily solved,

$$S_0(z_1,z_2) = -ih\ln\left(\frac{1}{z_1} - \frac{1}{z_2}\right), \qquad \varphi_2(z_1,z_2) = \left(\frac{z_{12}}{z_1 z_2}\right)^h \left[1 + \mathcal{O}(h^{-1})\right].$$

The last expression defines the holomorphic wave function in the leading order of the WKB expansion. Comparing it with the well-known expression for the wave function of the BFKL pomeron [7] we find that it is exact to all orders in $1/h$.

For $N \geqslant 3$ reggeon states, the Hamilton-Jacobi equations (1.7) can be solved in the separated variables [8]. Their solutions describe the compound N reggeon states as a system of N classical particles moving along quasiperiodic trajectories. The corresponding action-angle variables were constructed in [8], where the close relation between classical reggeon trajectories and finite-gap soliton solutions [9] was established. The main goal of the present paper is to apply algebro-geometrical methods [10] in order to obtain the explicit form of the reggeon trajectories on the 2-dimensional plane of transverse coordinates in terms of Riemann θ-functions [11].

2. Hamiltonian flows

In the leading order of the WKB expansion, we replace in (1.4) the momentum operators p_k by the corresponding classical functions to get the family of $N-1$ classical Hamiltonians $q_2,\dots,q_N$. For the system of N reggeons with the coordinates z_k, momenta p_k and the only non-trivial Poisson bracket $\{x_k,p_n\}=\delta_{kn}$, these Hamiltonians generate the hierarchy of the evolution equations

$$\frac{\partial z_n}{\partial t_\alpha}=\{z_n,q_\alpha\}=\frac{\partial q_\alpha}{\partial p_n}\,,\qquad \frac{\partial p_n}{\partial t_\alpha}=\{p_n,q_\alpha\}=-\frac{\partial q_\alpha}{\partial z_n}\,, \tag{2.1}$$

with t_α $(\alpha=2,\dots,N)$ the corresponding evolution times and $\partial_{t_\alpha}q_n=\{q_n,q_\alpha\}=0$. Their solutions define the reggeon trajectories $z_k=z_k(t_2,\dots,t_N)$ subject to the periodicity condition

$$z_{k+N}(t)=z_k(t)\,,\qquad p_{k+N}(t)=p_k(t)\,. \tag{2.2}$$

The system of evolution equations (2.1) has a sufficient number of the integrals of motion q_k to be completely integrable. It admits the Lax pair representation, which can be found using the equivalence of the system of N reggeons and the XXX Heisenberg magnet of SL_2 spin 0 [4,5]. Namely, for each reggeon we define the Lax operator as

$$L_k=\begin{pmatrix}1-Ez_kp_k & Ep_k\\ -Ez_k^2p_k & 1+Ez_kp_k\end{pmatrix}=\mathbb{1}+E\begin{pmatrix}1\\ z_k\end{pmatrix}\otimes(-z_k,1)\,p_k\,, \tag{2.3}$$

with E being an arbitrary complex spectral parameter. Then, the evolution equations (2.1) are equivalent to the matrix relation

$$\partial_{t_\alpha}L_k=\{L_k,q_\alpha\}=A_{k+1}^{(\alpha)}L_k-L_kA_k^{(\alpha)}\,, \tag{2.4}$$

where $A_k^{(\alpha)}(E)$ is a 2×2 matrix depending on the choice of the Hamiltonian. As an example, for the Hamiltonian q_N one can obtain

$$A_k^{(N)}=E\frac{q_N}{z_{k-1,k}}\begin{pmatrix}1\\ z_k\end{pmatrix}\otimes(z_{k-1},-1) \tag{2.5}$$

and the corresponding evolution equations look like

$$\partial_{t_N}z_k=\frac{q_N}{p_k}\,,\qquad \partial_{t_N}p_k=q_N\left(\frac{1}{z_{k-1,k}}-\frac{1}{z_{k,k+1}}\right).$$

The explicit form of $A_k^{(\alpha)}$ for the remaining Hamiltonians can be deduced from the Yang–Baxter equation for the Lax operator L_k and it will not be important in the sequel.

The integration of the evolution equations (2.1) and (2.4) is based on the Baker-Akhiezer function $\Psi_k(E;t_2,\dots,t_N)$. It is defined as a solution of the following system of matrix relations:

$$L_k(E)\,\Psi_k = \Psi_{k+1}\,, \qquad \partial_{t_\alpha}\Psi_k = A_k^{(\alpha)}(E)\,\Psi_k\,, \qquad \Psi_k = \begin{pmatrix}\phi_k\\ \chi_k\end{pmatrix}, \tag{2.6}$$

where ϕ_k and χ_k are scalar components. We construct the monodromy matrix

$$T(E) = L_N(E)\dots L_2(E)L_1(E) = \begin{pmatrix} A(E) & B(E)\\ C(E) & D(E)\end{pmatrix} \tag{2.7}$$

and verify using (2.4) that its eigenvalues are integrals of motion due to $\partial_{t_\alpha}T(E) = A_1^{(\alpha)}T - TA_1^{(\alpha)}$. The monodromy matrix generates the shift $T(E)\Psi_1 = \Psi_{N+1}$ and we impose the periodicity condition on the Baker-Akhiezer functions by requiring $\Psi_k(E)$ to be the Bloch-Floquet function

$$\Psi_{k+N}(E) = e^{P(E)}\,\Psi_k(E)\,. \tag{2.8}$$

Here, $P(E)$ is an eigenvalue of the monodromy matrix (2.7) and it satisfies the characteristic equation

$$\det(T(E) - e^{P(E)}) = e^{2P(E)} - \Lambda(E)\,e^{P(E)} + 1 = 0\,, \tag{2.9}$$

where $\Lambda(E) = \mathrm{tr}\,T(E)$ can be calculated using (2.7) and (2.3) to be a polynomial of degree N in the spectral parameter with the coefficients given by the integrals of motion

$$\Lambda(E) = 2 + q_2E^2 + q_3E^3 + \dots + q_NE^N\,.$$

Introducing the complex function $y(E) = (e^{P(E)} - e^{-P(E)})/E$ one rewrites (2.9) in the form of algebraic complex curve [2]

$$\begin{aligned}\Gamma_N:\quad y^2 &= E^{-2}(\Lambda^2(E) - 4)\\ &= (q_2 + q_3E + \dots + q_NE^{N-2})(4 + q_2E^2 + q_3E^3 + \dots + q_NE^N).\end{aligned} \tag{2.10}$$

For any complex E in general position Eq. (2.9) has two solutions for $P(E)$, or equivalently, $y(E)$ in (2.10). Under appropriate boundary conditions on $\Psi_k(E)$ (to be discussed later) each of them defines a branch of the Baker-Akhiezer function, $\Psi_k^{\pm}(E)$. Then, being a double-valued function on the complex E-plane, $\Psi_k(E)$ becomes a single-valued function on the Riemann surface corresponding to the complex curve Γ_N. This surface is constructed by gluing together two copies of the complex E-plane along the cuts $[e_2, e_3], \dots, [e_{2N-2}, e_1]$ running between the branching points e_j of the curve (see for details Fig. 1), defined as simple roots of the equation

$$\Lambda^2(e_j) = 4\,, \qquad j = 1,\dots,2N-2\,. \tag{2.11}$$

[2] To make a correspondence with the notations of Ref. [8], one has to redefine the local parameter on the curve as $E = 1/x$.

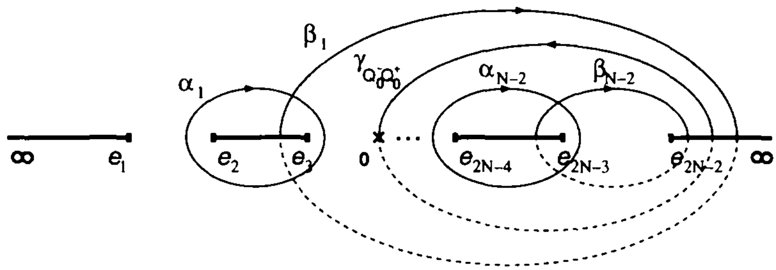

Fig. 1. The definition of the canonical basis of cycles on the Riemann surface Γ_N. The dotted line represents the part of the β-cycles belonging to the lower sheet. The cross denotes a projection of the points $Q_0^\pm \in \Gamma_N$ on the complex plane. The path $\gamma_{Q_0^- Q_0^+}$ does not cross the canonical cycles and goes from the point Q_0^- on the lower sheet to the point Q_0^+ on the upper sheet.

Their positions on the complex plane depend on the quantum numbers $q_2, q_3, \dots, q_N$ of the reggeon compound state. In general, thus defined Riemann surface has a genus $g = N - 2$ and it depends on the number of reggeons, e.g. it is a sphere for the $N = 2$ state, or BFKL pomeron, and a torus for the $N = 3$ state known as QCD odderon.

To distinguish points $Q = Q(E)$ on the Riemann surface lying above E on the upper and the lower sheets we give them a sign $Q^\pm = (\pm, E)$, where the upper $(+)$ sheet corresponds to the asymptotics $y \sim q_N E^{N-1}$ as $E \to \infty$. In particular, one finds from (2.9) the behaviour of the Bloch multiplier on different sheets of Γ_N for $E \to \infty$ as

$$e^{\pm P(E)} = q_N E^N \times \left(1 + \mathcal{O}(1/E)\right), \tag{2.12}$$

where $\pm$ corresponds to the choice of the sheet.

2.1. Boundary conditions

Being written in components, the first condition (2.6) on the Baker–Akhiezer function looks like

$$\chi_{n+1} - \chi_n = z_n(\varphi_{n+1} - \varphi_n) = E z_n p_n (\chi_n - z_n \varphi_n), \tag{2.13}$$

where φ_n and χ_n are single-valued functions of $Q(E)$ on the Riemann surface Γ_N. We notice that for $E = 0$ the solutions to this equation do not depend on the reggeon number n. Moreover, using (2.3) and (2.5) one can show that $L_k(E = 0) = \mathbb{1}$, $A_k^{(\alpha)}(E = 0) = 0$ and, therefore, there exist two special points on the Riemann surface $Q_0^\pm = (\pm, E = 0)$, at which the Baker–Akhiezer function takes constant values, $\Psi_k(Q_0^\pm; \{t\}) = \text{const}$. The choice of the constants implies the normalization of the Baker–Akhiezer function.

Let us show that the values $\Psi_k(Q_0^\pm)$ are fixed by the constraints (1.5). At the vicinity of $E = 0$ on one of the sheets of Γ_N the Baker–Akhiezer function can be expanded as

$$\Psi_n(Q; \{t\}) = \Psi^{(0)} + E\Psi_n^{(1)}(\{t\}) + \mathcal{O}(E^2),$$

where the coefficients depend on the choice of sheet and the leading term $\Psi^{(0)}$ is a constant. Let us substitute this identity into relation $\Psi_{1+N} = T(E)\Psi_1 = \exp(P(E))\Psi_1$

and expand both sides in powers of E. Taking into account the small-E expansion of the monodromy matrix (2.7),

$$T(E) = \mathbb{1} + iE\begin{pmatrix} L_3 & L_- \\ L_- & -L_3 \end{pmatrix} + \mathcal{O}(E^2)\,, \tag{2.14}$$

one obtains that $\Psi^{(0)}$ is an eigenvector of the following matrix:

$$\begin{pmatrix} -h & L_- \\ 0 & h \end{pmatrix}\Psi^{(0)} = -iP'(0)\Psi^{(0)}\,,$$

where conditions (1.5) were imposed. We find two solutions for $\Psi^{(0)}$ corresponding to $P'(0) = \pm ih$, which fix (up to an overall constant) the value of the Baker–Akhiezer functions at the points $Q_0^{\pm}$ as

$$\Psi_n(Q_0^+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Psi_n(Q_0^-) = \begin{pmatrix} \frac{L_-}{2h} \\ 1 \end{pmatrix}. \tag{2.15}$$

Here, h is the conformal weight of holomorphic wave function and $L_- = -i\sum_{k=1}^N p_k$. In the quantum case, the operator L_- commutes with the Hamiltonians q_k, but it does not take any definite value on the subspace (1.5), since $[L_-, L_3] = 2iL_-$ and $[L_-, L_+] = iL_3$. In the quasiclassical limit, L_- has a zero Poisson bracket with q_k and it does not depend on the evolution times t_k. Its value is fixed by the initial conditions as

$$L_- = -i\sum_{k=1}^N p_k(0) \equiv -iP_{\rm tot}\,,$$

with $P_{\rm tot}$ being the total holomorphic momentum of N reggeons.[3]

We notice, however, that the relations (2.6) and (2.13) are invariant under SL_2 transformations

$$p_k \to p_k(cz_k + d)^2\,, \qquad \Psi_n \to \begin{pmatrix} d & c \\ b & a \end{pmatrix}\Psi_n$$

with the coordinates z_k transformed as in (1.2). Since this transformation changes the value of L_-, one can choose its parameters in such a way that one puts $L_- = 0$ and keeps the relations $L_3 = -h$ and $L_+ = 0$ unchanged.[4] The relation between original and transformed reggeon coordinates looks as follows:

$$\frac{1}{z_n} = -i\frac{P_{\rm tot}}{2h} + \frac{1}{z_n^{(0)}}\,, \qquad p_n = p_n^{(0)}\left(1 - i\frac{P_{\rm tot}}{2h}z_n^{(0)}\right)^2. \tag{2.16}$$

It allows us to integrate the evolution equations (2.1) for $L_- = 0$ to get expressions for $z_n^{(0)}$ and $p_n^{(0)}$ and then restore the physical solutions using (2.16).

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[3] In hadronic scattering amplitudes, $P_{\rm tot}$ is defined as a holomorphic component of the total momentum transferred in the t-channel.

[4] These three conditions form a set of second-class constraints for the N reggeon system. Their quantization leads to the Baxter equation for the wave function of the N reggeon compound state [12].

Therefore, in what follows we put $L_- = 0$ and obtain the normalization conditions (2.15) for the Baker–Akhiezer function in the canonical form (up to overall constant factor)

$$\Psi_n^{(0)}(Q_0^+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Psi_n^{(0)}(Q_0^-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.17}$$

It easy to see that for $L_- = L_+ = 0$ the solutions of the evolution equations (2.1) and (1.5) are invariant under rescaling of the coordinates

$$z_n^{(0)} \to \rho\, z_n^{(0)}, \qquad p_n^{(0)} \to \rho^{-1}\, p_n^{(0)}. \tag{2.18}$$

For the Baker–Akhiezer function, this corresponds to a freedom in choosing overall normalization factors in (2.17). One can further simplify the analysis of the evolution equations (2.1) for $z_n^{(0)}$ and $p_n^{(0)}$ by noticing that $q_2 = -L_3L_3 = -h^2$ and the dependence on the "lowest" time t_2 is given by $\partial_{t_2} z_n^{(0)} = -2L_3 \frac{\partial L_3}{\partial p_n} = 2ihz_n^{(0)}$, and similarly for $p_n^{(0)}$, leading to

$$\begin{aligned} z_n^{(0)}(t_2, t_3, \dots, t_N) &= z_n^{(0)}(0, t_3, \dots, t_N)\, \mathrm{e}^{2iht_2}, \\ p_n^{(0)}(t_2, t_3, \dots, t_N) &= p_n^{(0)}(0, t_3, \dots, t_N)\, \mathrm{e}^{-2iht_2}. \end{aligned} \tag{2.19}$$

Thus, one can forget for a moment about the t_2 dependence of the reggeon coordinates and restore it in the final expressions using (2.19).

2.2. Singularities of the Baker–Akhiezer function

Let us show that the components of the Baker–Akhiezer function $\Psi_n(Q)$ are meromorphic functions on the Riemann surface Γ_N having $g + 1 = N - 1$ poles, the same number of zeros and essential singularities at two infinities $Q_\infty^\pm = (\pm, \infty)$ situated on the upper and lower sheets of Γ_N. Let us take the B-component of the monodromy matrix (2.7) and observe that $B(E)/E$ is a polynomial of degree $N - 1$ in the spectral parameter E and $B(E) = iL_-E + \mathcal{O}(E^2)$ as $E \to 0$ according to (2.14). Then, one defines the points $E_1, \dots, E_{N-1}$ as roots of this polynomial,

$$B(E_k)/E_k = 0, \qquad k = 1, \dots, N-1$$

and considers the relation $T(E)\Psi_1 = \exp(P(E))\Psi_1$ at the points $Q_k = (\pm, E_k)$ on the Riemann surface situated above E_k

$$\begin{pmatrix} A(E_k) & 0 \\ C(E_k) & D(E_k) \end{pmatrix} \begin{pmatrix} \varphi_1(Q_k) \\ \chi_1(Q_k) \end{pmatrix} = \mathrm{e}^{P(E_k)} \begin{pmatrix} \varphi_1(Q_k) \\ \chi_1(Q_k) \end{pmatrix}. \tag{2.20}$$

Solving it one obtains the values of $P(E_k)$ on two sheets of Γ_N. If the point $Q(E_k)$ belongs to the same sheet of Γ_N on which $D(E_k) = \mathrm{e}^{P(E_k)}$, then one gets from (2.20) that $\varphi_1(Q_k) = 0$. Thus, the upper component of the Baker–Akhiezer function $\Psi_1(Q)$ has $g + 1$ zeros on Γ_N above the points E_k defined as roots of the polynomial $B(E)/E$. In a similar way, $g + 1$ zeros of the lower component of $\Psi_1(Q)$ are related to the zeros of the polynomial $C(E)/E$ defined by the C-component of the monodromy matrix. In general, two components of the Baker–Akhiezer function, $\Psi_n^\alpha(Q;\{t\})$ ($\alpha = 1,2$), have different sets of $g+1$ zeros on Γ_N and their positions on Γ_N depend both on the reggeon number, n, and the evolution times, $\{t\}$. Relation (2.17) implies that one of the roots should be at the points $Q_0^\pm$. Indeed, it follows from (2.14) and (2.7) that the polynomials $B(E)/E$ and $C(E)/E$ vanish at $E=0$ for $L_- = 0$ and $L_+ = 0$, respectively, and therefore the components of the Baker–Akhiezer function vanish at the points $Q_0^\pm = (\pm, 0)$, in agreement with (2.13).

The remarkable property of the roots $E_k = E_k(z_1^{(0)}, p_1^{(0)}, \dots, z_N^{(0)}, p_N^{(0)})$ and the corresponding values of $P(E_k)$ is that they form a set of separated coordinates for the system of N reggeons [13]. Namely, they belong to the spectral curve

$$e^{P(E_k)} + e^{-P(E_k)} = \Lambda(E_k) = 2 + q_2 E_k^2 + \dots + q_N E_k^N, \quad k = 1, \dots, N-1 \tag{2.21}$$

and have the following Poisson brackets:

$$\{E_k, E_n\} = \{P(E_k), P(E_n)\} = 0, \qquad \{E_k^{-1}, P(E_n)\} = \delta_{kn}.$$

Inverting the relations (2.21) one can express q_k in terms of E_k and then obtain the equations of motion for zeros E_k of the Baker–Akhiezer function generated by Hamiltonians q_k in the following differential form [8]:

$$dt_k = -\sum_{j=1}^{N-1} \frac{dE_j\, E_j^{k-2}}{\sqrt{\Lambda^2(E_j) - 4}}, \qquad k = 2, 3, \dots, N. \tag{2.22}$$

These equations can be integrated by the Abel map and their solutions describe linear reggeon trajectories on the Jacobian variety of the Riemann surface Γ_N [8]. As we will show in Section 3, using the Baker–Akhiezer function one can perform the inverse Abel transformation and construct the explicit expressions for the reggeon trajectories on the 2-dimensional plane of transverse coordinates in terms of Riemann θ-functions.

For the Baker–Akhiezer function to be a well-defined meromorphic function on the Riemann surface the number of its zeros, $g+1 = N-1$, should match the number of simple poles, which we choose to be at the points $\gamma_1, \dots, \gamma_{g+1}$ in a general position on Γ_N. Moreover, considering the relations (2.6) and (2.13) at the points $Q = (\pm, E)$ close to γ_k on Γ_N one finds that the components of $\Psi_n^\alpha(Q;\{t\})$ share a common set of poles and the positions of $\gamma_1, \dots, \gamma_{g+1}$ on Γ_N do not depend either on n, or on the times t. This set can be considered as part of the initial data for the evolution equations (2.1).

Finally, let us examine the relation (2.13) in the neighborhood of punctures $Q_\infty^\pm = (\pm, \infty)$ on two sheets of Γ_N. Taking the limit $E \to \infty$ one obtains from (2.13) that the solutions have the following asymptotics on, say, the $(-)$ sheet: $\varphi_n(Q \to Q_\infty^-) = \mathcal{O}(E^n)$ and $\chi_n(Q \to Q_\infty^-) = \mathcal{O}(E^n)$. Let us require that the solutions to (2.13) should have a similar behaviour on the upper sheet, $\varphi_n(Q \to Q_\infty^+) = \mathcal{O}(E^{-n})$ and $\chi_n(Q \to Q_\infty^+) = \mathcal{O}(E^{-n})$. This behaviour is consistent with (2.13) provided that the r.h.s. of (2.13) scales as E^0 at large E, or equivalently $\chi_n(Q_\infty^+) - z_n \varphi_n(Q_\infty^+) = 0$, leading to the expression for the reggeon coordinates as a ratio of the components of the Baker–Akhiezer function at the point Q_∞^+. Thus, having constructed the Baker–Akhiezer function satisfying the desired asymptotic behaviour one will be able to determine the reggeon coordinates.

3. Construction of the Baker–Akhiezer function

The existence of the Baker–Akhiezer function with the properties established in the previous section can be deduced from the following well-known fact in the theory of finite-gap soliton solutions [10]. For a smooth hyperelliptic algebraic curve Γ_N of genus $g = N - 2$ with two punctures at $Q_\infty^\pm = (\pm, \infty)$ and a given set of $g+1$ points $(\gamma_1, \gamma_2, \dots, \gamma_{g+1})$ in general position there exists a unique function $\Psi_n^\alpha(Q; \{\tau\})$ such that

- Ψ_n is a meromorphic outside the punctures $Q_\infty^\pm$ and has simple poles at the points $\gamma_1, \dots, \gamma_{g+1}$;
- Ψ_n satisfies the normalization conditions (2.13) at the points $Q_0^\pm = (\pm, E = 0)$;
- At the vicinity of two punctures on the upper and lower sheets the components Ψ_n^α $(\alpha = 1, 2)$ have the following asymptotics:

$$\Psi_n^\alpha(Q \to Q_\infty^\pm; \{\tau\}) = E^{\mp n} \mathrm{e}^{\pm \sum_{j=1}^{N-2} \tau_j E^j} \left[\phi_\alpha^\pm(n, \{\tau\}) + \mathcal{O}\left(\frac{1}{E}\right)\right], \tag{3.1}$$

where $\phi_\alpha^\pm$ is some E-independent function.

The following comments are in order. In this definition, the Baker–Akhiezer function depends on the set of parameters $\{\tau\} = \tau_1, \dots, \tau_{N-2}$, which are different, in general, from the evolution times $\{t\}$ entering (2.1). However, as a function of $\{\tau\}$, $\Psi_n^\alpha(Q; \{\tau\})$ satisfies the system of first-order linear matrix equations similar to (2.6) and, as we will show in Section 4.1, the evolution times $\{t\}$ corresponding to the Hamiltonian flows are related to auxiliary parameters $\{\tau\}$ by a *linear* transformation, $t_{j+2} = \sum_k A_{jk}\tau_k$.

Finally, substituting expression (3.1) into (2.13) and comparing the asymptotic behaviour of different sides of the relation (2.13) for $Q \to Q_\infty^\pm$ we obtain the following consistency conditions:

$$z_n^{(0)} = \frac{\phi_2^+(n, \{\tau\})}{\phi_1^+(n, \{\tau\})} = \frac{\phi_2^-(n+1, \{\tau\})}{\phi_1^-(n+1, \{\tau\})} \tag{3.2}$$

and

$$p_n^{(0)} = \frac{1}{z_{n,n+1}^{(0)}} \frac{\phi_1^+(n, \{\tau\})}{\phi_1^+(n+1, \{\tau\})} = \frac{1}{z_{n-1,n}^{(0)}} \frac{\phi_1^-(n+1, \{\tau\})}{\phi_1^-(n, \{\tau\})}. \tag{3.3}$$

Together with (2.16) these relations provide the solution to the hierarchy of the evolution equations for reggeon coordinates and momenta, (2.1), in terms of the Baker–Akhiezer function.

3.1. Basis of differentials

To write down the expression for the Baker-Akhiezer function satisfying the above conditions one defines the canonical basis of cycles on Γ_N as shown in Fig. 1 and constructs the basis of normalized differentials as follows [11].

The unique set of holomorphic differentials on Γ_N, or differentials of the first kind, $d\omega_k$ ($k = 1, \dots, N-2$), is defined as

$$d\omega_k = \sum_{j=1}^{N-2} U_{kj} \frac{dE\, E^{j-1}}{y(E)} = \sum_{j=1}^{N-2} U_{kj} \frac{dE\, E^j}{\sqrt{\Lambda^2(E) - 4}} . \tag{3.4}$$

The coefficients U_{kj} are fixed by the normalization conditions

$$\oint_{\alpha_j} d\omega_k = 2\pi \delta_{jk} , \qquad j, k = 1, \dots, N-2 \tag{3.5}$$

and they can be calculated as an inverse to the following matrix:

$$\left(U^{-1}\right)_{jk} = \frac{1}{2\pi} \oint_{\alpha_k} \frac{dE\, E^j}{\sqrt{\Lambda^2(E) - 4}} . \tag{3.6}$$

The unique set of meromorphic differentials on Γ_N of the second kind, $d\Omega^{(j)}$ ($j \geqslant 1$), with the jth order pole at the punctures $Q_\infty^\pm$ is defined as

$$d\Omega^{(j)} = \frac{dE}{\sqrt{\Lambda^2(E) - 4}} \left[jq_N E^{N+j-1} + jq_{N-1} E^{N+j-2} + \dots + \mathcal{O}(E) \right] .$$

The coefficients in front of the remaining powers of E in the numerator are fixed by the normalization conditions

$$\oint_{\alpha_k} d\Omega^{(j)} = 0 , \qquad d\Omega^{(j)} \Big|_{Q \to Q_\infty^\pm} \overset{E \to \infty}{=} \pm \left(jE^{j-1} + \mathcal{O}(1/E^2) \right) dE . \tag{3.7}$$

The unique dipole differentials on Γ_N, or differentials of the third kind, $d\Omega_\infty$ and $d\Omega_0$, having simple poles at the points $Q_\infty^\pm$ and $Q_0^\pm$, respectively, and normalized by the conditions

$$\oint_{\alpha_j} d\Omega_\infty = \oint_{\alpha_j} d\Omega_0 = 0 , \qquad d\Omega_\infty \Big|_{Q \to Q_\infty^\pm} \overset{E \to \infty}{=} \mp \frac{dE}{E} , \qquad d\Omega_0 \Big|_{Q \to Q_0^\pm} \overset{E \to 0}{=} \pm \frac{dE}{E} \tag{3.8}$$

are defined as

$$d\Omega_\infty = -\frac{1}{N} \frac{dE\, \Lambda'(E)}{\sqrt{\Lambda^2(E) - 4}} = \frac{dE}{\sqrt{\Lambda^2(E) - 4}} \left[-q_N E^{N-1} + \dots + \mathcal{O}(E) \right] \tag{3.9}$$

and

$$d\Omega_0 = \frac{2ihdE}{\sqrt{\Lambda^2(E)-4}} + i\sum_{j=1}^{N-2}(U_0)_j \frac{dE\,E^j}{\sqrt{\Lambda^2(E)-4}}. \tag{3.10}$$

To satisfy the normalization conditions (3.8) the coefficients $(U_0)_j$ have to be equal to

$$(U_0)_k = -\frac{h}{\pi}\sum_{j=1}^{N-2} U_{jk} \oint_{\alpha_j} \frac{dE}{\sqrt{\Lambda^2(E)-4}} \tag{3.11}$$

with the matrix U_{jk} defined in (3.4) and (3.6).

Let us also define the following $(N-2)$-dimensional vectors V, $W^{(j)}$ and $A(Q)$:

$$V_k = -i\oint_{\beta_k} d\Omega_\infty, \qquad W_k^{(j)} = -i\oint_{\beta_k} d\Omega^{(j)}, \qquad A_k(Q) = \int_{\gamma_0}^{Q} d\omega_k, \tag{3.12}$$

where $k = 1,\dots,N-2$, γ_0 is an arbitrary reference point on Γ_N and integration in $A(Q)$ goes along some path on Γ_N between the points Q and γ_0.

3.2. Baker-Akhiezer function

The components of the Baker-Akhiezer function, $\Psi_n^\alpha(Q;\{\tau\})$ $(\alpha = 1,2)$, satisfying the conditions formulated in the beginning of Section 3 can be expressed in terms of the θ-function defined by the Riemann surface Γ_N (for definition see Appendix A) as follows [14]:

$$\Psi_n^\alpha(Q;\{\tau\}) = \frac{h_\alpha(Q)}{h_\alpha(\widetilde{Q}_\alpha)} \frac{\theta(A(Q)+Vn+\sum_{j=1}^{N-2}W^{(j)}\tau_j+Z_\alpha)\theta(Z_0)}{\theta(A(Q)+Z_\alpha)\theta(Vn+\sum_{j=1}^{N-2}W^{(j)}\tau_j+Z_0)}$$
$$\times\exp\left(n\int_{\widetilde{Q}_\alpha}^{Q} d\Omega_\infty + \sum_{j=1}^{N-2}\tau_j \int_{\widetilde{Q}_\alpha}^{Q} d\Omega^{(j)}\right). \tag{3.13}$$

Here, $\widetilde{Q}_1 = Q_0^+$ and $\widetilde{Q}_2 = Q_0^-$ are normalization points, the $(N-2)$-dimensional constant vectors Z_α and Z_0 are given by

$$Z_\alpha = Z_0 - A(\widetilde{Q}_\alpha), \qquad Z_0 = A(\widetilde{Q}_1) + A(\widetilde{Q}_2) - \sum_{n=1}^{g+1} A(\gamma_n) - \mathcal{K}$$

with $\mathcal{K}$ being the vector of Riemann constants. The function $h_\alpha(Q)$ is defined as

$$h_\alpha(Q) = \frac{\theta(A(Q)+Z_\alpha)\theta(A(Q)-T_\alpha)}{\theta(A(Q)-T_+)\theta(A(Q)-T_-)},$$

together with the constants

$$T_+ = \sum_{s=1}^{g-1} A(\gamma_s) + A(\gamma_g) + \mathcal{K}, \qquad T_- = \sum_{s=1}^{g-1} A(\gamma_s) + A(\gamma_{g+1}) + \mathcal{K},$$

$$T_\alpha = \sum_{\substack{j=1,2 \\ j\neq\alpha}} A(\widetilde{Q}_j) + \sum_{s=1}^{g-1} A(\gamma_s) + \mathcal{K}.$$

The following comments are in order. According to (3.12), the definition of the vectors $A(\widetilde{Q}_\alpha)$ and $A(\gamma_n)$ implies a certain choice of the integration contours connecting the points $\widetilde{Q}_\alpha$ and γ_n with the reference point γ_0. The same contours should enter into the definition of the contour integrals of meromorphic differentials in the exponent of (3.13), $\int_{\widetilde{Q}_\alpha}^{Q} d\Omega^{(j)} = \int_{\gamma_0}^{Q} d\Omega^{(j)} - \int_{\gamma_0}^{\widetilde{Q}_\alpha} d\Omega^{(j)}$. Only under this condition the Baker-Akhiezer function (3.13) does not depend either on the choice of the integration contours, or on the reference point γ_0. Indeed, deforming the integration contour between, say, points Q and $\widetilde{Q}_\alpha$ as $\gamma_{Q\widetilde{Q}_\alpha} \to \gamma_{Q\widetilde{Q}_\alpha} + \sum_j N_j\alpha_j + \sum_j M_j\beta_j$, and using the transformation properties (A.3) of the Riemann θ-function one gets that the variations of the exponent and the prefactor in (3.13) compensate each other, while the function $h_\alpha(Q)$ stays invariant.

Let us show that the thus defined Baker-Akhiezer function (3.13) satisfies the necessary normalization conditions. Considering the expression for $h_1(Q)$ and using the properties of the zeros of the θ-functions, (A.4), we notice that two θ-functions in the denominator of $h_1(Q)$ vanish at the points $\gamma_1, \dots, \gamma_{g-1}, \gamma_g$ and $\gamma_1, \dots, \gamma_{g-1}, \gamma_{g+1}$, respectively, while the numerator vanishes at the points $\gamma_1, \dots, \gamma_{g-1}, \widetilde{Q}_2$ plus additional g points coming from the first θ-function. In the expression for the Baker-Akhiezer function, (3.13), the latter cancels, however, against the same factor in the denominator of (3.13) and its zeros are replaced by zeros of the first θ-function in the numerator of (3.13). Therefore, the component $\Psi_n^1(Q;\{\tau\})$ has simple poles at the $g+1$ points $\gamma_1, \dots, \gamma_{g+1}$ and it vanishes at the point $\widetilde{Q}_2 \equiv Q_0^-$ plus additional g points $Q_1, \dots, Q_g$. Being solutions of the equation $\theta(A(Q_s) + Vn + \sum_{j=1}^{N-2} W^{(j)}\tau_j + Z_1) = 0$ they satisfy the relation (A.4), or

$$\sum_{s=1}^{N-2} A(Q_s) + A(Q_0^-) - \sum_{s=1}^{N-1} A(\gamma_s) = -Vn - \sum_{j=1}^{N-2} W^{(j)}\tau_j . \tag{3.14}$$

One checks that for $Q \to \widetilde{Q}_1 \equiv Q_0^+$ different factors in (3.13) cancel against each other leaving us with $\Psi_n^1(Q_0^+) = 1$. As to asymptotic behaviour of $\Psi_n^1(Q)$ for $Q \to Q_\infty^\pm$, it is controlled by the exponent in (3.13) and it is in agreement with (3.1) due to normalization conditions (3.7) and (3.8) for the meromorphic differentials. The generalization of this analysis to the component $\Psi_n^2(Q)$ is straightforward. We conclude that (3.13) gives a unique expression for the Baker-Akhiezer function corresponding to the N reggeon system.

4. Explicit solutions

We examine the asymptotics of the Baker-Akhiezer function (3.13) in the vicinity of infinity Q_∞^+ and compare it with (3.1) to find the explicit expression for the function ϕ_α^+. Substituting it into (3.2) and (3.3) we obtain the solution to the hierarchy of the evolution equations for the reggeon coordinates and momenta in the following form:

$$z_n^{(0)} = \rho \frac{\theta\left(Vn + \sum_{j=1}^{N-2} W^{(j)}\tau_j + Z_+ + i\Delta\right)}{\theta\left(Vn + \sum_{j=1}^{N-2} W^{(j)}\tau_j + Z_+\right)} \exp\left(iV_0 n + i\sum_{j=1}^{N-2} W_0^{(j)}\tau_j\right) \tag{4.1}$$

and

$$p_n^{(0)} z_{n,n+1}^{(0)} = q_N^{1/N} \frac{\theta\left(Vn + \sum_{j=1}^{N-2} W^{(j)}\tau_j + Z_+\right)\,\theta\left(V(n+1) + \sum_{j=1}^{N-2} W^{(j)}\tau_j + Z_0\right)}{\theta\left(Vn + \sum_{j=1}^{N-2} W^{(j)}\tau_j + Z_0\right)\,\theta\left(V(n+1) + \sum_{j=1}^{N-2} W^{(j)}\tau_j + Z_+\right)}. \tag{4.2}$$

Here, the constant ρ is given by a ratio of θ-functions and its value can be made arbitrary using the symmetry (2.18). The $(N-2)$-dimensional constant vector $(Z_+)_k$ is equal to

$$Z_+ = A(Q_\infty^+) + Z_1 = A(Q_\infty^+) - A(Q_0^+) + Z_0 = \int_{Q_0^+}^{Q_\infty^+} d\omega + Z_0 .$$

The phase shift Δ and the oscillator frequencies are defined as

$$\Delta_k = -i\int_{Q_0^-}^{Q_0^+} d\omega_k\,, \qquad V_0 = -i\int_{Q_0^-}^{Q_0^+} d\Omega_\infty\,, \qquad W_0^{(j)} = -i\int_{Q_0^-}^{Q_0^+} d\Omega^{(j)}\,, \tag{4.3}$$

where the normalized differentials are integrated along the path on Γ_N connecting the points $Q_0^\pm$ and going through the reference point γ_0. Although γ_0 enters into the definition (3.12) of the vector $A(Q)$, neither the Baker-Akhiezer function (3.13), nor the reggeon coordinates (4.1) and (4.2) depend on its particular choice. Prefactor, $q_N^{1/N}$, in (4.2) originates from the following expansion:

$$\exp\left(-\int_{Q_0^+}^{Q} d\Omega_\infty\right) \overset{Q\to Q_\infty^+}{=} \exp\left(\frac{1}{N}\int_0^E \frac{d\Lambda(E)}{\sqrt{\Lambda^2-4}}\right) = Eq_N^{1/N}(1+\mathcal{O}(1/E))$$

and it is consistent with the definition (1.4) of the Hamiltonian q_N, i.e. $\prod_{k=1}^N p_n^{(0)} z_{n,n+1}^{(0)} = q_N$. The expressions (4.1) and (4.2) involve $N-1$ free parameters – the prefactor ρ and the $(N-2)$-dimensional vector Z_+ (or equivalently Z_0).

Let us show that solutions (4.1) and (4.2) satisfy the periodicity conditions (2.2). Using the transformation properties of the θ-function, (A.3), the same conditions can be expressed as

$$V_k = -i\oint_{\beta_k} d\Omega_\infty = 2\pi\frac{m_k}{N}, \qquad V_0 = -i\int_{Q_0^-}^{Q_0^+} d\Omega_\infty = 2\pi\frac{m}{N},$$

with m_k and m being integers. To verify them we take into account the relation between the eigenvalues of the monodromy matrix, (2.9), and the dipole differential (3.9)

$$dP(E) = -N d\Omega_\infty, \tag{4.4}$$

and calculate from (2.9) and (2.11) the multi-valued function $P(E)$ at the branching points e_j of the curve Γ_N as $e^{P(e_k)} = \pm 1$ and at the points $Q_0^\pm$ above $E = 0$ as $e^{P(0)} = 1$.

4.1. Evolution times

We recall that the parameters τ_k entering (4.1) and (4.2) do not coincide with the evolution times t_k in (2.1). To find the relation between them we notice that $g+1$ points $Q_1, \dots, Q_g$, $Q_{g+1} \equiv Q_0^-$ obey (3.14) being zeros of the component $\Psi_n^1(Q)$ of the Baker–Akhiezer function. Moreover, the same points correspond to zeros of the B-component of the monodromy matrix and their t-evolution is described by (2.22). Comparing the relations (3.14) and (2.22) and using the definition (3.12) of the vector $A(Q_s)$ together with (3.4), one obtains that up to a unessential additive constant

$$\sum_{j=1}^{N-2} W_k^{(j)}\tau_j = \sum_{j=1}^{N-2} U_{kj} t_{j+2}, \tag{4.5}$$

where the matrix U was defined in (3.4) and (3.6). Let us also consider the τ-dependent part of the exponent in (4.1) and take into account the relations (B.3) and (4.5) to get

$$i\sum_{j=1}^{N-2} W_0^{(j)}\tau_j = -\frac{ih}{\pi}\sum_{j,k=1}^{N-2} W_k^{(j)}\tau_j \oint_{\alpha_k}\frac{dE}{\sqrt{\Lambda^2(E)-4}} = -\frac{ih}{\pi}\sum_{j,k=1}^{N-2} U_{kj}\oint_{\alpha_k}\frac{dE}{\sqrt{\Lambda^2(E)-4}} t_{j+2}.$$

Comparing this identity with (3.11) and (4.3) one obtains

$$\sum_{j=1}^{N-2} W_0^{(j)}\tau_j = \sum_{j=1}^{N-2} (U_0)_j t_{j+2}. \tag{4.6}$$

Substituting the transition formulas (4.5) and (4.6) into (4.1) and (3.13) we restore the dependence of the reggeon coordinates and the Baker–Akhiezer function on the evolution times t_k. Finally, combining (4.1), (4.5), (4.6) and (2.19) we obtain the expressions for the holomorphic reggeon coordinates as

$$z_n^{(0)}(t) = \rho\exp(2ih\, t_2 + iV_0\, n + iU_0\cdot t)\frac{\theta(Vn + U\cdot t + Z_+ + i\Delta)}{\theta(Vn + U\cdot t + Z_+)}, \tag{4.7}$$

where the following notations were introduced:

$$U_0 \cdot t = \sum_{j=1}^{N-2} (U_0)_j t_{j+2}, \qquad U_k \cdot t = \sum_{j=1}^{N-2} U_{kj} t_{j+2}.$$

The ratio of the θ-functions in (4.7) describes linear trajectories on the Jacobian $\mathcal{J}(\Gamma_N)$ of the curve (2.10). Relation (4.7) implies that the motion of reggeons becomes an image of these trajectories on the complex z-plane modulated by $U(1)$ rotations with the basic frequencies $(U_0)_k$.

4.2. Curves with real branching points

In the above consideration, the conserved charges q_k as well as the branching points e_k of the curve Γ_N were assumed to have arbitrary complex values. Let us consider the situation when all $2N-2$ branching points (2.11) of the algebraic curve Γ_N are distinct and real,

$$e_1 < \ldots < e_{2N-2}. \tag{4.8}$$

The branching points were defined in (2.11) as roots of the polynomial $\Lambda^2(E) - 4$ and in order for e_k to be real the values of charges q_k should also be real and, most importantly, they have to satisfy certain additional conditions. In particular, q_2 should be negative

$$q_2 < 0, \tag{4.9}$$

while the explicit form of the conditions on $q_3, \ldots, q_N$ depends on the number of reggeons N [6,8]. For example, for $N = 3$ states, they can be expressed as

$$0 \leqslant u_3^2 \leqslant \frac{1}{27}, \tag{4.10}$$

and for $N = 4$ states as

$$-4u_4 \leqslant u_3^2 \leqslant \frac{8}{9} \frac{u_4\left(\sqrt{48u_4+1}-2\right)^2}{\sqrt{48u_4+1}-1}, \tag{4.11}$$

where $u_3 = q_3/(-q_2)^{3/2}$ and $u_4 = q_4/q_2^2$. Let us identify the moduli of the hyperelliptic curve Γ_N as $(N-2)$-dimensional vector with the components $u_k = q_k/(-q_2)^{k/2}$ ($k = 3, \ldots, N$). The relations (4.10) and (4.11) define the regions on the moduli space, $\mathcal{M}(\Gamma_N)$, corresponding to curves Γ_N with real branching points. At the boundary of these regions two of the branching points merge and the curve Γ_N becomes singular.

There is a special interest in considering soliton solutions for curves with real branching points. We recall that the reggeon trajectories (4.1) appear as solutions of the evolution equations generated by the leading term S_0 of the WKB expansion of the wave function of N reggeon state (1.6). A natural question arises as to whether the WKB expansion gives a meaningful approximation to solutions of the Schrödinger equation (1.1). It turns out that for the quantum numbers q_k satisfying (4.10) and (4.11) one can construct the *exact* solutions to (1.1) within the Bethe ansatz approach [4,15]. A thorough analysis shows [6] that the WKB expansion is in a good agreement with the exact solutions. Thus, similar to electron orbits in the Bohr model of the hydrogen atom, the reggeon trajectory, corresponding to curves Γ_N with real branching points, describes the quasiclassical limit of the quantum states constructed in [4,15].

Let us consider the properties of solitons (4.1) under the additional condition (4.8). According to our definition of the canonical set of cycles shown in Fig. 1, the cycle α_j lies on Γ_N above the interval $[e_{2j}, e_{2j+1}]$, called forbidden zones, on which $\Lambda^2(E) > 4$. The Riemann surface defined by the curve Γ_N is constructed by gluing two copies of the complex E-plane along $2N-2$ finite intervals $[e_{2j}, e_{2j+1}]$ and one infinite interval $[e_{2N-2}, e_1]$ going through infinity. On the intervals $[e_{2j-1}, e_{2j}]$ $(j = 1, \dots, N-1)$, called allowed zones, one has $\Lambda^2(E) - 4 \leqslant 0$ and the eigenvalue of the monodromy matrix, $P(E)$, has pure imaginary values with $\frac{1}{N}\,\mathrm{Im}\,P(E)$ being a quasimomentum. In particular, since $\Lambda(0) = 2$, the point $E = 0$ belongs to one of the allowed zones, which we denote as $[e_{2K-1}, e_{2K}]$,

$$e_1 < \dots < e_{2K-1} < 0 < e_{2K} < \dots < e_{2N-2}. \tag{4.12}$$

The values of e_{2K-1} and e_{2K} can be defined as roots of (2.11) closest to the origin. The points $Q_0^{\pm}$ belong to the different sheets of Γ_N and one has to specify the integration path on Γ_N between the points Q_0^+ and Q_0^- entering (4.3). We would like to stress that although its choice in (4.3) can be arbitrary, in order to apply the relations (4.5) and (4.6) one has to require that the path should not cross the canonical cycles α_j and β_j. The latter condition fixes it uniquely as shown in Fig. 1. The path is trapped between the cycles β_{K-1} and β_K and its orientation on Γ_N is opposite to that of the β-cycles.

Under these definitions it is easy to see from (3.6), and (3.11) that the parameters U_{jk}, $(U_0)_k$, V, V_0 and Δ are real. Moreover, according to (4.4) and (3.12), the components of the vector V_n are given by the difference of the values of the quasimomentum at the branching points e_{2N-2} and e_{2n+1}, that is

$$V_n = \begin{cases} -\dfrac{2\pi}{N}(N-n)\,, & 1 \leqslant n \leqslant K-1 \\ -\dfrac{2\pi}{N}(N-n-1)\,, & K \leqslant n \leqslant N-2 \end{cases},$$

where the integer K is defined by configuration of the branching points. In a similar way, the parameter V_0 can be calculated as

$$V_0 = 2i\int_0^{e_{2K}} d\Omega_\infty + i\oint_{\beta_K} d\Omega_\infty = \frac{2\pi}{N} - V_K = \frac{2\pi}{N}(N-K)\,.$$

For the vector Δ_n we get

$$\Delta_n = 2i\int_0^{e_{2K}} d\omega_n + i\oint_{\beta_K} d\omega_n = \Delta_n^{(0)} + 2\pi i\tau_{Kn},$$

where τ_{Kn} is an element of Riemann matrix (A.1) and $\Delta_n^{(0)} = 2\sum_{j=1}^{N-2} U_{nj}\int_0^{e_{2K}} \frac{dE\, E^j}{\sqrt{4-\Lambda^2(E)}}$.

4.3. Solitons for $N = 3$ reggeon states

Let us consider in more detail the properties of the soliton waves for $N = 3$ reggeon states with the quantum numbers $q_2 = -h^2$ and q_3 satisfying the conditions (4.9) and (4.10). Choosing in (4.10) the branch $0 \leqslant q_3/h^3 \leqslant \frac{1}{\sqrt{27}}$ one matches the branching points of Γ_3 into (4.12) to get $K = 1$. The soliton solution (4.7) takes the form

$$z_n^{(0)}(t_2,t_3) = \rho \exp\left(2ih\,t_2 + \frac{4i\pi}{3}n + iU_0 t_3\right)\frac{\theta(-\frac{2\pi}{3}n + Ut_3 + Z_+ + i\Delta)}{\theta(-\frac{2\pi}{3}n + Ut_3 + Z_+)}, \qquad (4.13)$$

where ρ and Z_+ are arbitrary complex parameters defined by the initial conditions. The values of U_0, U and Δ can be calculated from (3.6), (3.11) and (4.3) as elliptic integrals

$$U = \frac{\pi}{\int_{e_2}^{e_3} \frac{dE\,E}{\sqrt{\Lambda^2-4}}}, \qquad U_0 = -2h\frac{\int_{e_2}^{e_3} \frac{dE}{\sqrt{\Lambda^2-4}}}{\int_{e_2}^{e_3} \frac{dE\,E}{\sqrt{\Lambda^2-4}}}, \qquad \Delta = 2U\int_{e_1}^{0} \frac{dE\,E}{\sqrt{4-\Lambda^2}} \qquad (4.14)$$

with $\Lambda(E) = 2 - h^2E^2 + q_3E^3$ and $\Lambda(e_1) = \Lambda(e_2) = \Lambda(e_3) = -2$. The modular parameter τ entering the definition of the Riemann θ-function takes pure imaginary values, $\mathrm{Im}\,\tau \geqslant 0$,

$$\tau = i\frac{\int_{e_3}^{e_4} \frac{dE\,E}{\sqrt{4-\Lambda^2}}}{\int_{e_2}^{e_3} \frac{dE\,E}{\sqrt{\Lambda^2-4}}}. \qquad (4.15)$$

For q_3 inside the region (4.10), the expression (4.13) defines quasiperiodic reggeon trajectories on the (x,y) plane of transverse coordinates, $z = x + iy$. An example of such trajectory corresponding to the quantum numbers $q_2 = -1$ and $q_3 = 1/7$ is shown in Fig. 2. An interesting property of the trajectories (4.13) is that at any moment of time t_2 and t_3 the reggeon coordinates satisfy the condition

$$\frac{z_1 - z_2}{z_1 + z_2} + \frac{z_2 - z_3}{z_2 + z_3} + \frac{z_3 - z_1}{z_3 + z_1} = -\frac{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}{(z_2 + z_3)(z_2 + z_3)(z_3 + z_1)} = i\frac{h^3}{q_3}. \qquad (4.16)$$

It follows from the expression (1.4), $q_3 = z_{12}z_{23}z_{31}p_1p_2p_3$, after one excludes the momenta using $L_3 = -h$ and $L_+ = L_- = 0$.

For $q_3 = 0$ and $q_3 = h^3/\sqrt{27}$ the branching points of the curve Γ_3 merge leading to $\tau = 0$ and $\tau = i\infty$, respectively. The curve Γ_3 becomes singular and the properties of the soliton solutions are changed.

4.3.1. Singularity at $q_3 = h^3/\sqrt{27}$

In this case, the branching points are located at $e_1 = -\sqrt{3}/h$, $e_2 = e_3 = 2\sqrt{3}/h$, $e_4 = 3\sqrt{3}/h$ and the α-cycle on Γ_3 is shrinking into a point. Applying relations (4.14) and (4.15) one gets

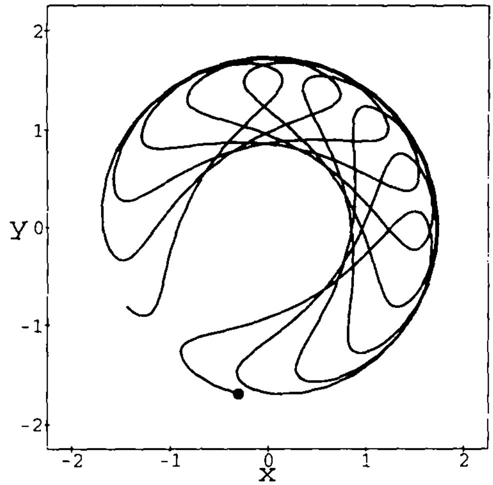

Fig. 2. The trajectory of the reggeon on the (x, y) plane described by Eq. (4.13) for $t_2 = 0$ and $n = 1$. The parameters have the following values: $\rho = 1$, $h = 1$ and $Z_+ = 3 - i$. The dot denotes the initial position of the reggeon at $t_3 = 0$.

$$U_0 = -U = -\frac{h^2}{\sqrt{3}}, \qquad \Delta = -2\ln 2, \qquad \tau \to i\infty. \tag{4.17}$$

We notice that in the limit $\tau \to i\infty$ and $Z_+ = \mathcal{O}(\tau^0)$ the θ-functions in (4.13) are replaced by 1, since only one term with $m = 0$ survives in the infinite sum (A.2). The corresponding solution, $z_n^{(0)}(t_2, t_3) = \rho\exp(2iht_2 + \frac{4\pi i}{3}n - i\frac{h^2}{\sqrt{3}}t_3)$, describes three reggeons located in the vertices of equilateral triangle and rotating around the origin as a whole. In order to avoid such rigid reggeon configurations, one has to adjust the constant Z_+ as

$$Z_+ = \pi\tau + Z_\infty + \mathcal{O}(\tau^{-1})$$

and use the asymptotic behaviour of the θ-function (A.2)

$$\theta(u + \pi\tau; \tau) = 1 + e^{-iu}, \qquad \text{as } \tau \to i\infty.$$

Finally, the soliton solution for $q_3 = h^3/\sqrt{27}$ becomes

$$z_n^{(0)}(t_2, t_3) = \rho\, e^{2iht_2 + \frac{2\pi i}{3}n}\, \frac{4 + e^{i\Phi}}{4(1 + e^{-i\Phi})}, \qquad \Phi = \frac{2\pi}{3}n - \frac{h^2}{\sqrt{3}}t_3 - Z_\infty \tag{4.18}$$

with ρ and Z_∞ being arbitrary constants. This relation defines periodic reggeon trajectories on the plane and example of such a trajectory is shown in Fig. 3.

4.3.2. Singularity at $q_3 = 0$

In this case, $e_1 = -e_2 = -2/h$ and two remaining branching points merge at infinity, $e_3 = e_4 = \mathcal{O}(1/q_3)$. Integration in (4.14) and (4.15) yields

$$U = -i\tau h^2, \qquad \Delta = i\tau\pi, \qquad \tau \to 0.$$

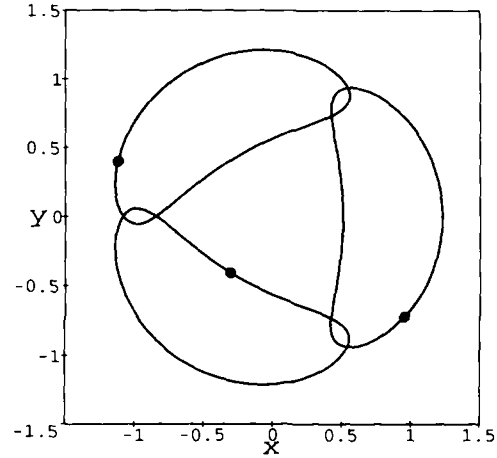

Fig. 3. The reggeon trajectories on the (x, y) plane described by Eq. (4.18) for $t_2 = 0$. The parameters have the following values: $\rho = 1$, $h = 1$ and $Z_\infty = 1 + i$. Dots denote the initial positions of the reggeons at $t_3 = 0$.

To analyze (4.13) at small τ one uses the duality property of the θ-function [11]

$$\theta(z;\tau) = (-i\tau)^{-1/2} \exp\left(-\frac{iz^2}{4\pi\tau}\right) \theta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right), \tag{4.19}$$

which maps $\tau \to 0$ limit into asymptotic behaviour at $\tau \to i\infty$. Taking the constant Z_+ in the form $Z_+ = Z_0 + \tau Z_1$ with Z_0 and Z_1 arbitrary one gets from (4.19) and (4.13) that in the limit $\tau \to 0$ the solution (4.13) becomes t_3-independent, $z_n^{(0)}(t_2, t_3) = (-)^n \rho \exp(2hit_2 + \frac{i}{2} Z_0)$. To get a non-trivial solution one has to adjust the constant Z_+ as

$$Z_+ = \frac{\pi}{3} + \tau Z_1 + \mathcal{O}(\tau^2).$$

The resulting expression for reggeon coordinates looks like

$$z_1^{(0)}(t_2, t_3) = -z_3^{(0)}(t_2, t_3) = \rho\, e^{2iht_2},$$
$$z_2^{(0)}(t_2, t_3) = z_3^{(0)}(t_2, 0) \tanh\left(\frac{h^2}{2} t_3 - \frac{i}{2} Z_1\right). \tag{4.20}$$

Here, the positions of the reggeons with the numbers $n = 1$ and $n = 3$ are frozen in time t_3 and the $n = 2$nd reggeon moves between them, $z_2^{(0)}(t_2, -\infty) = z_1^{(0)}(t_2, 0)$ and $z_2^{(0)}(t_2, \infty) = z_3^{(0)}(t_2, 0)$. It is easy to see that the trajectory (4.20) is consistent with (4.16).

5. Multi-soliton solutions

As we have seen in the previous section, the soliton solutions corresponding to the degenerate singular spectral curve Γ_N can be written for $N = 3$ in terms of trigonometric functions. In the finite-gap theory they are called multi-soliton solutions [9]. Let us show that the same property holds for any N. Although multi-soliton solutions can be obtained as limits of general periodic solutions (4.7) for the degenerate curve Γ_N, there is a simpler way to write them directly without going through a complicated degeneration procedure.

The multi-soliton solutions correspond to a degenerate case, when all but two roots of Eq. (2.11) are double, that is

$$\Lambda^2(E) - 4 = q_N^2 (E - e_-)(E - e_+) E^2 \prod_{j=1}^{N-2} (E - e_j)^2 , \tag{5.1}$$

with $e_- \neq e_+$. In this limit, the branching points of the curve Γ_N merge and the definition (2.10) of Γ_N takes the form

$$\Gamma_N^{\rm sing} : \qquad \widetilde{y}^2 = (E - e_-)(E - e_+) , \qquad y = \widetilde{y} \times q_N \prod_{j=1}^{N-2} (E - e_j) . \tag{5.2}$$

Here, $\widetilde{y}$ is a rational curve and the Riemann surface corresponding to $\Gamma_N^{\rm sing}$ has a genus $g = 0$, a sphere. It is constructed by gluing together two sheets along the cuts $(\infty, e_-]$ and $[e_+, \infty)$. One can introduce a global complex parameter κ on Γ_N and parameterize the solutions to (5.2) as

$$E = \beta\left(\kappa + \frac{1}{\kappa}\right) + \alpha , \qquad \widetilde{y} = \beta\left(\kappa - \frac{1}{\kappa}\right) \tag{5.3}$$

with $\beta = \frac{1}{4}(e_+ - e_-)$ and $\alpha = \frac{1}{2}(e_- + e_-)$. In this parameterization, the points $\kappa = \infty$ and $\kappa = 0$ correspond to the punctures $Q_\infty^\pm$ on the upper and lower sheets, respectively. The permutation of sheets of Γ_N corresponds to involution in the κ-plane, $\kappa \to 1/\kappa$, which does not change the value of E,

$$E(\kappa) = E(1/\kappa) . \tag{5.4}$$

All meromorphic functions (and differentials) on Γ_N introduced before can be now rewritten as functions of the parameter κ only. In particular, consider the function $w = e^{P(E)}$, the Bloch multiplier, defined in (2.8). According to (2.12), it has a pole of order N at Q_∞^+ and a zero of degree N at Q_∞^-. Therefore, as a function of κ it has a unique form $w = a\kappa^N$. At the vicinity of the punctures $Q_\infty^\pm$ one substitutes $\kappa \sim \beta/E \to 0$ and $\kappa \sim E/\beta \to \infty$ into $w(\kappa)$ and matches the asymptotics of $w(E)$ with (2.12) to get the values of the constants, $a = 1$ and $\beta = q_N^{-1/N}$, leading to

$$w = e^{P(E)} = \kappa^N .$$

To find the positions of the branching points of $\Gamma_N^{\rm sing}$ we substitute this relation into (2.9),

$$\Lambda(E) = \kappa^N + \kappa^{-N} = 2 + q_2 E^2(\kappa) + \ldots + q_N E^N(\kappa) , \tag{5.5}$$

and notice that, according to (5.1), $\Lambda^2(E) = 4$ at the branching points and $\Lambda(E) = 2$ for $E = 0$. Thus, $\kappa^{2N} = 1$ at the branching points $e_\pm$ and e_j, while $\kappa^N = 1$ at two points $Q_0^\pm$ on $\Gamma_N^{\rm sing}$ above $E = 0$. These conditions define $2N$ points on the complex κ-plane

$$\kappa_j = \mathrm{e}^{\frac{i\pi}{N}j} \quad (j = 1, \dots, 2N-1) \tag{5.6}$$

and we identify the corresponding values $E(\kappa_j) = 2q_N^{-1/N}\cos\left(\frac{\pi j}{N}\right) + \alpha$ as $2N$ roots of $\Lambda^2 - 4$ in (5.1). Since $1/\kappa_j = \kappa_{2N-j}$, it follows from (5.4) that, in agreement with (5.1), $2N-2$ roots are degenerate, $E(\kappa_j) = E(\kappa_{2N-j})$ and $j \neq 0, N$. The remaining two roots, $j = 0, N$, correspond to the branching points $e_+ = E(\kappa_0)$ and $e_- = E(\kappa_N)$. Among $N-1$ pairs κ_j, κ_{2N-j} of the double roots one has to specify the pair that corresponds to two points $Q_0^\pm$ on $\Gamma_N^{\rm sing}$ above $E = 0$. Let $j = j_0$ be the corresponding index. Since $e_\pm \neq E(\kappa_{j_0}) = 0$ and $\kappa_{j_0}^N = 1$ the values of j_0 are constrained to be positive even

$$0 < j_0 < N, \qquad j_0/2 = \mathbb{Z}. \tag{5.7}$$

This index parameterizes different multi-soliton solutions. For given j_0 one solves the equation $E(\kappa_{j_0}) = 0$ to find the constant $\alpha = -2q_N^{-1/N}\cos(\pi j_0/N)$ and obtain the branching points of $\Gamma_N^{\rm sing}$ as

$$e_j = 2q_N^{-1/N}\left[\cos\left(\frac{\pi j}{N}\right) - \cos\left(\frac{\pi j_0}{N}\right)\right], \qquad e_\pm = 2q_N^{-1/N}\left[\pm 1 - \cos\left(\frac{\pi j_0}{N}\right)\right]$$

with $j = 1, \dots, N-1$ and $j \neq j_0$. The branching points fix uniquely the curve $\Gamma_N^{\rm sing}$ and allow us to evaluate the corresponding values of the quantum numbers q_k. Substituting the first relation (5.3) into (5.5) and comparing the coefficients in front of the various powers of κ in both sides of (5.5) one obtains the system of equations on q_k and, in particular,

$$q_2 q_N^{-\frac{2}{N}} = -\frac{N^2}{4\sin^2\left(\frac{\pi j_0}{N}\right)}, \quad \dots \quad q_{N-1} q_N^{-\frac{N-1}{N}} = 2N\cos\left(\frac{\pi j_0}{N}\right).$$

For $N = 3$ and $N = 4$ the solution to (5.7) is $j_0 = 2$ leading to $q_2 q_3^{-2/3} = -3$ and $(q_2 q_4^{-1/2} = -4,\ q_3 = 0)$, respectively.

Let us consider the Baker-Akhiezer function Ψ_n^α on $\Gamma_N^{\rm sing}$. Its properties formulated in Section 3 imply that being functions of complex κ it can be written in the form

$$\Psi_n^\alpha(\kappa; \{\tau\}) = \kappa^{-n}\, \mathrm{e}^{\sum_{j=1}^{N-2} \tau_j (\kappa^j - \kappa^{-j})\, q_N^{-j/N}} \times \frac{R_\alpha(\kappa; n, \{\tau\})}{R_0(\kappa)}, \tag{5.8}$$

where R_0 and R_α ($\alpha = 1, 2$) are polynomials of degree $N-1$ in κ. The zeros of $R_0(\kappa)$ become poles of Ψ_n^α and in what follows can be considered as parameters of the multi-soliton solutions, $R_0(\kappa) = \prod_{s=1}^{N-1}(\kappa - \gamma_s)$. To verify the asymptotic behaviour (3.1) one has to take into account the relation

$$E(\kappa) = q_N^{-1/N}\left(\kappa + \frac{1}{\kappa} - 2\cos\left(\frac{\pi j_0}{N}\right)\right) \tag{5.9}$$

and perform the limits $\kappa \to 0$ and $\kappa \to \infty$. The so far unknown polynomials R_α are uniquely defined by the normalization conditions (2.17),

$$\begin{aligned} &\Psi_n^1(1/\kappa_{j_0}) = 1\,, \qquad \Psi_n^1(\kappa_{j_0}) = 0\,, \\ &\Psi_n^2(1/\kappa_{j_0}) = 0\,, \qquad \Psi_n^2(\kappa_{j_0}) = 1 \end{aligned} \tag{5.10}$$

and the additional $N-2$ linear relations

$$\Psi_n^\alpha(1/\kappa_j) = \Psi_n^\alpha(\kappa_j)\,, \qquad j = 1, \dots, N-1\,, \quad j \neq j_0\,. \tag{5.11}$$

The meaning of the last condition is that the parameters κ_j and $1/\kappa_j$ define the same point on the curve. It belongs simultaneously to both sheets of $\Gamma_N^{\rm sing}$ and the value of the functions Ψ_n^α should be the same. Let us choose the polynomial $R_2(\kappa; n\{\tau\})$ in the following form:

$$R_2(\kappa; n, \{\tau\}) = A \times \left(\kappa - 1/\kappa_{j_0}\right) \left[\prod_{\substack{j=1 \\ j \neq j_0}}^{N-1} (\kappa - \kappa_j) + (\kappa - \kappa_{j_0}) \sum_{l=0}^{N-3} a_l \kappa^{N-3-l} \right],$$

with A and a_l ($l = 0, \dots, N-3$) being some functions of n and τ. One verifies that Ψ_n^2 satisfies the normalization conditions (5.10) provided that A is given by

$$\begin{aligned} A(n, \{\tau\}) &= \frac{R_0(\kappa_{j_0})}{(\kappa_{j_0} - 1/\kappa_{j_0}) \prod_{\substack{j=1 \\ j \neq j_0}}^{N-1} (\kappa_{j_0} - \kappa_j)} \kappa_{j_0}^n \, \mathrm{e}^{-\sum_{s=1}^{N-2} \tau_s \left(\kappa_{j_0}^s - \kappa_{j_0}^{-s}\right) q_N^{-s/N}} \\ &= \text{const.} \times \exp\left[in \frac{\pi j_0}{N} - 2i \sum_{s=1}^{N-2} \tau_s \sin\left(\frac{\pi j_0 s}{N}\right) q_N^{-s/N} \right]. \end{aligned}$$

The conditions (5.11) lead to the linear system of equations for a_l that can be expressed as

$$\sum_{s=0}^{N-3} a_s \sin\left(\frac{1}{2}\Phi_j + \frac{\pi j}{N} s\right) = \frac{i}{2} \mathrm{e}^{\frac{i}{2}\Phi_j} \frac{\prod_{l=1}^{N-1} (1 - \kappa_j \kappa_l)}{(1 - \kappa_j \kappa_{j_0})^2}\,, \qquad j = 1, \dots, N-2\,. \tag{5.12}$$

Here, the parameters κ_j are given by (5.6) and the phases are defined as

$$\Phi_j(n, \{\tau\}) = n \frac{2\pi j}{N} - 4 \sum_{s=1}^{N-2} \tau_s \sin\left(\frac{\pi j s}{N}\right) q_N^{-s/N} + Z_j \tag{5.13}$$

with Z_j being some constants depending on zeros $\gamma_1, \dots, \gamma_{N-1}$. The solution to (5.12) can be written explicitly as a ratio of determinants. Comparing (3.1) with (5.8) in the limit $\kappa \to \infty$ we get

$$\phi_2^+(n, \{\tau\}) = A \times (1 + a_0) = \text{const} \times \mathrm{e}^{\frac{i}{2}\Phi_{j_0}} \left[1 + a_0(\mathrm{e}^{i\Phi_1}, \dots, \mathrm{e}^{i\Phi_{N-1}}) \right], \tag{5.14}$$

where a_0 is rational function of all phases except $\mathrm{e}^{i\Phi_{j_0}}$. Repeating a similar analysis for the polynomial R_1 one arrives at the relation

$$\Phi_1^+(n,\{\tau\}) = A^{-1} \times (1+\bar{a}_0) = \text{const.}\, e^{-\frac{i}{2}\Phi_{j_0}}\left[1+\bar{a}_0(e^{-i\Phi_1},\dots,e^{-i\Phi_{N-1}})\right], \tag{5.15}$$

with $\bar{a}_0$ being a function complex conjugated to a_0.

To finish the description of the multi-soliton solutions one has to turn to the evolution times t_k using the transition formulas (4.5) and (4.6). Let us consider the meromorphic differential of the second kind entering the definitions (3.12) and (4.3). It follows from (5.9) and (3.7), that on the degenerate curve Γ_N^{sing} the differential $d\Omega^{(j)}$ has an Nth order pole at the points $\kappa = 0$ and $\kappa = \infty$. Therefore it can be written as

$$d\Omega^{(j)} = q_N^{-j/N} d\left(\kappa^j - \kappa^{-j}\right)$$

with the prefactor fixed by the asymptotics (3.7). Substituting this identity into (3.12) and (4.3) one gets

$$W_k^{(j)} = 4q_N^{-j/N}\sin\left(\frac{\pi jk}{N}\right), \qquad W_0^{(j)} = 4q_N^{-j/N}\sin\left(\frac{\pi j_0 j}{N}\right), \tag{5.16}$$

where $k = 1,\dots,N-1$ and $k \neq j_0$. To evaluate the r.h.s. of (4.5) and (4.6) one has to examine the E-expansion of the differentials $d\omega_k$ and $d\Omega_0$ on Γ_N^{sing} and identify the coefficients U_{kj} and $(U_0)_k$. We observe that due to (5.1) and (3.4) the differential $d\omega_k$ has $N-2$ additional poles on Γ_N^{sing} located at the points $E = e_j$. The normalization condition (3.5) implies that it has a vanishing residue at all points except of $E = e_k$, where its residue is equal to i. This gives the following system of equations on U_{kj}:

$$\sum_{j=1}^{N-2} U_{kj}E^j = 2q_N^{1-1/N}\sin\left(\frac{\pi k}{N}\right)\prod_{\substack{i=1\\ i\neq j_0}}^{N-1}(E-e_i), \tag{5.17}$$

valid for any E. The analysis of the differential $d\Omega_0$ can be performed in a similar way with the only difference that, according to (3.8), the residue of $d\Omega_0$ vanishes at all points $E = e_j$ $(j \neq j_0)$ and its residue at $E = e_{j_0} = 0$ is equal to 1,

$$2h + \sum_{j=1}^{N-2}(U_0)_j E^j = 2q_N^{1-1/N}\sin\left(\frac{\pi j_0}{N}\right)\prod_{\substack{i=1\\ i\neq j_0}}^{N-1}(E-e_i). \tag{5.18}$$

The relations (5.17) and (5.18) allow us to calculate the coefficients U_{kj} and $(U_0)_k$ in terms of the branching points e_j and, in particular,

$$(U_0)_{N-2} = 2\sin\left(\frac{\pi j_0}{N}\right)q_N^{1-1/N}, \qquad U_{k,N-2} = 2\sin\left(\frac{\pi k}{N}\right)q_N^{1-1/N}.$$

One checks that for $N = 3$, leading to $j_0 = 2$, these expressions are in agreement with (4.17). Observing the relation $U_{j_0,k} = (U_0)_k$, we substitute (5.16) into the transition formulas (4.5) and (4.6) to obtain the phases (5.13) in the following form:

$$\Phi_k(n,\{t\}) = n\frac{2\pi k}{N} - \sum_{j=1}^{N-2} U_{kj}t_{j+2} + Z_k, \qquad 1 \leqslant k \leqslant N-1,$$

with the coefficients U_{kj} calculated from (5.17). Finally, substitution of (5.14) and (5.15) into (3.2) yields the multi-soliton solutions for the reggeon coordinates,

$$z_n^{(0)}(t) = \rho\, e^{2iht_2 + i\Phi_{n0}} \frac{1 + a_0(e^{i\Phi_1}, \dots, e^{i\Phi_{N-1}})}{1 + \bar{a}_0(e^{-i\Phi_1}, \dots, e^{-i\Phi_{N-1}})} \tag{5.19}$$

with ρ being a constant. For $N = 3$ this expression coincides with (4.18). Solving (5.12) for arbitrary N one obtains the multi-soliton solutions (5.19) in the form of rational functions of N exponentials, $e^{i\Phi_j}$.

6. Conclusions

In the present paper we studied the asymptotic solutions of the Schrödinger equation (1.1) for the color-singlet reggeon compound states in multi-color QCD. In the Regge limit, a non-trivial QCD dynamics affects only transversal reggeon degrees of freedom and the N reggeon compound states look like a system of N interacting particles on the 2-dimensional plane of transverse reggeon coordinates. This system possesses a large enough family of conserved charges q_k for the Schrödinger equation to be completely integrable. We identified the eigenvalue of the "lowest" charge q_2 as a parameter of the asymptotic expansion playing the role of the effective Planck constant.

In the leading order of the asymptotic expansion, quantum fluctuations are frozen and quantum mechanical motion of reggeons is restricted to the classical trajectories driven by the "action" function S_0. The conserved charges q_k are replaced by classical functions on the phase space of N reggeons. They have a mutually vanishing Poisson brackets and generate the hierarchy of the evolution equations for the reggeon coordinates. For $N = 2$ reggeon states, the solution to the evolution equations describes two reggeons rotating around the center-of-mass with the angular velocity given by the conformal weight h. For $N \geqslant 3$, integrals of motion $q_3, \dots, q_N$ generate new modes of the classical reggeon motion which we identified as soliton waves propagating on the chain of N particles with periodic boundary conditions. Applying the methods of the finite-gap theory we constructed the explicit form of the reggeon trajectories in terms of Riemann θ-functions and studied their properties.

The orbits of the classical reggeon motion are parameterized by the eigenvalues of the conserved charges q_k ($k \geqslant 3$). They define the moduli of the hyperelliptic curve Γ_N and appear as parameters of the soliton solutions. In the leading order of the asymptotic expansion the eigenvalues of q_k could take arbitrary complex values. Quantization of $q_3, \dots, q_N$ appears as a result of imposing the Bohr–Sommerfeld quantization conditions on the classical orbits of reggeon motion. It was shown [6] that for $N = 3$ reggeon compound states the results of the WKB expansion for eigenvalue of q_3 are in a good agreement with the exact expressions [4,15] obtained by means of the algebraic Bethe ansatz [16] for integer positive h. Solutions to the Bohr–Sommerfeld conditions [6] give the quantized values of q_k as functions of the conformal weight h and some additional set of integers $\{l\} = (l_1, \dots, l_{N-2})$ parameterizing different families of curves on the moduli space, $q_k = q_k(h; \{l\})$ $(k = 3, \ldots, N)$. These curves can be found as solutions of the Whitham equations [8] describing the adiabatic perturbation [17] of the reggeon soliton waves.

Acknowledgments

We are most grateful to L.D. Faddeev for stimulating discussions. The work of I.K. was supported in part by RFFI grant 95-01-00755. He also thanks l'Institut Henri Poincaré for the hospitality.

Appendix A

Let us recall the definition of the Riemann θ-function and some of its properties [11]. We choose the basis of cycles α_j, β_j on the Riemann surface Γ_N with the canonical matrix of intersections as shown in Fig. 1. Then, we construct the basis of normalized holomorphic differentials (3.4) and evaluate the $g \times g$ Riemann matrix of their β-periods,

$$\frac{1}{2\pi} \oint_{\beta_j} d\omega_k = \tau_{jk} , \qquad \frac{1}{2\pi} \oint_{\alpha_j} d\omega_k = \delta_{jk} . \tag{A.1}$$

The τ-matrix is symmetric and it has a positive definite imaginary part. The basic g-dimensional vectors $(e_j)_k = 2\pi\delta_{jk}$ and $(b_j)_k = 2\pi\tau_{jk}$ generate the lattice $\mathcal{L}$ and allow us to construct the torus $\mathcal{J}(\Gamma_N) = \mathbb{C}^g/\mathcal{L}$ called the Jacobian of the Riemann surface Γ_N.

One defines the g-dimensional Riemann θ-function as

$$\theta(u;\tau) = \sum_{m\in\mathbb{Z}^g} \exp\left(i\pi\langle\tau m, m\rangle + i\langle u, m\rangle\right), \tag{A.2}$$

where $u = (u_1, \ldots, u_g)$ is a complex vector on the Jacobian $\mathcal{J}(\Gamma_N)$, $m = (m_1, \ldots, m_g)$ is a vector with integer components, $\langle\tau m, m\rangle = \sum_{j,k=1}^{g} \tau_{jk} m_j m_k$ and $\langle u, m\rangle = \sum_{j=1}^{g} u_j m_j$.

The θ-function has the following periodicity properties. For any integer $l \in \mathbb{Z}^g$ and complex $u \in \mathbb{C}^g$ one has

$$\theta(u + 2\pi l) = \theta(u) , \qquad \theta(u + 2\pi\tau l) = \exp\left(-i\pi\langle\tau l, l\rangle - i\langle u, l\rangle\right)\theta(u) . \tag{A.3}$$

The zeros of the θ-function satisfy the following condition. Let us fix a reference point γ_0 on Γ_N and define the vector $A_k(Q) = \int_{\gamma_0}^{Q} d\omega_k$ for any $Q \in \Gamma_N$. Then, for an arbitrary complex $Z = (Z_1, \ldots, Z_g) \in \mathbb{C}^g$, the function $\theta(A(Q) - Z)$ either vanishes identically or it has exactly g zeros at the points $Q_1, \ldots, Q_g$ such that

$$A(Q_1) + \ldots + A(Q_g) = Z - \mathcal{K} \tag{A.4}$$

with $\mathcal{K} = \mathcal{K}(\Gamma_N, \gamma_0)$ being the vector of Riemann constants.

Appendix B

Let us establish some useful relations between the parameters Δ_k and $W_0^{(j)}$ defined in (4.3) and the periods of the normalized differentials on Γ_N. Their derivation is based on the following identity (Stokes theorem) valid for any two unnormalized differentials $d\omega$ and $d\omega'$ on Γ_N,

$$\sum_{j=1}^{g}(a_j b_j' - a_j' b_j) = \oint_C d\omega'(Q) \int_{\gamma_0}^{Q} d\omega = -\oint_C d\omega(Q) \int_{\gamma_0}^{Q} d\omega' . \tag{B.1}$$

Here, $a_j = \oint_{\alpha_j} d\omega$, $b_j = \oint_{\beta_j} d\omega$ and similarly for $d\omega'$ are the periods of the differentials, γ_0 is a point on Γ_N in a general position, C is an oriented closed contour on Γ_N which encircles all singularities of differentials. The contour C should not cross the cycles α_s and β_s $(s = 1, \dots, g)$. The same condition is imposed on the integration path between points γ_0 and Q in (B.1).

Applying the identity (B.1) to a pair of the normalized differentials $d\omega_k$ and $d\Omega_0$ defined in (3.4) and (3.10) one uses the normalization condition on the l.h.s., calculates the r.h.s. by taking the residue at the poles $Q_0^{\pm}$ and gets the well-known relation [11]

$$\int_{Q_0^-}^{Q_0^+} d\omega_k = -i \oint_{\beta_k} d\Omega_0 .$$

Let us consider the following unnormalized dipole differential:

$$d\widehat{\Omega}_0 = \frac{2ih\, dE}{Ey(E)} = \frac{2ih\, dE}{\sqrt{\Lambda^2(E) - 4}} ,$$

which appears as a leading term in the expansion (3.10) of the normalized dipole differential $d\Omega_0$. It has simple poles at the points $Q_0^{\pm}$ with the residue ± 1, respectively, and its asymptotics at the vicinity of puncture Q_∞^+ on the upper sheet is

$$d\widehat{\Omega}_0 \stackrel{E\to\infty}{=} \frac{2ih}{q_N} dE\, E^{-N}\left(1 + \mathcal{O}(1/E)\right) . \tag{B.2}$$

Applying the identity (B.1) to a pair of $d\widehat{\Omega}_0$ and normalized meromorphic differential of the second kind, $d\Omega^{(j)}$, taking into account the normalization condition (3.8), calculating the r.h.s. by taking the residue at the simple poles $Q_0^{\pm}$ of the differential $d\widehat{\Omega}_0$ and jth order poles $Q_\infty^{\pm}$ of the meromorphic differential $d\Omega^{(j)}$, one arrives at the following relation:

$$\sum_{k=1}^{N-2} \oint_{\alpha_k} d\widehat{\Omega}_0 \oint_{\beta_k} d\Omega^{(j)} = -2\pi i \int_{Q_0}^{Q_0^+} d\Omega^{(j)} + 2 \cdot 2\pi i \frac{1}{(j-1)!} \left(\frac{d}{dz}\right)^{j-1} \frac{d\widehat{\Omega}_0}{dz}\bigg|_{z=1/E\to 0} .$$

We notice that the derivative in the last term vanishes for $j = 1, \dots, N-2$ due to (B.2) and one gets the identity

$$iW_0^{(j)} = \int_{Q_0^-}^{Q_0^+} d\Omega^{(j)} = -\frac{ih}{\pi}\sum_{k=1}^{N-2} W_k^{(j)} \oint_{\alpha_k} \frac{dE}{\sqrt{\Lambda^2(E)-4}}, \qquad 1 \leqslant j \leqslant N-2, \qquad \text{(B.3)}$$

with the vector $W^{(j)}$ defined in (3.12).

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