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Elliptic Solutions to Difference Non-Linear Equations and Related Many-Body Problems

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Abstract: We study algebro-geometric (finite-gap) and elliptic solutions of fully discretized KP or 2D Toda equations. In bilinear form they are Hirota's difference equation for τ-functions. Starting from a given algebraic curve, we express the τ-function and the Baker–Akhiezer function in terms of the Riemann theta function. We show that the elliptic solutions, when the τ-function is an elliptic polynomial, form a subclass of the general algebro-geometric solutions. We construct the algebraic curves of the elliptic solutions. The evolution of zeros of the elliptic solutions is governed by the discrete time generalization of the Ruijsenaars-Schneider many body system. The zeros obey equations which have the form of nested Bethe-ansatz equations, known from integrable quantum field theories. We discuss the Lax representation and the action-angle-type variables for the many body system. We also discuss elliptic solutions to discrete analogues of KdV, sine-Gordon and 1D Toda equations and describe the loci of the zeros.

1. Introduction

Among a vast class of solutions to classical non-linear integrable equations elliptic solutions play a special role. First, these solutions occupy a distinguished place among all algebro-geometric (also called finite-gap) solutions, i.e. solutions constructed out of a given algebraic curve. The general formulas in terms of Riemann theta-functions become much more effective – in this case the Riemann theta-function splits into a product of Weierstrass σ-functions associated to an elliptic curve. Second, there exists a remarkable connection between the motion of poles (zeros) of the elliptic solutions and certain integrable many body systems.

The pole dynamics of elliptic solutions to the Korteweg–de Vries (KdV) equation and the Calogero–Moser system of particles were linked together in the paper [1] (see also [2]). It has been shown in [3],[4] that this relation becomes an isomorphism if one considers elliptic solutions of the Kadomtsev–Petviashvili (KP) equation. More recently, these results were generalized to elliptic solutions of the matrix KP and the matrix 2D Toda lattice equations (see [5] and [6], respectively). The dynamics of their poles obeys the spin generalization of the Ruijsenaars–Schneider (RS) model [7].

Let us recall some elements of the elliptic solutions for the standard example of the KP equation $3u_{yy} = (4u_t + 6uu_x - u_{xxx})_x$ for a function $u = u(x, y, t)$. An elliptic solution in the variable x is given by

$$u(x, y, t) = \text{const } + 2 \sum_{i=1}^{N} \wp(x - x_i(y, t)), \tag{1.1}$$

where $\wp(x)$ is the Weierstrass $\wp$-function. The self-consistency of this ansatz is a manifestation of integrability. It has been shown in [3],[4] that the dynamics of poles as functions of y obeys the Calogero-Moser many body system with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - 2 \sum_{i \neq j} \wp(x_i - x_j). \tag{1.2}$$

This system in its turn is known to be integrable. There is an involutive set of conserved quantities $H^{(j)}$ – the Hamiltonian (1.2) and the total momentum are $H^{(2)}$ and $H^{(1)}$. The equations of motion are

$$\partial_y^2 x_i = 4 \sum_{j=1, \neq i}^{N} \wp'(x_i - x_j). \tag{1.3}$$

The t-dynamics is described by $H^{(3)}$.

The reduction to the KdV equation restricts the particles to the *locus* $\mathbf{L}_N$ in the phase space:

$$\mathbf{L}_N = \left\{ \; (p_i, x_i) \;\middle|\; p_i = 0, \; \sum_{i \neq j} \wp'(x_i - x_j) = 0 \; \right\} \tag{1.4}$$

(here $p_i = \partial_y x_i$). In spite of interesting developments, an analysis of the locus structure is far from completed.

In this paper we extend these results to the fully discretized version of the KP equation or 2D Toda lattice. Being fully discretized they become the same equation. In bilinear form they are known as Hirota's bilinear difference equation (HBDE) [8], [9] (see [10] for a review). This is a bilinear equation for a function $\tau(l, m, n)$ (called τ-function) of three variables:

$$\begin{aligned} &\lambda\tau(l + 1, m, n)\tau(l, m + 1, n + 1) + \mu\tau(l, m + 1, n)\tau(l + 1, m, n + 1) \\ &+ \nu\tau(l, m, n + 1)\tau(l + 1, m + 1, n) = 0, \end{aligned} \tag{1.5}$$

where $\lambda, \; \mu, \; \nu$ are complex parameters and the three variables are not necessarily integer. In what follows we call them *discrete times* stressing the difference with continuous KP-flows. Let us introduce a lattice spacing η for one of the variables, say, l and denote $x \equiv \eta l$. By *elliptic solutions* (in the variable x) to this equation we mean the following ansatz for the τ-function:

$$\tau(l, m, n) \equiv \tau^{m,n}(x) = \prod_{j=1}^{N} \sigma(x - x_j^{m,n}), \tag{1.6}$$

where $\sigma(x)$ is the Weierstrass σ-function. We refer to the r.h.s. of (1.6) as *elliptic polynomials* in x. For brevity, we call solutions of this type elliptic though the τ-function itself is not double-periodic. However, suitable ratios of these τ-functions, for instance,

$$A^{m,n}(x) = \frac{\tau^{m,n}(x)\tau^{m+1,n}(x+\eta)}{\tau^{m+1,n}(x)\tau^{m,n}(x+\eta)} \tag{1.7}$$

are already elliptic functions.

In this paper we derive equations of motion for poles of $A^{m,n}(x)$ or zeros of $\tau^{m,n}(x)$ for discrete times m, n and thus obtain a fully discretized Calogero-Moser many body problem. This appears to be the discrete time version of the Ruijsenaars-Schneider (RS) model proposed in the seminal paper [11]. Remarkably, the discrete equations of motion have the form of Bethe equations of the hierarchical (nested) Bethe ansatz. The discrete time runs over "levels" of the nested Bethe ansatz.

We also consider stationary reductions of HBDE. In this case the initial configuration of poles (zeros) is not arbitrary but constrained to a stable locus as in the continuous case (1.4). For the most important examples we give equations defining the loci.

A renewed interest in soliton difference equations, and especially in their elliptic solutions is caused by the revealing *classical* integrable structures present in integrable models of *quantum* field theory. It turns out that Hirota's Eq. (1.5) is the universal fusion rule for a family of quantum transfer matrices. Their eigenvalues (as functions of spectral parameters) obey a set of functional Eqs. [12] which can be recast into the bilinear Hirota Eq. [13] (see also [14] and [15] for less technical reviews). Furthermore, it turned out that most of the ingredients of the Bethe ansatz and the quantum inverse scattering method are hidden in the elliptic solutions of the entirely classical discrete time soliton Eqs. [13]. In particular, the discrete dynamics of poles of $A^{m,n}(x)$ or zeros of (1.6) has the form of Bethe ansatz equations, where the discrete time runs over "nested" levels.

The theory of elliptic solutions has direct applications to the algebraic Bethe ansatz and to Baxter's T-Q-relation, which we plan to discuss elsewhere.

Here we attempt to develop a systematic approach to the elliptic solutions of the integrable difference equations. The basic concept of the approach is the Baker–Akhiezer functions on algebraic curves. We prove that all solutions to HBDE of the form (1.6) are of the algebro-geometric type and present them in terms of Riemann theta functions.

The plan of the paper is as follows.

In Sect. 2 we describe general algebro-geometric (finite-gap) solutions to HBDE. We start from the Baker–Akhiezer function constructed from a complex algebraic curve of genus g with marked points. This function satisfies an overcomplete set of linear difference equations. Their consistency is equivalent to Hirota's equation. In this way, one obtains a $(4g + 1)$-parametric family of quasiperiodic solutions to HBDE in terms of the Riemann theta-functions. Solitonic degenerations of these solutions are discussed in Sect. 2.4.

Section 3 is devoted to elliptic solutions. They are shown to be a particular subclass of the algebro-geometric family of solutions of Sect. 2. We derive equations of motion for zeros of the τ-function (the Bethe ansatz equations) and their Lax representation. We discuss variables of the action-angle type and write down equations for the stable loci for the most important reductions of Hirota's equation.

2. Algebro-Geometric Solutions to Hirota's Equation

In this section we construct algebro-geometric solutions of Hirota's equation out of a given algebraic curve. The general method of constructing such solutions of integrable equations is standard. As soon as the bilinear equation can be represented as a compatibility condition for an overdetermined system of linear problems, the first step is to pass to common solutions Ψ to the *linear* problems. Given a linear multi-dimensional difference operator with quasiperiodic coefficients, one associates with it a *spectral curve* defined by the generalized dispersion relations for quasimomenta of Bloch eigenfunctions of the linear operator. The Bloch solutions Ψ are parametrized by points of this curve. Solutions to the initial non-linear equation are encoded in the analytical properties of Ψ as a function on the curve. Spectral curves of general linear operators with quasiperiodic coefficients are transcendental and, therefore, intractable. However, soliton theory mostly deals with the *inverse* problem: to characterize specific operators whose spectral curves are *algebraic curves of finite genus*. Such operators are called algebro-geometric or finite-gap. Their coefficients yield solutions to HBDE, that we are going to study.

Finite-gap multi-dimensional linear difference operators were constructed in the paper [16] by one of the authors. We present the corresponding construction in a form adequate for our purposes.

2.1. The Baker–Akhiezer function. As usual, we begin with the axiomatization of analytical properties of Bloch solutions Ψ. The Baker–Akhiezer function is an abstract version of the Bloch function. Since we solve the inverse problem, the primary objects are Ψ-functions rather than linear operators.

Let Γ be a smooth algebraic curve of genus g. We fix the following data related to the curve:

– A finite set of marked points (punctures) $P_\alpha \in \Gamma$, $\alpha = 0, 1, \ldots, M$;
– Local parameters w_α in neighbourhoods of P_α: $w_\alpha(P_\alpha) = 0$;
– A set of cuts $C_{\alpha\beta}$ between the points P_α, P_β for some pairs α, β (it is implied that different cuts do not have common points other than their endpoints at the punctures);
– A set D of g (distinct) points $\gamma_1, \ldots, \gamma_g \in \Gamma$.

Further, we introduce the following complex parameters $l_{\alpha\beta}$ (times or flows):

– To each cut $C_{\alpha\beta}$ is associated a complex number $l_{\alpha\beta}$ (it is convenient to assume that $l_{\beta\alpha} = -l_{\alpha\beta}$).

Consider a linear space $\mathcal{F}(l; D)$ of functions $\Psi(l; P)$, $P \in \Gamma$, such that:

1. The function $\Psi(l; P)$ as a function of the variable $P \in \Gamma$ is meromorphic outside the cuts and has at most simple poles at the points γ_s;
2. The boundary values $\Psi^{\pm,(\alpha\beta)}$ of this function at opposite sides of the cut $C_{\alpha\beta}$ satisfy the relation
$$\Psi^{+,(\alpha\beta)}(l; P) = \Psi^{-,(\alpha\beta)}(l; P)e^{2\pi i l_{\alpha\beta}}; \tag{2.1}$$
3. In a neighbourhood of the point P_α it has the form
$$\Psi(l; P) = w_\alpha^{-L_\alpha}\left(\xi_0^{(\alpha)}(l) + \sum_{s=1}^{\infty} \xi_s^{(\alpha)}(l) w_\alpha^s\right), \quad L_\alpha = \sum_\beta l_{\alpha\beta}. \tag{2.2}$$

Note that if $l_{\alpha\beta}$ are integers, then Ψ is a meromorphic function having simple poles at γ_s and having zeros or poles of orders $|L_\alpha|$ at the points P_α.

Any function Ψ obeying conditions 1 - 3 is called a *Baker–Akhiezer function*. In our approach, these functions are central objects of the theory. In some cases (especially in the matrix generalizations of the theory) the notion of the *dual Baker–Akhiezer function* $\Psi^\dagger$ is also important (see e.g. [5], [6]). We omit its definition because it can be easily restored using [5], [6].

Let us prepare some notation. Fix a canonical basis of cycles a_i, b_i on Γ and denote the canonically normalized holomorphic differentials by $d\omega_i$, $i = 1, 2, \ldots, g$. We have

$$\oint_{a_i} d\omega_j = \delta_{ij}\,, \qquad \oint_{b_i} d\omega_j = B_{ij}\,,$$

where B is the period matrix. Given a period matrix B, the g-dimensional Riemann theta-function is defined by

$$\Theta(\vec{X}) = \Theta(\vec{X}|B) = \sum_{\vec{n}\in\mathbf{Z}^g} \exp\left[\pi i(B\vec{n},\vec{n}) + 2\pi i(\vec{n},\vec{X})\right].$$

Here $\vec{X} = (X_1, \ldots, X_g)$ is a g-component vector.

For each pair of points $P_\alpha, P_\beta \in \Gamma$, let $d\Omega^{(\alpha\beta)}$ be the unique differential of the third kind holomorphic on Γ but having simple poles at the points P_α, P_β with residues -1 and 1 and zero a-periods.

To write an explicit form of Baker–Akhiezer functions, let us choose one of the marked points, say P_0, and by $\vec{A}(P) = (A_1(P), \ldots, A_g(P))$,

$$A_i(P) = \int_{P_0}^{P} d\omega_i,$$

denote the Abel map. The Baker–Akhiezer function is given by the following theorem.

Theorem 2.1. *If the points $\gamma_1, \ldots, \gamma_g$ are in a general position (i.e. D is a non-special divisor), then $\mathcal{F}(l;D)$ is a one-dimensional space generated by the function $\Psi(l;P) \in \mathcal{F}(l;D)$,*

$$\Psi(l;P) \;=\; \frac{\Theta(\vec{A}(P)+\vec{X}(l)+\vec{Z}|B)\Theta(\vec{Z}|B)}{\Theta(\vec{A}(P)+\vec{Z}|B)\Theta(\vec{X}(l)+\vec{Z}|B)} \exp\left(\sum_{(\alpha\beta)} l_{\alpha\beta}\int_{Q_0}^{P} d\Omega^{(\alpha\beta)}\right). \qquad (2.3)$$

Here $Q_0 \in \Gamma$ is an arbitrary point in the vicinity of P_0, $Q_0 \neq P_0$, belonging to the integration path $P_0 \to P$ in the Abel map. Further,

$$\vec{Z} = -\vec{K} - \sum_{i=1}^{g} \vec{A}(\gamma_i), \qquad \vec{X}(l) = \sum_{(\alpha\beta)} \vec{U}^{(\alpha\beta)} l_{\alpha\beta}, \qquad (2.4)$$

where $\vec{K}$ is the vector of Riemann's constants and the components of the vectors $\vec{U}^{(\alpha\beta)}$ are

$$(\vec{U}^{(\alpha\beta)})_j = \frac{1}{2\pi i}\oint_{b_j} d\Omega^{(\alpha\beta)} = A_j(P_\beta) - A_j(P_\alpha). \qquad (2.5)$$

The proof of theorems of this kind as well as the explicit formula for Ψ in terms of Riemann theta-functions are standard in finite-gap theory (see e.g. [17]). The last equality in (2.5) follows from Riemann's relations.

Remark 2.1. Although the explicit formula (2.3) requires a fixed basis of cycles, the Baker–Akhiezer function is modular invariant.

Remark 2.2. Since abelian integrals in (2.3) have logarithmic singularities at the punctures, one can define a single valued branch of Ψ only after cutting the curve along $C_{\alpha\beta}$.

Remark 2.3. The choice of the initial point of the Abel map is in fact not essential. It can be chosen not to be one of the punctures. In particular, it may be Q_0, which slightly simplifies the theorem. However, our choice simplifies the linear equations for Ψ below. Moreover, below we assume that the integration paths in the Abel maps $\vec{A}(P_\alpha)$ go along the cuts $C_{0\alpha}$.

Remark 2.4. The theorem implies that any function from $\mathcal{F}(l;D)$ has the form $r(l)\Psi(l;P)$, where $r(l)$ is an arbitrary function of l but does not depend on P. It is convenient to choose it such that at the point P_0 the first regular term $\xi_0^{(0)}(l)$ in (2.2) equals 1.

Coefficients $\xi_s^{(\alpha)}$ of the asymptotical behaviour of the Ψ (2.2) can be expressed through the τ-function

$$\tau(l) \equiv \tau(\{l_{\alpha\beta}\}) = \Theta(\vec{X}(l) + \vec{Z}). \tag{2.6}$$

In particular,

$$\frac{\xi_0^{(\alpha)}(l)}{\xi_0^{(\beta)}(l)} = \chi_{\alpha\beta}\frac{\tau(l_{0\alpha}+1, l_{0\beta})}{\tau(l_{0\alpha}, l_{0\beta}+1)}, \quad \alpha, \beta \neq 0, \tag{2.7}$$

where $\chi_{\alpha\beta}$ are l-independent constants. Here and thereafter we skip unshifted arguments.

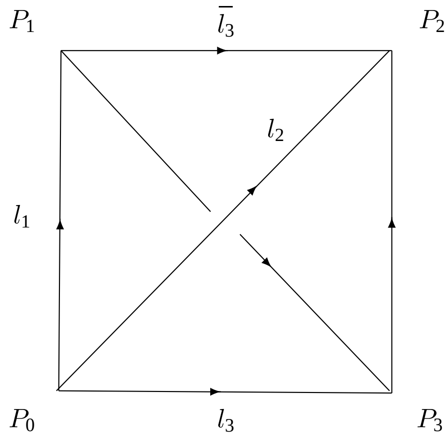
Remark 2.5. If the graph of cuts includes a closed cycle, then a shift of variables $l_{\alpha\beta} \to l_{\alpha\beta}+1$ does not change the τ-function but multiplies the Ψ-function by a cycle dependent constant. For instance, if the cycle consists of three links $C_{\alpha\beta},\ C_{\beta\gamma},\ C_{\gamma\alpha}$, then

$$\begin{aligned} &\tau(l_{\alpha\beta}+1, l_{\beta\gamma}+1, l_{\gamma\alpha}+1) = \tau(l_{\alpha\beta}, l_{\beta\gamma}, l_{\gamma\alpha}), \\ &\Psi(l_{\alpha\beta}+1, l_{\beta\gamma}+1, l_{\gamma\alpha}+1; P) = \text{const}\,\Psi(l_{\alpha\beta}, l_{\beta\gamma}, l_{\gamma\alpha}; P). \end{aligned} \tag{2.8}$$

This follows from (2.1), (2.3).

In the sequel, we do not need the above construction in its full generality. For our purposes it is enough to consider the case of four punctures $P_0, \dots, P_3$ and a general graph of cuts as is in the figure. Cuts connect each pair of points. Any three links (not forming a cycle) give rise to a bilinear equation of the Hirota type. They have different forms, but are in fact equivalent due to (2.8). For further convenience we specify

$$l_{01} = l_1, \quad l_{02} = l_2, \quad l_{03} = l_3, \quad l_{12} = \bar{l}_3. \tag{2.9}$$



The general case of more punctures yields higher Hirota equations (i.e. the discretized KP or 2D Toda lattice hierarchies).

2.2. Difference equations for the Baker–Akhiezer function. The Baker–Akhiezer function $\Psi(l;P)$ obeys certain linear difference equations with respect to the variables $l_{\alpha\beta}$. Coefficients of the equations are fixed by the analytical properties of $\Psi(l;P)$ as a function of $P \in \Gamma$. We restrict ourselves to the case of four punctures and use the notation introduced at the end of the previous subsection. The general case of more punctures can be treated in a similar way.

Theorem 2.2. *Let $\Psi(l;P)$ be the Baker–Akhiezer function normalized so that $\xi_0^{(0)} = 1$. Then it satisfies the following linear difference equations:*

$$\Psi(l_\alpha + 1, l_\beta; P) - \Psi(l_\alpha, l_\beta + 1; P) + A_{\alpha\beta}(l_\alpha, l_\beta)\Psi(l_\alpha, l_\beta; P) = 0 \tag{2.10}$$

with

$$A_{\alpha\beta}(l_\alpha, l_\beta) = \frac{\xi_0^{(\alpha)}(l_\alpha, l_\beta + 1)}{\xi_0^{(\alpha)}(l_\alpha, l_\beta)} \tag{2.11}$$

for any $\alpha, \beta = 1, 2, 3$, $\alpha \neq \beta$.

The proof is standard in finite gap theory. Denote the l.h.s. of Eq. (2.10) by $\tilde{\Psi}$. This function has the same analytical properties as Ψ. At the same time the leading term at the point P_1 is zero: $\tilde{\xi}_0^{(1)} = 0$ for any l_α, l_β. From the uniqueness of the Baker–Akhiezer function it follows that $\tilde{\Psi} = 0$. Equations (2.18), (2.20) are proved in the same way.

Remark 2.6. The dual Baker–Akhiezer function $\Psi^\dagger$ obeys difference equations obtained from Eqs. (2.10), (2.18) by conjugating the difference operators in the right hand sides.

The coefficient functions of eq. (2.10) are given by the leading nonsingular term $\xi_0^{(\alpha)}$ of the Baker–Akhiezer function at the punctures. They can be found from Eq. (2.3) and are expressed through the τ-function (2.6):

$$A_{\alpha\beta}(l_\alpha, l_\beta) = -\lambda_{\alpha\beta} \frac{\tau(l_\alpha, l_\beta)\tau(l_\alpha + 1, l_\beta + 1)}{\tau(l_\alpha, l_\beta + 1)\tau(l_\alpha + 1, l_\beta)}. \tag{2.12}$$

The constants $\lambda_{\alpha\beta}$ are expressed through the constant terms $r_\gamma^{(\alpha\beta)}$ in expansion of the abelian integrals

$$\int_{Q_0}^{P} d\Omega^{(\alpha\beta)}\Big|_{P\to P_\gamma} = (\delta_{\gamma\beta} - \delta_{\gamma\alpha})\log w_\gamma + r_\gamma^{(\alpha\beta)} + O(w_\gamma), \tag{2.13}$$

as follows:

$$\lambda_{\alpha\beta} = -\exp\left(r_\alpha^{(0\beta)} - r_0^{(0\beta)}\right). \tag{2.14}$$

It can be shown that $\lambda_{\beta\alpha} = -\lambda_{\alpha\beta}$ for $\alpha \neq \beta$ and

$$\frac{\lambda_{\alpha\beta}}{\lambda_{\beta\gamma}} = -\exp\left(\int_{P_\gamma}^{P_\alpha} d\Omega^{(0\beta)}\right) \tag{2.15}$$

for any cyclic permutation of $\{\alpha\beta\gamma\} = \{123\}$. The integration path goes from P_γ to the neighbourhood of P_0 along the cut $C_{0\gamma}$ (in the opposite direction), passes through the point Q_0 and then goes along the cut $C_{0\alpha}$.

Eqautions (2.10), (2.11) can be viewed as linear problems for the discretized KP equation. Choosing another triplet of variables, say $l_1, l_3, \bar{l}_3$ and using Eq. (2.8), i.e.

$$\tau(l_1+1, l_2, l_3; \bar{l}_3+1) = \tau(l_1, l_2+1, l_3; \bar{l}_3), \tag{2.16}$$

$$\tau(l_1, l_2, l_3; \bar{l}_3) = \Theta\left(\sum_{\alpha=1}^{3} \vec{A}(P_\alpha)l_\alpha + \left(\vec{A}(P_2) - \vec{A}(P_1)\right)\bar{l}_3 + \vec{Z}\right), \tag{2.17}$$

one may rewrite Eqs. (2.10), (2.11) in a form suitable for discretization of the 2D Toda lattice. In this case eq. (2.10) for $\alpha, \beta = 1, 3$ remains the same. Another linear equation,

$$\Psi(l_1, \bar{l}_3; P) - \Psi(l_1, \bar{l}_3+1; P) + B(l_1, \bar{l}_3)\Psi(l_1-1, \bar{l}_3; P) = 0\,, \tag{2.18}$$

with

$$B(l_1, \bar{l}_3) = \frac{\xi_0^{(1)}(l_1, \bar{l}_3+1)}{\xi_0^{(1)}(l_1-1, \bar{l}_3)} = -\lambda_{12}\frac{\tau(l_1+1; \bar{l}_3+1)\tau(l_1-1; \bar{l}_3)}{\tau(l_1; \bar{l}_3+1)\tau(l_1; \bar{l}_3)} \tag{2.19}$$

follows from (2.10) for $\alpha, \beta = 1, 2$ as a result of the change of variables. The third equation,

$$\Psi(l_3+1, \bar{l}_3+1; P) - \tilde{A}(l_3, \bar{l}_3)\Psi(l_3, \bar{l}_3+1; P) = \Psi(l_3+1, \bar{l}_3; P) - \tilde{B}(l_3, \bar{l}_3)\Psi(l_3, \bar{l}_3; P)\,, \tag{2.20}$$

with

$$\tilde{A}(l_3, \bar{l}_3) = \frac{\xi_0^{(1)}(l_3+1, \bar{l}_3+1)}{\xi_0^{(1)}(l_3, \bar{l}_3+1)} = -\lambda_{12}\frac{\tau(l_1+1, l_3+1; \bar{l}_3+1)\tau(l_1, l_3; \bar{l}_3+1)}{\tau(l_1, l_3+1; \bar{l}_3+1)\tau(l_1+1, l_3; \bar{l}_3+1)}, \tag{2.21}$$

$$\tilde{B}(l_3, \bar{l}_3) = \frac{\xi_0^{(2)}(l_3+1, \bar{l}_3)}{\xi_0^{(2)}(l_3, \bar{l}_3)} = \frac{\lambda_{12}\lambda_{32}}{\lambda_{13}}\,\frac{\tau(l_1+1, l_3+1; \bar{l}_3+1)\tau(l_1, l_3; \bar{l}_3)}{\tau(l_1, l_3+1; \bar{l}_3)\tau(l_1+1, l_3; \bar{l}_3+1)}, \tag{2.22}$$

is a linear combination of two other equations (2.10) written in terms of the new variables. The constant prefactors in (2.21), (2.22) are derived using the reciprocity law for differentials of the third kind [18].

Alternatively, Eqs. (2.18), (2.20) can be proved in the same way as in Theorem 2.2.

2.3. Bilinear equations for the τ-function. We have shown that the Baker–Akhiezer function satisfies an overdetermined system of linear equations. For compatibility of this system the coefficient functions must obey certain non-linear relations. In terms of the τ-function all these relations have a bilinear form.

Theorem 2.3. *The τ-function obeys the Hirota bilinear difference equation*

$$\lambda_{23}\tau(l_1+1,l_2,l_3)\tau(l_1,l_2+1,l_3+1) - \lambda_{13}\tau(l_1,l_2+1,l_3)\tau(l_1+1,l_2,l_3+1) + \lambda_{12}\tau(l_1,l_2,l_3+1)\tau(l_1+1,l_2+1,l_3) = 0 \tag{2.23}$$

with the constants $\lambda_{\alpha\beta}$ defined in (2.14).

Proof. First we examine the compatibility of Eqs. (2.10) for $(\alpha\beta) = (12)$ and $(\alpha\beta) = (13)$. The variable l_1 is common for both linear problems. Equations (2.10) can be rewritten as

$$\Psi(l_\alpha+1;P) = M_1^{(\alpha)}\Psi(l_\alpha;P), \qquad \alpha = 2,3, \tag{2.24}$$

where $M_1^{(\alpha)}$ is the difference operator in l_1:

$$M_1^{(\alpha)} = e^{\partial_{l_1}} - \lambda_{1\alpha}\frac{\tau(l_1,l_\alpha)\tau(l_1+1,l_\alpha+1)}{\tau(l_1,l_\alpha+1)\tau(l_1+1,l_\alpha)}. \tag{2.25}$$

Equation (2.24) has a family of linearly independent solutions parametrized by points of the curve Γ. Whence the compatibility is equivalent to commutativity of the operators

$$\hat{M}_1^{(\alpha)} = e^{-\partial_{l_\alpha}}M_1^{(\alpha)}, \tag{2.26}$$

i.e.,

$$[\hat{M}_1^{(2)},\ \hat{M}_1^{(3)}] = 0, \tag{2.27}$$

which is a discrete zero curvature condition. Commuting the operators (2.25), we find after some algebra that this condition is equivalent to the relation

$$\lambda_{13}\tau(l_1,l_2+1,l_3)\tau(l_1+1,l_2,l_3+1) - \lambda_{12}\tau(l_1,l_2,l_3+1)\tau(l_1+1,l_2+1,l_3) + H_1(l_1;l_2,l_3)\tau(l_1+1,l_2,l_3)\tau(l_1,l_2+1,l_3+1) = 0\,, \tag{2.28}$$

where H_1 is an arbitrary function such that $H_1(l_1+1;l_2,l_3) = H_1(l_1;l_2,l_3)$.

All this can be repeated for other two pairs of linear problems, i.e., for $(\alpha\beta) = (21), (23)$ and $(\alpha\beta) = (31), (32)$ in (2.10). This leads to bilinear relations similar to (2.28) for the same τ-function but with different functions H_2 and H_3. To be consistent, all three bilinear relations must be identical. This determines $H_1 = -\lambda_{23}$, etc, which proves the theorem. □

Remark 2.7. In the class of the algebro-geometric solutions the Hirota Eq. (2.23) is equivalent to Fay's trisecant identity [18].

The coefficients in Eq. (2.23) may be hidden by the transformation

$$\tau(l_1,l_2,l_3) \to \left(\frac{\lambda_{13}}{\lambda_{12}}\right)^{l_1l_2}\left(\frac{\lambda_{13}}{\lambda_{23}}\right)^{l_2l_3}\tau(l_1,l_2,l_3) \tag{2.29}$$

bringing the Hirota equation into its *canonical form*:

$$\tau(l_1+1,l_2,l_3)\tau(l_1,l_2+1,l_3+1)-\tau(l_1,l_2+1,l_3)\tau(l_1+1,l_2,l_3+1)$$
$$+\tau(l_1,l_2,l_3+1)\tau(l_1+1,l_2+1,l_3)=0. \tag{2.30}$$

The formula

$$\tau(l_1,l_2,l_3)=\exp\left(l_1l_2\int_{P_3}^{P_2}d\Omega^{(01)}+l_2l_3\int_{P_1}^{P_2}d\Omega^{(03)}\right)\Theta\left(\sum_{\alpha=1}^{3}\vec{A}(P_\alpha)l_\alpha+\vec{Z}\right) \tag{2.31}$$

thus provides the family of algebro-geometric solutions to Eq. (2.30).

This family has $4g+1$ continuous parameters. Indeed, for $g>1$ the solution depends on $3g-3$ moduli of the curve, on g points γ_i and on the 4 marked points P_α. The dependence on the choice of local parameters is not essential.

In terms of the variables $l_1, l_3, \bar{l}_3$ the bilinear equations has the form

$$\lambda_{13}\tau(l_1,l_3;\bar{l}_3+1)\tau(l_1,l_3+1;\bar{l}_3)-\lambda_{23}\tau(l_1,l_3;\bar{l}_3)\tau(l_1,l_3+1;\bar{l}_3+1)$$
$$=\lambda_{12}\tau(l_1+1,l_3;\bar{l}_3+1)\tau(l_1-1,l_3+1;\bar{l}_3). \tag{2.32}$$

(Alternatively, this equation is a result of the compatibility of Eqs. (2.10), (2.18) and (2.20).) In contrast to Eq. (2.23), the variable l_2 is not shifted and skipped. The Hirota equation in the form (2.23) can be considered as a discrete KP, whereas the form (2.32) is a fully discretized 2D Toda lattice. Let us stress again that in the fully discretized setup the KP equation and the 2D Toda lattice become equivalent.

2.4. Degenerate cases. Degenerations of the curve Γ lead to important classes of solutions. Among them are multi-soliton and rational solutions. Some examples of soliton solutions to Eqs. (2.30), (2.32) were found by R.Hirota [8]. Here we outline the algebro-geometric construction of the general soliton solutions.

Let us concentrate on multi-soliton solutions. In this case all N "handles" of the Riemann surface of genus N become infinitely thin. In other words, the algebraic curve of genus N degenerates into the complex plane with a set of $2N$ marked points $p_i,\ q_i$:

$$\Gamma\to\{p_i,\ q_i,\ \ i=1,\dots,N\},$$

where $p_i,\ q_i$ are the ends of the i^{th} handle. The Baker–Akhiezer function has the same value at each pair $p_i,\ q_i$. The punctures P_α are replaced by points z_α with local parameters $w_\alpha=z-z_\alpha$. In this case the meromorphic differentials of the third kind have the form:

$$d\Omega^{(\alpha\beta)}=\frac{(z_\beta-z_\alpha)dz}{(z-z_\beta)(z-z_\alpha)}.$$

Let $F_0(z)$ be a polynomial of degree N:

$$F_0(z)=z^N+\sum_{j=1}^{N}b_jz^{N-j}=\prod_{i=1}^{N}(z-\gamma_i).$$

Its zeros γ_i will stand for poles of the Baker–Akhiezer function.

Let us concentrate on the particular case when there are four punctures and three cuts from z_0 to z_α ($\alpha=1,2,3$) on the complex z-plane. Here z_α are the z-coordinates of the marked points P_α: $z_\alpha=z(P_\alpha)$. In this case the general definition of the Baker–Akhiezer function suggests the ansatz:

$$\Psi(l;z) = \frac{F(l;z)}{F_0(z)}\Psi_0(l;z) \tag{2.33}$$

with

$$\Psi_0(l;z) = \prod_{\alpha=1}^{3}\left(\frac{z-z_\alpha}{z-z_0}\right)^{l_\alpha}, \qquad l \equiv (l_1, l_2, l_3) \tag{2.34}$$

and the polynomial

$$F(l;z) = \sum_{j=0}^{N} a_j(l) z^{N-j} \tag{2.35}$$

with yet undetermined l-dependent coefficients. These coefficients are determined by N conditions

$$\Psi(l;p_i) = \Psi(l;q_i), \qquad l = 1, \dots, N. \tag{2.36}$$

They are equivalent to the system of N linear equations for $N+1$ unknown coefficients a_j in Eq. (2.35):

$$\sum_{j=0}^{N} K_{ij} a_j = 0, \quad i = 1, \dots, N, \quad j = 0, 1, \dots, N,$$

where

$$K_{ij} = \frac{p_i^{N-j}}{F_0(p_i)}\Psi_0(l;p_i) - \frac{q_i^{N-j}}{F_0(q_i)}\Psi_0(l;q_i). \tag{2.37}$$

Solving the system of linear equations we represent the Baker–Akhiezer function in the form

$$\Psi(l;z) = c(l)\frac{\Delta(l;z)}{\Delta(0;z)}\Psi_0(l;z), \tag{2.38}$$

where

$$\Delta(l;z) = \det_{ij}(\hat{K}_{ij})$$

and $\hat{K}$ is the $(N+1)\times(N+1)$-matrix with entries

$$\hat{K}_{0j} = z^{N-j}, \quad \hat{K}_{ij} = K_{ij}, \quad i = 1, \dots, N.$$

The normalization factor $c(l)$ is fixed by the asymptotics

$$\Psi(l;z) = (z-z_0)^{-l_1-l_2-l_3}(1 + O(z-z_0))$$

near the point z_0. It gives

$$c(l) = \frac{\Delta(0;z_0)}{\Delta(l;z_0)}\prod_{\alpha=1}^{3}(z_0-z_\alpha)^{-l_\alpha}.$$

We also point out the identity

$$\Delta(l_\alpha; z_\alpha) = \Delta(l_\alpha + 1; z_0), \qquad \alpha = 1, 2, 3. \tag{2.39}$$

The degeneration of the τ-function (2.17) is

$$\tau(l) = \Delta(l;z_0). \tag{2.40}$$

It satisfies the bilinear Eq. (2.23) with

$$\lambda_{\alpha\beta} = \frac{z_\alpha - z_\beta}{(z_0 - z_\alpha)(z_0 - z_\beta)}.$$

The continuous parameters of the solution (2.40) are $2N$ points p_i, q_i, N points γ_i and 4 points z_α. However, the τ-function is invariant (up to an irrelevant constant) under the simultaneous fractional-linear transformation $z \to (az + b)/(cz + d)$, $ad - bc = 1$, of all these parameters, so we are left with $3N + 1$ parameters.

In the case of rational degeneration the points p_i and q_i merge and the condition (2.36) becomes

$$(\hat{\mathcal{D}}_i \Psi)(p_i) = 0, \qquad i = 1, \ldots, N, \tag{2.41}$$

where $\hat{\mathcal{D}}_i$ are some differential operators in z with constant coefficients. This changes the form of the matrix K but Eqs. (2.38), (2.39) remain the same.

3. Elliptic Solutions

General finite-gap solutions of Hirota's equation for an arbitrary algebraic curve Γ with punctures P_α are *quasi-periodic* functions of all variables l_α. Below we construct a special important class of solutions for which the quantity (1.7) is *doubly periodic* in one of the variables. The τ-function in this case is an elliptic polynomial (1.6). We call them elliptic solutions. We show that the elliptic solutions also imply a spectral algebraic curve and are therefore a subclass of the algebro-geometric solutions of Sect.2.

Among all algebro-geometric solutions described in the previous section the elliptic solutions in one of the variables or their linear combination are characterized as follows. Let

$$\partial_x = \sum X_{\alpha\beta} \partial_{l_{\alpha\beta}} \tag{3.1}$$

be a vector field in the space of variables $l_{\alpha\beta}$ and let $\vec{U} = \sum_{\alpha\beta} X_{\alpha\beta} \vec{U}^{(\alpha\beta)}$. Let us transport the τ-function (2.6) along the vector field ∂_x and denote it by

$$\tau(x) = \Theta\left(\vec{U}x + \vec{X}(l) + \vec{Z}\right). \tag{3.2}$$

Consider a set of algebraic curves with punctures P_α and cuts such that the vector $\vec{U}$ has the property:

- There exist two constants ω_1, ω_2, Im $(\omega_2/\omega_1) \neq 0$, such that $2\omega_a \vec{U}$, $a = 1, 2$, belong to the lattice of periods of the holomorphic differentials on Γ:

  $$\tau(x + 2\omega_a) = e^{r_a x + s_a} \tau(x),$$

  where r_a, s_a are constants.

The τ-function is then an elliptic polynomial in the variable x. Due to the commensurability of $\vec{U}$ and the lattice of periods, solutions to the equation

$$\Theta(\vec{U}x + \vec{Z}) = 0 \tag{3.3}$$

are $x_i(\vec{Z}) + 2J_1\omega_1 + 2J_2\omega_2$, where $x_i(\vec{Z})$ belong to the fundamental domain of the lattice generated by $2\omega_1$, $2\omega_2$ and J_1, J_2 run over all integers. Therefore,

$$\Theta(\vec{U}x + \vec{Z}) = e^{a_1 x + a_2 x^2} \prod_{i=1}^{N} \sigma\big(x - x_i(\vec{Z})\big) \tag{3.4}$$

with x-independent a_1, a_2 and $N \geq g$.

The requirement of the ellipticity imposes $2g$ constraints on $4g + 1$ parameters. Therefore the dimension of the family of elliptic solutions is $2g + 1$.

Below we concentrate on a specific case where the ellipticity is imposed along $l_{01} \equiv l_1$ by setting $\vec{U} = \eta^{-1}\vec{U}^{(01)}$ and $x = \eta l_1$, where η is a complex constant.

The logic of this section is opposite to the one of Sect. 2. Here we solve the *direct problem* and show that *all* solutions which are elliptic in any "direction" $l_{\alpha\beta}$ of the form (1.6) with a time-independent degree N, are of the finite gap type (3.2).

3.1. Equations of motion for zeros of the τ-function. Let us show how to obtain the equations of motion for zeros of elliptic solutions by elementary methods.

First of all, let us rename variables to emphasize the "direction" of ellipticity:

$$l_1 \equiv \eta x, \quad l_2 \equiv n, \quad l_3 = m, \quad \overline{l_3} \equiv \bar{m}, \tag{3.5}$$

so that the τ-function has the form (1.6). Let us now consider one of Eqs. (2.10), say for $\alpha, \beta = 1, 2$. In this equation all variables except x and m are parameters and we skip them wherever it does not cause confusion: $x_i^{m,n} \to x_i^m,\ \tau^{m,n}(x) \to \tau^m(x)$,

$$\psi^m(x + \eta) - \lambda_{13} \frac{\tau^m(x)\tau^{m+1}(x + \eta)}{\tau^{m+1}(x)\tau^m(x + \eta)} \psi^m(x) = \psi^{m+1}(x)\,. \tag{3.6}$$

Let us look for solutions of the form

$$\psi^m(x) = \frac{\rho^m(x)}{\tau^m(x)}$$

with some function $\rho^m(x)$. Eq. (3.6) then reads

$$\tau^{m+1}(x)\rho^m(x + \eta) - \lambda_{13}\tau^{m+1}(x + \eta)\rho^m(x) = \tau^m(x + \eta)\rho^{m+1}(x). \tag{3.7}$$

We are interested in the case when $\tau^m(x)$ is an elliptic polynomial in x for any m:

$$\tau^m(x) = \prod_{j=1}^{N} \sigma(x - x_j^m). \tag{3.8}$$

The "equations of motion" for its roots x_j^m in the "discrete time" m can be easily obtained from Eq. (3.7) in the following way. Substituting $x = x_j^{m+1}$, $x = x_j^{m+1} - \eta$, $x = x_j^m - \eta$, we get the relations

$$\begin{aligned}
&-\lambda_{13}\tau^{m+1}(x_j^{m+1} + \eta)\rho^m(x_j^{m+1}) = \tau^m(x_j^{m+1} + \eta)\rho^{m+1}(x_j^{m+1}),\\
&\tau^{m+1}(x_j^{m+1} - \eta)\rho^m(x_j^{m+1}) = \tau^m(x_j^{m+1})\rho^{m+1}(x_j^{m+1} - \eta),\\
&\tau^{m+1}(x_j^m - \eta)\rho^m(x_j^m) = \lambda_{13}\tau^{m+1}(x_j^m)\rho^m(x_j^m - \eta),
\end{aligned} \tag{3.9}$$

respectively. Combining these relations, we eliminate ρ and obtain a system of equations for the roots x_i^m:

$$\prod_{k=1}^{N}\frac{\sigma(x_j^m-x_k^{m-1})\sigma(x_j^m-x_k^m+\eta)\sigma(x_j^m-x_k^{m+1}-\eta)}{\sigma(x_j^m-x_k^{m-1}+\eta)\sigma(x_j^m-x_k^m-\eta)\sigma(x_j^m-x_k^{m+1})}=-1. \tag{3.10}$$

Note that these equations involve only one discrete variable m while the direct substitution of the ansatz (3.8) into the non-linear Eq. (2.30) would give a system of equations involving two discrete variables. It is not easy to see that they in fact decouple. The decoupling becomes transparent if one starts with the auxiliary linear problem (3.6).

Similar equations hold for the discrete $l_3 \equiv \bar{m}$-dynamics. From the linear problem (2.18) we obtain the equations for the $\bar{m}$ dependence of x_i:

$$\prod_{k=1}^{N}\frac{\sigma(x_j^{\bar m}-x_k^{\bar m-1}-\eta)\sigma(x_j^{\bar m}-x_k^{\bar m}+\eta)\sigma(x_j^{\bar m}-x_k^{\bar m+1})}{\sigma(x_j^{\bar m}-x_k^{\bar m-1})\sigma(x_j^{\bar m}-x_k^{\bar m}-\eta)\sigma(x_j^{\bar m}-x_k^{\bar m+1}+\eta)}=-1. \tag{3.11}$$

To avoid confusion, let us stress that the x_i depend on all discrete times. However, the variables are separated in the equations of motions.

Remark 3.1. Another choice of the "direction" of ellipticity gives rise to different equations of motion. To illustrate this, let us require $\tau(\underline{l})$ to be an elliptic polynomial in the direction orthogonal to $s = l_3 + \bar{l}_3$ and $a = l_1 + l_3 - \bar{l}_3$, so that the zeros of the τ-function depend on a and s. Then the vector field in (3.1) is $\partial_x = \eta\partial_{l_1} - \frac{1}{2}\partial_{l_3} + \frac{1}{2}\partial_{\bar{l}_3}$. In terms of $T^{a,s}(x) \equiv \tau(l_1, l_3, \bar{l}_3)$ the bilinear Eq. (2.32) reads:

$$\lambda_{12}T^{a,s}(x+\eta)T^{a,s}(x-\eta)+\lambda_{23}T^{a,s+1}(x)T^{a,s-1}(x)=\lambda_{13}T^{a+1,s}(x)T^{a-1,s}(x),$$

and the zeros $x_i^{a,s}$ of $T^{a,s}(x)$ obey the system of coupled equations:

$$\frac{T^{a+1,s+1}(x_i^{a,s})T^{a-1,s}(x_i^{a,s}-\eta)T^{a,s-1}(x_i^{a,s}+\eta)}{T^{a-1,s-1}(x_i^{a,s})T^{a+1,s}(x_i^{a,s}+\eta)T^{a,s+1}(x_i^{a,s}-\eta)}=-1\,,$$

$$\frac{T^{a+1,s-1}(x_i^{a,s})T^{a-1,s}(x_i^{a,s}-\eta)T^{a,s+1}(x_i^{a,s}+\eta)}{T^{a-1,s+1}(x_i^{a,s})T^{a+1,s}(x_i^{a,s}+\eta)T^{a,s-1}(x_i^{a,s}-\eta)}=-1\,.$$

These equations are more complicated than (3.10). In contrast to the previous case, evolutions in a and s are not separated.

3.2. Double-Bloch solutions to the linear problems. In order to further examine the elliptic solutions, we need the notion of double-Bloch functions. A meromorphic function $f(x)$ is said to be *double-Bloch* if it enjoys the following monodromy properties:

$$f(x+2\omega_a)=B_a f(x), \qquad a=1,2. \tag{3.12}$$

The complex numbers B_a are called *Bloch multipliers*. A non-trivial double-Bloch function can be represented as a linear combination of elementary ones:

$$f(x)=\sum_{i=1}^{N}c_i\Phi(x-x_i,\zeta)k^{x/\eta}, \tag{3.13}$$

where [6]

$$\Phi(x,\zeta)=\frac{\sigma(\zeta+x+\eta)}{\sigma(\zeta+\eta)\sigma(x)}\left[\frac{\sigma(\zeta-\eta)}{\sigma(\zeta+\eta)}\right]^{x/(2\eta)} \tag{3.14}$$

and complex parameters ζ and k are related to the Bloch multipliers by the formulas

$$B_a = k^{2\omega_a/\eta} \exp(2\zeta(\omega_a)(\zeta + \eta)) \left(\frac{\sigma(\zeta - \eta)}{\sigma(\zeta + \eta)}\right)^{\omega_a/\eta} \tag{3.15}$$

($\zeta(x) = \sigma'(x)/\sigma(x)$ is the Weierstrass ζ-function).

Let us point out some properties of the function $\Phi(x, \zeta)$. Considered as a function of ζ, $\Phi(x, \zeta)$ is double-periodic:

$$\Phi(x, \zeta + 2\omega_a) = \Phi(x, \zeta).$$

For general values of x one can define a single-valued branch of $\Phi(x, \zeta)$ by cutting the elliptic curve between the points $\zeta = \pm\eta$. In the fundamental domain of the lattice generated by $2\omega_a$ the function $\Phi(x, \zeta)$ has a unique pole at the point $x = 0$:

$$\Phi(x, \zeta) = \frac{1}{x} + O(1)\,.$$

In the next subsection we need the identity:

$$\Phi(x, z)\Phi(y, z) = \Phi(x + y, z)\big(\zeta(x) + \zeta(y) + \zeta(z + \eta) - \zeta(x + y + z + \eta)\big) \tag{3.16}$$

which is equivalent to the well known 3-term bilinear functional equation for the σ-function.

Recall the notion of equivalent Bloch multipliers [6]. The "gauge transformation" $f(x) \to \tilde{f}(x) = f(x)e^{bx}$ (b is an arbitrary constant) does not change the poles of any function and transforms a double-Bloch function into another double-Bloch function. If B_a are Bloch multipliers for f, then the Bloch multipliers for $\tilde{f}$ are $\tilde{B}_1 = B_1 e^{2b\omega_1}$, $\tilde{B}_2 = B_2 e^{2b\omega_2}$. Two such pairs of Bloch multipliers B_a and $\tilde{B}_a$ are said to be ***equivalent***. (In other words, they are equivalent if the product $B_1^{\omega_2} B_2^{-\omega_1}$ is the same for both pairs.)

This definition implies that any double-Bloch function can be represented as a ratio of two elliptic polynomials of the same degree multiplied by an exponential function and a constant:

$$f(x) = c'(k')^{x/\eta} \prod_{i=1}^{N} \frac{\sigma(x - y_i)}{\sigma(x - x_i)}. \tag{3.17}$$

The Bloch multipliers are

$$B_a = (k')^{2\omega_a/\eta} \exp\left(2\zeta(\omega_a) \sum_{j=1}^{N} (x_j - y_j)\right).$$

Equations (3.13) represents a Bloch function by its poles and residues, whereas Eq. (3.17) represents a Bloch function by its poles and zeros.

3.3. The Lax representation. The coefficients in Eq. (3.6) are elliptic functions, i.e. double-periodic with periods $2\omega_a$. Therefore, the equation has double-Bloch solutions . Similarly to the case of the Calogero-Moser model and its spin generalizations the dynamics of poles of the elliptic coefficient in the linear problem is determined by the fact that Eq. (3.6) has an infinite number of double-Bloch solutions.

In what follows we always assume that the poles are in a generic position, i.e. $x_i^m - x_j^m \neq 0, \pm\eta$ and $x_i^m - x_j^{m\pm1} \neq 0, \pm\eta$ for any pair $i \neq j$. Exceptional cases are also of interest but must be treated separately.

Theorem 3.1. *Let $\tau^m(x)$ be an elliptic polynomial of degree N. Equation (3.6) has N linearly independent double-Bloch solutions with simple poles at the points x_i^m and equivalent Bloch multipliers if and only if zeros x_i^m of the τ-function satisfy "equations of motion" (3.10).*

Theorem 3.2. *If Eq. (3.6) has N linearly independent double-Bloch solutions with equivalent Bloch multipliers, then it has infinite number of them. All these solutions have the form*

$$\psi^m(x) = \sum_{i=1}^{N} c_i(m,\zeta,k)\Phi(x - x_i^m, \zeta)k^{x/\eta} \tag{3.18}$$

($\Phi(x,\zeta)$ is defined in (3.14)). The set of corresponding pairs (ζ, z) is parametrized by points of an algebraic curve.

These theorems are proved by the same arguments as in [6]. Here we present the main steps.

N linearly independent double-Bloch solutions with equivalent Bloch multipliers may be written in the form (3.18) with some values of the parameters $\zeta_r, k_r,\ \ s = 1,\ldots,N$. Equivalence of the multipliers implies that the ζ_r can be chosen to be equal $\zeta_r = \zeta$.

Let us substitute the function $\psi^m(x)$ of the form (3.18) with this particular value of ζ into Eq. (3.6). Since any double-Bloch function (except equivalent to a constant) has at least one pole, it follows that the equation is satisfied if its left hand side has zero residues at the points $x = x_i^m - \eta$ and $x = x_i^{m+1}$. The cancelation of poles at these points gives the conditions

$$kc_i(m,\zeta,k) - f_i(m)\sum_{j=1}^{N} c_j(m,\zeta,k)\Phi(x_i^m - x_j^m - \eta, \zeta) = 0\,, \tag{3.19}$$

$$c_i(m+1,\zeta,k) = g_i(m)\sum_{j=1}^{N} c_j(m,\zeta,k)\Phi(x_i^{m+1} - x_j^m, \zeta)\,, \tag{3.20}$$

where

$$f_i(m) = \lambda_{13}\frac{\prod_{s=1}^{N}\sigma(x_i^m - x_s^m - \eta)\sigma(x_i^m - x_s^{m+1})}{\prod_{s=1,\neq i}^{N}\sigma(x_i^m - x_s^m)\prod_{s=1}^{N}\sigma(x_i^m - x_s^{m+1} - \eta)}\,, \tag{3.21}$$

$$g_i(m) = -\lambda_{13}\frac{\prod_{s=1}^{N}\sigma(x_i^{m+1} - x_s^{m+1} + \eta)\sigma(x_i^{m+1} - x_s^m)}{\prod_{s=1,\neq i}^{N}\sigma(x_i^{m+1} - x_s^{m+1})\prod_{s=1}^{N}\sigma(x_i^{m+1} - x_s^m + \eta)}\,. \tag{3.22}$$

Introducing a vector $C(m)$ with components $c_i(m,\zeta,z)$ we can rewrite these conditions in the form

$$(\mathcal{L}(m) - kI)C(m) = 0\,, \tag{3.23}$$

$$C(m+1) = \mathcal{M}(m)C(m)\,, \tag{3.24}$$

where I is the unit matrix. The matrix elements of $\mathcal{L}(m)$ and $\mathcal{M}(m)$ are:

$$\mathcal{L}_{ij}(m) = f_i(m)\Phi(x_i^m - x_j^m - \eta, \zeta), \tag{3.25}$$

$$\mathcal{M}_{ij}(m) = g_i(m)\Phi(x_i^{m+1} - x_j^m, \zeta). \tag{3.26}$$

The compatibility condition of (3.19) and (3.20),

$$\mathcal{L}(m+1)\mathcal{M}(m) = \mathcal{M}(m)\mathcal{L}(m) \tag{3.27}$$

has a form of the discrete Lax equation. The Lax Eq. (3.27) appeared in ref. [11], where Eqs. (3.10) were proposed as a time discretezation of the RS model.

Lemma 3.1. *For matrices $\mathcal{L}$ and $\mathcal{M}$ defined by (3.25), (3.26) the discrete Lax Eq. (3.27) is equivalent to the equations (3.10).*

The proof is along the lines of ref. [11]. We have

$$\begin{aligned} F_{ij} &\equiv (x,\zeta)(\mathcal{M}(m)\mathcal{L}(m) - \mathcal{L}(m+1)\mathcal{M}(m))_{ij} \\ &= f_i(m+1)\sum_s g_s(m)\Phi(x_i^{m+1} - x_s^{m+1} - \eta, \zeta)\Phi(x_s^{m+1} - x_j^m, \zeta) \\ &+ g_i(m)\sum_s f_s(m)\Phi(x_i^{m+1} - x_s^m, \zeta)\Phi(x_s^m - x_j^m - \eta, \zeta). \end{aligned} \tag{3.28}$$

The coefficient in front of the leading singularity at $\zeta = -\eta$ is proportional to

$$f_i(m+1)\sum_s g_s(m) + g_i(m)\sum_s f_s(m).$$

On the other hand,

$$\sum_s \big(f_s(m) - g_s(m)\big) = 0 \tag{3.29}$$

(because this is the sum of residues of the elliptic coefficient in Eq. (3.6)). Therefore,

$$f_i(m+1) = g_i(m), \qquad i = 1, \dots, N. \tag{3.30}$$

These equations are equivalent to (3.10).

To show that (3.28) is identically zero provided Eqs. (3.30) hold, we use the identity (3.16):

$$F_{ij}(x,\zeta) = -g_i(m)\Phi(x_i^{m+1} - x_j^m - \eta, \zeta)\, G,$$

$$\begin{aligned} G = &-\sum_s f_s(m)\big(\zeta(x_s^{m+1} - x_s^m) + \zeta(x_s^m - x_j^m) - \eta)\big) \\ &+ \sum_s g_s(m)(\zeta(x_i^{m+1} - x_s^{m+1} - \eta) + \zeta(x_s^{m+1} - x_j^m)\big). \end{aligned} \tag{3.31}$$

Noting that G is proportional to the sum of residues of the elliptic function

$$\big[\zeta(x_i^{m+1} - \eta - x) + \zeta(x - x_j^m)\big]\prod_{i=1}^{N}\frac{\sigma(x - x_i^m)\sigma(x - x_i^{m+1} + \eta)}{\sigma(x - x_i^m + \eta)\sigma(x - x_i^{m+1})}$$

at the points $x = x_i^{m+1}$ and $x = x_i^m - \eta$, we conclude that $G = 0$.

It was already proved that Eq. (3.6) has N linearly independent solutions if Eqs. (3.10) or the Lax Eq. (3.27) hold for some value of the spectral parameter ζ. It then follows from Lemma 3.1 that the Lax equation holds for any value of ζ. Therefore, for each ζ there exists a double-Bloch solution given by (3.18), where the c_i are components of the common solution to (3.23), (3.24).

Theorem 3.3. *All elliptic solutions of Eq. (2.23) of the form (3.8) are of the algebro-geometric type and x_i^m are given implicitly by the equation*

$$\Theta\left(\eta^{-1}\vec{A}(P_1)x_i^m + m\vec{A}(P_3) + \vec{Z}\right) = 0, \tag{3.32}$$

where the Riemann theta-function corresponds to the algebraic curve Γ defined by the characteristic equation

$$R(k,\zeta) \equiv \det(\mathcal{L}(m) + kI) = k^N + \sum_{i=1}^{N} r_i(\zeta)k^{N-i} = 0\,. \tag{3.33}$$

The matrix $\mathcal{L}$ is given by Eqs. (3.25), (3.21) and the coefficients $r_i(\zeta)$ have the form

$$r_i(\zeta) = \frac{\sigma(\zeta+\eta)^{(i-2)/2}\sigma(\zeta-(i-1)\eta)}{\sigma(\zeta-\eta)^{i/2}} I_i$$

where I_i are integrals of motion. The characteristic Eq. (3.33) and I_i are functions of x_i^m and x_i^{m+1} but stay the same for all m. The spectral curve Γ determined by Eq. (3.33) is an algebraic curve realized as a ramified covering of the elliptic curve. The function (3.18) is the Baker–Akhiezer function on Γ.

We call the spectral curve Γ defined in Theorem 3.3 the Ruijsenaars-Schneider (RS) curve. The RS curve is identical to the spectral curve for the continuous time RS model studied in ref. [6]. The proof of the theorem is omitted. It is as in ref. [6].

The matrix $\mathcal{L}$ is defined by fixing $x_i^{m_0}$ and $x_i^{m_0+1}$. These Cauchy data uniquely define the RS curve Γ, the vectors $\vec{A}(P_\alpha)$, $\alpha = 1,2$ and $\vec{Z}$ in Eq. (3.32). The curve and the vectors $\vec{A}(P_\alpha)$, $\alpha = 1,2$ do not depend on the choice of m_0. They are action-type variables. The vector $\vec{Z}$ depends linearly on this choice and its components are thus angle-type variables.

Remark 3.2. The discrete time dynamics defined in Theorem 3.3 is time-reversible, i.e. the Cauchy data $x_i^{m_0}$, $x_i^{m_0+1}$ completely determine the dynamics for both time directions up to permutation of the "particles". The $\mathcal{L}$-$\mathcal{M}$-pair for the backward time motion is obtained from the difference equations for the dual Baker–Akhiezer function (see Remark 2.6) with an ansatz similar to (3.18). An alternative way to derive equations of motion (3.10) is to require the spectral Eq. (3.23) to be identical to the similar equation for the backward time motion.

Remark 3.3. The form of equations for the dynamics in $l_2 \equiv n$ is identical to the equations (3.10) of the dynamics in $m \equiv l_3$. The Cauchy data for m-dynamics, i.e., values of x_i^m at $m = 0$ and $m = 1$ completely determine an evolution and Cauchy data in n (as well as all other flows). Comparing $\mathcal{L}$-operators for each flow, one finds relations between the Cauchy data:

$$\prod_{s=1}^{N} \frac{\sigma(x_i^{0,0} - x_s^{1,0})\sigma(x_i^{0,0} - x_s^{0,1} - \eta)}{\sigma(x_i^{0,0} - x_s^{0,1})\sigma(x_i^{0,0} - x_s^{1,0} - \eta)} = \frac{\lambda_{12}}{\lambda_{13}}, \qquad i = 1,2,\ldots,N. \tag{3.34}$$

Similar connections exist for the initial data of the $\bar{m}$-flow.

3.4. Loci equations. The values of x_i^0, x_i^1 may be arbitrary if no other reduction (apart from the elliptic one) is imposed. If there is an additional reduction, then the x_i^0, x_i^1 are constrained to belong to a submanifold of $\mathbf{C}^{2N}$, the *reduction locus*. An example of such a locus in the continuous setup is the KP → KdV locus (1.4). Here we present equations defining the loci for three important reductions of Hirota's difference equation. In these cases spectral curves of algebro-geometric solutions are hyperelliptic. As before, x_i^0, x_i^1 are assumed to be in generic position.

A) Discrete KdV equation [18]. The discrete KdV equation appears as the reduction

$$\tau(l_1, l_2 + 1, l_3 + 1) = \tau(l_1, l_2, l_3), \qquad x \equiv \eta l_1$$

of the general 3-dimensional Hirota Eq. (2.23). In the notation (3.5) the equation is

$$\lambda_{23}\tau^m(x+\eta)\tau^m(x) - \lambda_{13}\tau^{m-1}(x)\tau^{m+1}(x+\eta) + \lambda_{12}\tau^{m+1}(x)\tau^{m-1}(x+\eta)\,. \tag{3.35}$$

For algebro-geometric solutions the reduction means that $\vec{U}^{(02)} + \vec{U}^{(03)}$ belongs to the lattice of periods. Therefore, the function

$$\varepsilon(P) = \exp\left(\int_{Q_0}^{P} (d\Omega^{(02)} + d\Omega^{(03)})\right) \tag{3.36}$$

is meromorphic on the curve Γ. From the definition of the abelian integrals it follows that this function has a double pole at P_0 and simple zeros at P_2 and P_3. Outside these points $\varepsilon(P)$ is holomorphic and is not zero. The existence of such a function means that the spectral curve is *hyperelliptic*. A hyperelliptic curve of genus g can be defined by the equation

$$y^2 = \prod_{i=1}^{2g+1}(\varepsilon - \varepsilon_i). \tag{3.37}$$

This is a two-fold covering of the complex plane of the variable ε. The projection of Γ onto the ε-plane defines ε as a meromorphic function on Γ. This function has a double pole on Γ at the branch point P_∞ (above $\varepsilon = \infty$) and two simple zeros at the points $P_0^{(\pm)}$ (above $\varepsilon = 0$). The identification of this notation for the punctures with our previous ones is

$$P_\infty = P_0, \quad P_0^{(-)} = P_2, \quad P_0^{(+)} = P_3.$$

The branch points ε_i in (3.37) may not be arbitrary since the curve should simultaneously be of the RS type. Correspondingly, the Cauchy data x_i^0, x_i^1 with respect to the l_3-flow obey certain constraints. Using the equations of motion (3.10) and (3.34), we obtain a system of $2N$ coupled equations on allowed values of x_i^0, x_i^1 ("equilibrium locus"):

$$\prod_{s=1}^{N} \frac{\sigma(x_i^1 - x_s^1 + \eta)\sigma(x_i^1 - x_s^0 - \eta)}{\sigma(x_i^1 - x_s^1 - \eta)\sigma(x_i^1 - x_s^0 + \eta)} = -\frac{\lambda_{13}}{\lambda_{12}}, \tag{3.38}$$

$$\prod_{s=1}^{N} \frac{\sigma(x_i^0 - x_s^0 + \eta)\sigma(x_i^0 - x_s^1 - \eta)}{\sigma(x_i^0 - x_s^0 - \eta)\sigma(x_i^0 - x_s^1 + \eta)} = -\frac{\lambda_{12}}{\lambda_{13}} \tag{3.39}$$

for $i = 1, 2, \dots, N$. With the help of Eq. (2.15) the right hand sides can be expressed through abelian integrals. The relation between the number of zeros N and genus g of the spectral curve is a subtle question. We do not discuss it here.

Each of the systems (3.38), (3.39) has the form of Bethe equations for an N-site spin chain of spin 1 at each site. One may treat x_s^0 (for instance) as arbitrary input parameters while Bethe's quasimomenta x_s^1 are to be determined by Eqs. (3.38). However, the system of locus equations (3.38), (3.39) determines x_i^0 and x_i^1 simultaneously.

In the continuous time limit we set $x_i^0 = x_i$, $x_i^1 = x_i + \epsilon \dot{x}_i + \frac{1}{2}\epsilon^2 \ddot{x}_i$, $\dot{x}_i = \partial_t x_i$, $\epsilon \to 0$. Assuming $\lambda_{13}/\lambda_{12} \to -(1 + C\epsilon)$ with a constant C, we get from (3.38), (3.39):

$$\sum_{k=1}^{N} \dot{x}_k \left[\zeta(x_i - x_k - \eta) - \zeta(x_i - x_k + \eta)\right] = C, \tag{3.40}$$

$$\sum_{k=1}^{N} \dot{x}_k \left[\wp(x_i - x_k - \eta) - \wp(x_i - x_k + \eta)\right] = 0. \tag{3.41}$$

In the leading order in ϵ the systems (3.38), (3.39) coincide with each other and yield the first system (3.40) while the second one (3.41) follows from the higher order terms. These equations define the equilibrium locus for the Ruijsenaars-Schneider system of particles.

B) 1D Toda chain in discrete time [8]. The reduction condition in this case is:

$$\tau(l_1, l_2, l_3 + 1, \overline{l_3} + 1) = \tau(l_1, l_2, l_3, \overline{l_3}).$$

The discrete time 1D Toda chain in the bilinear form reads

$$\lambda_{13}\tau^{m-1}(x)\tau^{m+1}(x) - \lambda_{12}\tau^{m-1}(x+\eta)\tau^{m+1}(x-\eta) = \lambda_{23}(\tau^m(x))^2\,, \tag{3.42}$$

where we have excluded $\overline{l_3}$ and have passed to the notation of Example A). In this case $\vec{U}^{(03)} + \vec{U}^{(12)}$ belongs to the lattice of periods. The corresponding curve is given by Eq. (3.37) with a polynomial of *even* degree in the r.h.s.

The Cauchy data x_i^0, x_i^1 with respect to the l_3-flow satisfy the locus equations:

$$\prod_{s=1}^{N} \frac{\sigma(x_i^1 - x_s^1 + \eta)\sigma^2(x_i^1 - x_s^0)}{\sigma(x_i^1 - x_s^1 - \eta)\sigma^2(x_i^1 - x_s^0 + \eta)} = -\frac{\lambda_{12}}{\lambda_{13}}. \tag{3.43}$$

$$\prod_{s=1}^{N} \frac{\sigma(x_i^0 - x_s^0 + \eta)\sigma^2(x_i^0 - x_s^1 - \eta)}{\sigma(x_i^0 - x_s^0 - \eta)\sigma^2(x_i^0 - x_s^1)} = -\frac{\lambda_{13}}{\lambda_{12}} \tag{3.44}$$

The continuous time limit can be taken similar to the way of the previous example. In this case, however, we have to assume $\lambda_{13}/\lambda_{12} \to \tilde{C}\epsilon^{-2}$ as $\epsilon \to 0$ with a constant $\tilde{C}$. We get

$$\sigma^2(\eta) \prod_{k=1, \neq i}^{N} \frac{\sigma(x_i - x_k + \eta)\sigma(x_i - x_k - \eta)}{\sigma^2(x_i - x_k)} = \tilde{C}\dot{x}_i^2, \tag{3.45}$$

$$\sum_{k=1,\neq i}^{N} (\dot{x}_i + \dot{x}_k)\,[\zeta(x_i - x_k - \eta) + \zeta(x_i - x_k + \eta) - 2\zeta(x_i - x_k)] = 0. \qquad (3.46)$$

These equations follow also from Eqs. (4.58), (4.59) of the paper [6]. They define the stable locus for the RS system with respect to another flow than in Eqs. (3.40), (3.41).

C) Discrete sine-Gordon equation[1]. The reduction condition is

$$\tau(l_1, l_2, l_3 + 1, \bar{l_3}) = \tau(l_1 + 2, l_2, l_3, \bar{l_3} + 1),$$

so, passing to the same independent variables as in the previous examples, we get the equation

$$\lambda_{13}\tau^m(x-\eta)\tau^m(x+\eta) - \lambda_{23}\tau^{m-1}(x+\eta)\tau^{m+1}(x-\eta) = \lambda_{12}(\tau^m(x))^2\,. \qquad (3.47)$$

Now it is the vector $\vec{U}^{(01)} + \vec{U}^{(32)}$ that belongs to the lattice of periods. The continuous SG equation is reproduced in the limit $P_3 \to P_0$, $P_2 \to P_1$. The spectral curves are again hyperelliptic.

The locus equations for the Cauchy data x_i^0, x_i^1 with respect to the m-flow are

$$\prod_{s=1}^{N} \frac{\sigma(x_i^0 - x_s^1)\sigma(x_i^0 - x_s^1 - 2\eta)}{\sigma^2(x_i^0 - x_s^1 - \eta)} = \frac{\lambda_{12}}{\lambda_{13}}, \qquad (3.48)$$

$$\prod_{s=1}^{N} \frac{\sigma(x_i^1 - x_s^0)\sigma(x_i^1 - x_s^0 + 2\eta)}{\sigma^2(x_i^1 - x_s^0 + \eta)} = \frac{\lambda_{12}}{\lambda_{13}}. \qquad (3.49)$$

Note that the structure of these equations is different compared to the previous examples.

3.5. Remarks on trigonometric solutions. Trigonometric solutions are degenerations of the elliptic solutions when one of the periods tends to infinity. They form a particular subfamily in the variety of soliton solutions described in Sect. 2.4. The trigonometric solutions admit a very explicit description in terms of the data defining the singular curve.

Let us set the period to be 2π:

$$\tau^m(x + 2\pi) = \tau^m(x)\,,$$

so an elliptic polynomial becomes a Laurent polynomial in e^{ix}. The Bethe-like equations on motion (3.10) preserve its form, but the Weierstrass function $\sigma(x)$ is replaced by $\sin x$.

It follows from the periodicity that the function Ψ_0 (2.34) obeys

$$\frac{\Psi_0(\eta x + 2\pi\eta, m; p_j)}{\Psi_0(\eta x + 2\pi\eta, m; q_j)} = \frac{\Psi_0(\eta x, m; p_j)}{\Psi_0(\eta x, m; q_j)}, \qquad j = 1, 2, \ldots, N,$$

or, explicitly

$$\frac{p_j - z_1}{p_j - z_0} = \frac{q_j - z_1}{q_j - z_0} e^{-i\eta J_j}, \qquad (3.50)$$

[1] This version of the discrete SG equation is different from the one considered in [20], [21]. The latter is closer to a special degeneracy of the discrete KdV at $\lambda_{12} \to \lambda_{13}$.

where J_j are integer numbers. This condition restricts admissible singular curves (Riemann spheres with N double points). Minimal Laurent polynomials correspond to the choice $J_j = \pm 1$. This gives N conditions for p_j, q_j, so the number of continuous parameters in the trigonometric case is $2N + 1$ – the same as for the non-degenerate curves of genus N.

Here we do not discuss the trigonometric degeneration of the $\mathcal{L}$-$\mathcal{M}$-pair and dependence of of p_j and q_j on initial data for Ruijsenaars-Schneider particles. This can be done along the lines of the paper [5].

The trigonometric loci can be characterized alternatively by imposing a relation on p_j and q_j in addition to (3.50):

$$E(p_i) = E(q_i), \qquad p_i \neq q_i, \qquad i = 1, 2, \ldots, N. \tag{3.51}$$

For the examples A)-C) of Sect. 3.4 the functions $E(z)$ are

$$\begin{aligned} &A) \qquad E(z) = \frac{(z - z_2)(z - z_3)}{(z - z_0)^2}\,, \\ &B) \qquad E(z) = \frac{(z - z_2)(z - z_3)}{(z - z_1)(z - z_0)}\,, \\ &C) \qquad E(z) = \frac{(z - z_1)(z - z_2)}{(z - z_3)(z - z_0)}\,. \end{aligned} \tag{3.52}$$

Conditions (3.50), (3.51) leave us with a discrete set of admissible pairs p_i, q_i. The continuous parameters γ_s give then an implicit parametrization of the loci.

4. Conclusion

We have shown that the main body of finite-gap theory and the theory of elliptic solutions to nonlinear integrable equations is also applicable to finite-difference (discrete) integrable equations. Discrete equations includes the continuous theory as the result of a limiting procedure. Furthermore, discrete equations reveal some symmetries lost in the continuum limit. We have shown that all elliptic solutions with a constant number of zeros in the evolution (compare to [13]), are of the algebro-geometric type. Moreover, their algebraic curves are spectral curves for L-operators of the Ruijsenaars-Schneider model. Each point of this curve gives rise to discrete time dynamics of zeros of the τ-function.

The structure of equilibrium loci equations of reductions of Hirota's Eq. (analogues of the known KdV-locus (1.4)) is expected to be richer than in the continuous case and requires further study. It would be very interesting to extend the algebro-geometric approach to elliptic solitons of KdV of ref. [22] to the difference case as well as to understand difference elliptic solitons in terms of the Weierstrass reduction theory [23].

To the two main motivations pointed out at the beginning of the paper we can now add yet another one: an intriguing intimate connection between the elliptic solutions to soliton equations and quantum integrable models solved by the Bethe ansatz. In our opinion, the very fact that the zeros dynamics and equilibrium loci are described by Bethe-like equations is remarkable and suggests hidden parallels between quantum and classical integrable equations.

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