Trivalent graphs and solitons

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Until recently, non-linear integrable systems were studied only on the lattices $\mathbb{Z}$ and $\mathbb{Z}^2$: ((L, A)-pairs of the type of the Toda lattice for $\mathbb{Z}$ and (L, A, B)-triples for $\mathbb{Z}^2$, likewise discrete spectral symmetries of a second-order linear operator L such as the Euler-Darboux and Laplace transformations [1]). Note that the trivalent tree Γ_3 is a discrete model of hyperbolic geometry (the Lobachevskii plane) as is $\mathbb{Z}^2$ for the Euclidean plane. No isospectral deformation of a second-order operator L on Γ_3 has been discovered, even in the form of an (L, A, B)-triple $\dot{L} = LA - BL$ deforming only one spectral level $L\Psi = 0$ (see [2]–[4]).

By the order of an equation $L\Psi = 0$, where $(L\Psi)_P = \sum_Q b_{PQ}\Psi_Q$, we mean the maximal diameter $\max_P d(Q_1, Q_2)$, where $b_{PQ_1} \neq 0$, $b_{PQ_2} \neq 0$ or $b_{Q_1Q_2} \neq 0$. The metric on a graph is defined by setting the length of each edge equal to 1, and Ψ_P is a function of the vertices P. We consider graphs where each edge has exactly two vertices and three edges meet at each vertex.

Theorem 1. *A general real self-adjoint operator L of order 4 on Γ_3 has isospectral deformations of one energy level $L\Psi = 0$ in the form of an (L, A, B)-triple*:

$$\dot{L} = LA - BL$$

with

$$(L\Psi)_P = \sum b_{PP''}\Psi_{P''} + b_{PP'}\Psi_{P'} + w_P\Psi_P,$$

where P, P', P'' are vertices, $d(P, P'') = 2$, $d(P, P') = 1$, and we assume that $b_{PP''} > 0$. Here, $B = -A^t$, $(A\Psi)_P = \sum c_{PP'}\Psi_{P'}$.

To express the coefficients $c_{PP'}$ of the nearest neighbours P, P' we choose an initial vertex P_0 of Γ_3. Take a minimal path γ, with edges R_i, joining P_0 and P and oriented from P_0 to P. Let R'_{i_1}, R'_{i_2} be the edges entering the initial vertex of R_i and R''_{i_1}, R''_{i_2} those emanating from its terminal vertex. Consider the multiplicative 1-cocycle on Γ_3 given by

$$\chi(R_i) = -\frac{\left(b_{R''_{i_1}R_i} \cdot b_{R''_{i_2}R_i}\right)}{\left(b_{R'_{i_1}R_i} \cdot b_{R'_{i_2}R_i}\right)}$$

and define

$$c_R = -\frac{1}{b_{R'_1R'_2}}\left(\prod_{R_i\in\gamma}\chi(R_i)\right), \qquad R = PP'.$$

These formulae are obtained from the condition that the operator $LA + A^tL$ has order at most 4. Then the dynamical system $\dot{L} = LA + A^tL$ is well defined and has the form

$$\begin{aligned}\dot{b}_{PP''} &= b_{P'P''}c_{P'P} + c_{P'P}b_{P'P};\\ \dot{b}_{PP'} &= b_{P'P''_i}c_{P''_iP'} + c_{P^*_\alpha P}b_{P^*_\alpha P'} + w_Pc_{PP'} + w_{P'}c_{P'P};\\ \dot{w}_P &= 2b_{PP'}c_{P'P}, \quad i, \alpha = 1, 2,\end{aligned}$$

where $P^*_\alpha PP'P''_i$ are the shortest paths of length $d = 3$ containing the segment $PP' = R$.

Remark 1. For any trivalent graph Γ the coefficients $c_{PP'}$ of the operator A are defined on the Abelian covering of Γ determined by the above 1-cocyle χ along the 1-cycles.

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Theorem 2. *A general real self-adjoint operator L of order 4 on Γ_3 admits a one-parameter family of factorizations of the form*

$$L = Q^t Q + u_P, \quad \textit{where} \quad (Q\psi)_P = \sum_Q d_{PQ}\psi_Q + v_P\psi_P,$$

with

$$b_{PP''} = d_{P'P}d_{P'P''}; \qquad b_{PP'} = d_{P'P}v_{P'} + d_{PP'}v_P,$$
$$w_P = v_P^2 + \sum_{P'} d_{P'P}^2 + u_P \quad (\textit{for } d_{PQ} > 0).$$

Here the coefficients d_{PQ} are determined uniquely and v_P is defined by one parameter, its value at $P_0 \in \Gamma_3$. These factorizations determine a Laplace-type transformation

$$\widetilde{L} = Qu_P^{-1}Q^t + 1, \qquad \tilde{\psi} = Q\psi,$$

where $\widetilde{L}\tilde{\psi} = 0$ if $L\psi = 0$. The self-adjoint operator $\widetilde{L}$ is defined up to a transformation

$$\widetilde{L} \to f_P^{-1} \cdot \widetilde{L} \cdot f_P, \qquad \tilde{\psi} \to f_P^{-1} \cdot \tilde{\psi}.$$

It is convenient to choose $f_P = u_P^{1/2}$. Then we have $\widetilde{L} = \widetilde{Q}^t\widetilde{Q} + u_P$, where

$$\widetilde{Q} = u_P^{-1/2}Q^t u_P^{1/2}, \qquad \tilde{\psi} = u_P^{-1/2}Q\psi$$

(compare [5] for $\mathbb{Z}^2$).

Remark 2. The factorization of L depends only on the solubility of the linear equation $b_{PQ} = d_{QP}v_Q + d_{PQ}v_P$. Incidentally, this operator has a non-trivial (one-dimensional) kernel if and only if the above cocycle χ is cohomologous to zero on Γ.

Bibliography

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