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The τ-Function for Analytic Curves

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ABSTRACT. We review the concept of the τ-function for simple analytic curves. The τ-function gives a formal solution to the two-dimensional inverse potential problem and appears as the τ-function of the integrable hierarchy which describes conformal maps of simply-connected domains bounded by analytic curves to the unit disk. The τ-function also emerges in the context of topological gravity and enjoys an interpretation as a large N limit of the normal matrix model.

1. Introduction

Recently, it has been realized [1; 2] that conformal maps exhibit an integrable structure: conformal maps of compact simply connected domains bounded by analytic curves provide a solution to the dispersionless limit of the two-dimensional Toda hierarchy. As is well known from the theory of solitons, solutions of an integrable hierarchy are represented by τ-functions. The dispersionless limit of the τ-function emerges as a natural object associated with the curves. In this paper we discuss the τ-function for simple analytic curves and its connection to the inverse potential problem, area preserving diffeomorphisms, the Dirichlet boundary problem, and matrix models.

2. The Inverse Potential Problem

Define a *closed analytic curve* as a curve that can be parametrized by a function $z \equiv x+iy = z(w)$, analytic in a domain that includes the unit circle $|w| = 1$. Consider a closed analytic curve γ in the complex plane and denote by D_+ and D_- the interior and exterior domains with respect to the curve. The point $z = 0$ is assumed to be in D_+. Assume that the domain D_+ is filled homogeneously with electric charge, with a density that we set to 1. The potential Φ created by the charge obeys the equation

$$-\partial_z \partial_{\bar z} \Phi(z, \bar z) = \begin{cases} 1 & \text{if} \quad z = x + iy \in D_+, \\ 0 & \text{if} \quad z = x + iy \in D_-. \end{cases} \tag{2–1}$$

The potential Φ can be written as an integral over the domain D^+:

$$\Phi(z, \bar z) = -\frac{2}{\pi} \int_{D_+} d^2 z' \, \log |z - z'| \tag{2–2}$$

In the exterior domain D_-, the potential is the harmonic function whose asymptotic expansion as $z \to \infty$ is given by

$$\Phi^-(z, \bar z) = -2t_0 \log |z| + 2 \operatorname{Re} \sum_{k>0} \frac{v_k}{k} z^{-k}, \tag{2–3}$$

where

$$v_k = \frac{1}{\pi} \int_{D_+} z^k d^2 z \qquad (k > 0) \tag{2–4}$$

are the harmonic moments of the interior domain D_+ and

$$\pi t_0 = \int_{D_+} d^2 z$$

is its area. In the interior domain D_+, the potential (2–2) is equal to a function Φ^+, which is harmonic up to the term $-|z|^2$. The expansion of this function around $z = 0$ is

$$\Phi^+(z, \bar z) = -|z|^2 - v_0 + 2 \operatorname{Re} \sum_{k>0} t_k z^k. \tag{2–5}$$

Here

$$t_k = -\frac{1}{\pi k} \int_{D_-} z^{-k} d^2 z \qquad (k > 0) \tag{2–6}$$

are the harmonic moments of the exterior domain D_- and

$$v_0 = \frac{2}{\pi} \int_{D_+} \log |z| \, d^2 z.$$

The two sets of moments (2–4) and (2–6) are related by the conditions $\Phi^+ = \Phi^-$ and $\partial_z \Phi^+ = \partial_z \Phi^-$ on the curve γ.

The inverse potential problem is to determine the form of the curve γ given one of the functions Φ^+ or Φ^-, i.e., given one of the infinite sets of moments. We will choose as independent variables the area πt_0 and the moments of the exterior t_k, for $k \geq 1$. Under certain conditions, they completely determine the form of the curve as well as the moments v_k, for $k \geq 0$ [3]. More precisely, $\{t_k\}_{k=0}^{\infty}$ is a good set of local coordinates in the space of analytic curves. For simplicity we assume in this paper that only a finite number of t_k are nonzero. In this case the series (2–5) is a polynomial in z, $\bar z$ and, therefore, it gives the function Φ^+ for $z \in D_+$. Note that t_0, v_0 are real quantities while all other moments are in general complex variables.

3. Variational Principle

Consider the energy functional describing a charge with a density $\rho(z, \bar{z})$ in the background potential created by the homogeneously distributed charge with the density $+1$ inside the domain D_+ (2–1):

$$\mathcal{E}\{\rho\} = -\frac{1}{\pi^2} \iint d^2z\, d^2z'\ \rho(z,\bar{z})\ \log|z-z'|\,\rho(z',\bar{z}')\ -\frac{1}{\pi}\int d^2z\ \rho(z,\bar{z})\ \Phi(z,\bar{z}).$$

The first term is the two-dimensional "Coulomb" energy of the charge while the second one is the energy due to the background charge. Clearly, the distribution of the charge neutralizing the background charge gives the minimum to the functional: $\rho_0 = -1$ inside the domain and $\rho_0 = 0$ outside. At the minimum the functional is equal to minus electrostatic energy E of the background charge :

$$-E = \min_\rho \mathcal{E}\{\rho\} = \frac{1}{\pi^2}\int_{D_+} d^2z \int_{D_+} d^2z'\ \log|z-z'| = -\frac{1}{2\pi}\int_{D_+} d^2z\ \Phi(z,\bar{z}).$$

Varying over ρ and then setting $\rho = -1$ inside the domain, we obtain (2–5).

The first corollary of the variational principle is that the E is a potential function for the moments. Equation (2–5) suggests treating v_0 and t_k as independent variables, so moments of the interior, v_k, $k \geq 1$, and t_0 are functions of v_0 and t_k. Differentiate E or $-\mathcal{E}\{\rho\}$ at the extremum with respect to the parameters v_0, t_k. Since ρ_0 minimizes the functional, the derivative is equivalent to the partial derivative of $\mathcal{E}$ at the fixed extremum ρ. This gives

$$\frac{\partial E}{\partial t_k} = v_k, \quad \frac{\partial E}{\partial \bar{t}_k} = \bar{v}_k, \quad \frac{\partial E}{\partial v_0} = -t_0, \tag{3–1}$$

where the partial derivative with respect to t_k is taken at fixed v_0 and t_j, for $j \neq 0, k$. Therefore the differential dE reads

$$dE = \sum_{k>0}(v_k dt_k + \bar{v}_k d\bar{t}_k) - t_0 dv_0.$$

The variational principle may be formulated in a number of different ways. One particular variational principle is suggested by the matrix model discussed in Section 9. In this case one considers a charged liquid in the potential

$$V(z,\bar{z}) = z\bar{z} + v_0 - \sum_{k>0}\left(t_k z^k + \bar{t}_k \bar{z}^k\right) \tag{3–2}$$

defined everywhere on the plane and v_0 and t_k are parameters. The energy of the charged liquid,

$$\mathcal{E}\{\rho, V\} = -\frac{1}{\pi^2}\int d^2z \int d^2z'\ \rho(z,\bar{z})\ \rho(z',\bar{z}')\ \log|z-z'| + \frac{1}{\pi}\int d^2z\ \rho(z,\bar{z})\ V(z,\bar{z}), \tag{3–3}$$

reaches its minimum if the liquid forms a drop with the density $\rho_0 = -1$ bounded by the curve determined by parameters of the potential v_0 and t_k. For another version of the variational principle see [4].

4. The τ-Function

It is more natural to treat the total charge t_0 rather than v_0 as an independent variable, i.e., to consider the variational principle at a fixed total charge $t_0 = \int \rho d^2 z$. This is achieved via the Legendre transformation. Introduce the function $F = E + t_0 v_0$, whose differential is

$$dF = \sum_{k>0} (v_k dt_k + \bar{v}_k d\bar{t}_k) + v_0 dt_0.$$

We define the τ-function as $\tau = e^F$, so that

$$\log \tau = \frac{1}{2\pi} \int_{D_+} d^2 z\ \Phi(z, \bar{z}) + t_0 v_0 = -\frac{1}{\pi^2} \int\!\!\int_{D_+} \log \left| \frac{1}{z} - \frac{1}{z'} \right| d^2 z\, d^2 z'. \tag{4–1}$$

The τ-function is a real function of the moments $\{t_0, t_1, t_2, \ldots\}$. Under the assumption that only a finite number of them are nonzero, we can substitute (2–5) into (4–1) and perform the term-wise integration. Taking into account that $\frac{1}{\pi} \int_{D_+} |z|^2 d^2 z = \frac{1}{2} t_0^2 + \frac{1}{2} \sum_{k>0} k(t_k v_k + \bar{t}_k \bar{v}_k)$ (a simple consequence of the Stokes formula), we get the expression for the τ-function in terms of t_k and v_k:

$$2 \log \tau = -\frac{1}{2} t_0^2 + t_0 v_0 - \frac{1}{2} \sum_{k>0} (k-2)(t_k v_k + \bar{t}_k \bar{v}_k).$$

Rephraising (3–1) we get the main property of the τ-function, which was used in [2] as its definition:

$$\frac{\partial \log \tau}{\partial t_k} = v_k, \quad \frac{\partial \log \tau}{\partial \bar{t}_k} = \bar{v}_k, \quad \frac{\partial \log \tau}{\partial t_0} = v_0, \tag{4–2}$$

where the derivative with respect to t_k is taken at fixed t_j $(j \neq k)$.

Two immediate consequences of the very existence of the potential function are symmetry relations for the moments

$$\frac{\partial v_k}{\partial t_n} = \frac{\partial v_n}{\partial t_k}, \quad \frac{\partial v_k}{\partial \bar{t}_n} = \frac{\partial \bar{v}_n}{\partial t_k}$$

and the quasi-homogeneity condition for the τ-function:

$$4 \log \tau = -t_0^2 + 2 t_0 \frac{\partial \log \tau}{\partial t_0} - \sum_{n>0} (n-2) \Big(t_n \frac{\partial \log \tau}{\partial t_n} + \bar{t}_n \frac{\partial \log \tau}{\partial \bar{t}_n} \Big).$$

Apart from the term $-t_0^2$, this formula reflects the scaling of moments as $z \to \lambda z$: $t_k \to \lambda^{2-k} t_k$ $(k \geq 0)$, $v_k \to \lambda^{2+k} v_k$ $(k \geq 1)$.

As an illustration we present the τ-function of ellipse [2]. In this case only the first two moments t_1 and t_2 are nonzero:

$$\log\tau = -\frac{3}{4}t_0^2 + \frac{1}{2}t_0^2 \log\left(\frac{t_0}{1-4|t_2|^2)}\right) + \frac{t_0}{1-4|t_2|^2}\left(|t_1|^2 + t_1^2\bar{t}_2 + \bar{t}_1^2 t_2\right).$$

(The τ-function for the ellipse (at $t_1 = 0$) appeared in [5] as the limit of the Laughlin wave function or a planar limit of the free energy of normal matrix models; see Section 9.)

5. The Schwarz Function and the Generating Function of the Conformal Map

Consider a univalent conformal map of the exterior domain D_- to the exterior of the unit disk and expand it in a Laurent series:

$$w(z) = \frac{1}{r}z + \sum_{j=0}^{\infty} p_j z^{-j},$$

where the coefficient r is chosen to be real and positive. The series for the inverse map (from the exterior of the unit disk to D_-) has a similar form:

$$z(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j}. \tag{5–1}$$

Chosen w on the unit circle, (5–1) gives a parametrization of the curve. By the definition of an analytic curve, the map can be analytically continued to a strip-like neighborhood of the curve belonging to D_+. The continuation is given by the Riemann–Schwarz reflection principle (see [6], for example):

$$w = (\bar{w}(S(z)))^{-1},$$

where $S(z)$ is the point reflected relative to the curve, and where the bar notation has the following meaning: Given an analytic function $f(z) = \sum_j f_j z^j$, we set $\bar{f}(z) = \sum_j \bar{f}_j z^j$. Following [7], we call $S(z)$ *the Schwarz function* of the curve. We recall its construction. Write the equation for the curve $F(x,y) = 0$ in complex coordinates, $F\left(\frac{1}{2}(z+\bar{z}),\, \frac{1}{2i}(z-\bar{z})\right) = 0$, and solve it with respect to $\bar{z}$. One gets the Schwarz function: $\bar{z} = S(z)$. The Schwarz function is analytic in a strip-like domain that includes the curve. On the curve the Schwarz function is equal to the complex conjugate argument. The main property of the Schwarz function is the obvious but important *unitarity condition*

$$\bar{S}(S(z)) = z$$

(the inverse function coincides with the complex conjugate function). In terms of a conformal map the Schwarz function is

$$S(z) = rw^{-1}(z) + \sum_{j=0}^{\infty} \bar{u}_j w^j(z). \tag{5–2}$$

Using the Schwarz function one can write the moments of the exterior and the interior domains (2–4), (2–6) as contour integrals

$$t_n = \frac{1}{2\pi i n} \oint_\gamma z^{-n} S(z)\, dz, \quad v_n = \frac{1}{2\pi i} \oint_\gamma z^n S(z)\, dz. \tag{5–3}$$

(This follows from the more general statement

$$\int_{D_\pm} f(z)\, d^2 z = \pm \frac{1}{2i} \oint_\gamma f(z) S(z)\, dz,$$

where $f(z)$ is an analytic function in the domain $D_\pm$.) Equation (5–3) yields the Laurent expansion of the Schwarz function

$$S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}. \tag{5–4}$$

We now define the *generating function* $\Omega(z)$, related to the Schwarz function by

$$S(z) = \partial_z \Omega(z).$$

The latter is given, according to (5–4), by the Laurent series

$$\Omega(z) = \sum_{k=1}^{\infty} t_k z^k - \frac{1}{2} v_0 + t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k}.$$

It can be represented as $\Omega(z) = \Omega^{(+)}(z) + \Omega^{(-)}(z) - \frac{1}{2}v_0$, where $\Omega^{(\pm)}(z)$ are analytic in $D_\pm$:

$$\Omega^{(+)}(z) = \frac{1}{\pi} \int_{D_-} \log\Bigl(1 - \frac{z}{z'}\Bigr) d^2 z' = \sum_{k=1}^{\infty} t_k z^k,$$

$$\Omega^{(-)}(z) = \frac{1}{\pi} \int_{D_+} \log(z - z') d^2 z' = t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k}.$$

From (2–3) and (2–5) we see that $\Phi^-(z,\bar{z}) = -2\,\mathrm{Re}\,\Omega^{(-)}(z)$ and $\Phi^+(z,\bar{z}) = 2\,\mathrm{Re}\,\Omega^{(+)}(z) - v_0 - |z|^2$. Contrary to the potentials $\Phi^\pm$, the analytical functions Ω^+ and $-\Omega^-$ do not match each other on the curve. The discontinuity gives the value of the generating function restricted to the curve

$$\Omega(z) = \frac{1}{2}|z|^2 + 2iA(z), \quad z \in \gamma,$$

where $A(z)$ is the area of the interior domain bound by the ray $\varphi = \arg z$ and the real axis. As a corollary, it is easy to show that variations of the $\Omega(z)$ on the curve with respect to the *real* parameters t_0, $\operatorname{Re} t_k$ and $\operatorname{Im} t_k$ are purely imaginary. This allows one to apply the Riemann–Schwarz reflection principle to analytical continuation of

$$H_k(z) = \partial_{t_k}\Omega(z), \quad \bar{H}_k(z) = -\partial_{\bar{t}_k}\Omega(z),$$

and to prove the fundamental relations

$$\partial_{t_0}\Omega(z) = \log w(z), \tag{5–5}$$

$$\partial_{t_k}\Omega(z) = \big(z^k(w)\big)_+ + \tfrac{1}{2}\big(z^k(w)\big)_0, \tag{5–6}$$

$$\partial_{\bar{t}_k}\Omega(z) = \big(S^k(z(w))\big)_- + \tfrac{1}{2}\big(S^k(z(w))\big)_0. \tag{5–7}$$

The symbols $(f(w))_\pm$ stand for the truncated Laurent series, preserving only terms with positive or negative powers of w, as the case may be; $(f(w))_0$ is the constant term of the series. The derivatives in (5–5) and (5–7) are taken at fixed z.

To prove (5–5), we first notice that

$$\begin{aligned}\partial_{t_0}\Omega(z(w)) &= \log z - \tfrac{1}{2}\partial_{t_0}v_0 + \text{negative powers in } z\\ &= \log wr - \tfrac{1}{2}\partial_{t_0}v_0 + \text{negative powers in } w.\end{aligned}$$

Independently, one can show that $\partial_{t_0}v_0 = 2\log r$.

Then, using the Riemann–Schwarz reflection principle, we may also write $\partial_{t_0}\Omega(z(w))$ in the form $\partial_{t_0}\bar{\Omega}(S(z(w))$. Expanding this in $S(z)$ and then using the expansion of (5–2) in w, we have

$$\begin{aligned}\partial_{t_0}\bar{\Omega}(S(z(w))) &= \log S(z) - \tfrac{1}{2}\partial_{t_0}v_0 + \text{negative powers in } S(z)\\ &= \log w + \text{positive powers in } w.\end{aligned}$$

Comparing both expansions, we conclude that $\partial_{t_0}\Omega(z) = \log w(z)$. Similar arguments are used in the proof of (5–6) and (5–7).

6. Dispersionless Hirota Equation and the Dirichlet Boundary Problem

Using the representation (4–2) of the moments v_k as derivatives of the τ-function, one can express the conformal map $w(z)$ (5–5) through the τ-function:

$$\log w = \log z - \partial_{t_0}\left(\frac{1}{2}\partial_{t_0} + \sum_{k\geq 1}\frac{z^{-k}}{k}\partial_{t_k}\right)\log\tau. \tag{6–1}$$

With the help of the τ-function, equations (5–6) and (5–7) can be similarly encoded as follows:

$$\partial_z\partial_\zeta \log\big(w(z){-}w(\zeta)\big) = \frac{1}{(z{-}\zeta)^2} + \left(\sum_{k\geq 1} z^{-k-1}\partial_{t_k}\right)\left(\sum_{n\geq 1}\zeta^{-n-1}\partial_{t_n}\right)\log\tau, \tag{6–2}$$

$$-\partial_z\partial_{\bar\zeta}\log\left(w(z)\bar w(\bar\zeta)-1\right)=\Big(\sum_{k\geq 1}z^{-k-1}\partial_{t_k}\Big)\Big(\sum_{n\geq 1}\bar\zeta^{-n-1}\partial_{\bar t_n}\Big)\log\tau. \tag{6–3}$$

The derivation is similar to the one given in [8; 9] for the case of the KP hierarchy. Moreover, these equations in the integrated form are most conveniently written in terms of the differential operators

$$D(z)=\sum_{k\geq 1}\frac{z^{-k}}{k}\partial_{t_k},,\quad \bar D(\bar z)=\sum_{k\geq 1}\frac{\bar z^{-k}}{k}\partial_{\bar t_k}. \tag{6–4}$$

From (6–2) and (6–3) one obtains:

$$\log\frac{w(z)-w(\zeta)}{z-\zeta}=-\tfrac12\partial_{t_0}^2\log\tau+D(z)D(\zeta)\log\tau, \tag{6–5}$$

$$-\log\left(1-\frac{1}{w(z)\bar w(\bar\zeta)}\right)=D(z)\bar D(\bar\zeta)\log\tau. \tag{6–6}$$

Combining (6–1) and (6–5), one obtains the dispersionless Hirota equation (or the dispersionless Fay identity) for two-dimensional Toda lattice hierarchy [2]:

$$(z-\zeta)e^{D(z)D(\zeta)\,\log\tau}=ze^{-\partial_{t_0}D(z)\log\tau}-\zeta e^{-\partial_{t_0}D(\zeta)\log\tau} \tag{6–7}$$

This equation, after being expanded in powers of z and ζ, generates an infinite set of relations between the second derivatives $\partial_{t_n}\partial_{t_m}\log\tau$ of the τ-function. Using (6–6) instead of (6–5), a similar equation for the mixed derivatives $\partial_{t_n}\partial_{\bar t_m}\log\tau$ can be written:

$$1-e^{-D(z)\bar D(\bar\zeta)\log\tau}=\frac{1}{z\bar\zeta}e^{\partial_{t_0}(\partial_{t_0}+D(z)+\bar D(\bar\zeta))\log\tau}$$

We conclude this section with two other forms of the dispersionless Hirota equation for the conformal map. They emphasize a relation between the Hirota equation and two fundamental objects of the classical analysis: the Green function of the Dirichlet problem (which was pointed out to us by L. Takhtajan) and the Schwarz derivative.

The Green function of the Dirichlet boundary problem for the Laplace operator in D_- expressed through the conformal map $w(z)$ is:

$$G(z,\zeta)=\log\left|\frac{w(z)-w(\zeta)}{w(z)\bar w(\bar\zeta)-1}\right|.$$

Combining (6–5) and (6–6), and using the notation (6–4), we represent the Green function as follows:

$$2G(z,\zeta)=2\log|z^{-1}-\zeta^{-1}|+\left(\partial_{t_0}+D(z)+\bar D(\bar z)\right)\left(\partial_{t_0}+D(\zeta)+\bar D(\bar\zeta)\right)\log\tau. \tag{6–8}$$

This formula generalizes (6–1), since (6–8) becomes the real part of (6–1) as $\zeta\to\infty$. (As ζ approaches infinity, $G(z,\zeta)$ tends to $-\log|w(z)|$.) The real part of (6–1) can be written in the form

$$\Phi(z,\bar{z}) = -2t_0 \log|z| + \big(D(z) + \bar{D}(\bar{z})\big)\log\tau,$$

where Φ is the potential (2–2) and $z \in D_-$.

The left-hand side of (6–5) generalizes the Schwarz derivative of the conformal map

$$T(z) \equiv \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = 6 \lim_{z\to\zeta} \partial_z\partial_\zeta \log\frac{w(z)-w(\zeta)}{z-\zeta}.$$

Taking the limit $\zeta \to z$ of both sides of (6–5), we get a relation between the Schwarz derivative and the τ-function:

$$T(z) = 6\, z^{-2} \sum_{k,n\geq 1} z^{-k-n} \frac{\partial^2 \log\tau}{\partial t_k \partial t_n}$$

This can be used as an alternative definition of the τ-function.

7. Integrable Structure of Conformal Maps and Area-Preserving Diffeomorphisms

Equations (5–5)–(5–7) allow one to say that the differential

$$d\Omega = S\,dz + \log w\,dt_0 + \sum_{k=1}^{\infty}(H_k\,dt_k - \bar{H}_k\,d\bar{t}_k)$$

generates the set of Hamiltonian equations for deformations of the curve due to variation of t_k:

$$\partial_{t_k} S(z) = \partial_z H_k(z), \quad \partial_{\bar{t}_k} S(z) = -\partial_z \bar{H}_k(z), \tag{7–1}$$

where we set $H_0(z) = \log w(z)$. The equations are consistent due to commutativity of the flows:

$$\big(\partial_{t_j} H_k\big)_z = (\partial_{t_k} H_j)_z = \partial_{t_j}\partial_{t_k}\Omega(z).$$

Equations (7–1) are more transparent when written in terms of canonical variables. The differential $d\Omega$ suggests that the pairs $\log w,\ t_0$ and $z(w),\ S(z(w))$ are canonical and establishes the *symplectic structure for conformal maps*. Indeed, treating w as an independent variable, one rewrites (5–5) as

$$\{z(w),\, S(z(w))\} = 1, \tag{7–2}$$

where the Poisson bracket $\{\,\cdot\,,\cdot\,\}$ is with respect to $\log w$ and the area t_0 is defined as

$$\{f,\, g\} = w\frac{\partial f}{\partial w}\frac{\partial g}{\partial t_0} - w\frac{\partial g}{\partial w}\frac{\partial f}{\partial t_0},$$

where the derivatives with respect to t_0 are taken at fixed t_k and w.

The other flows read

$$\frac{\partial z(w)}{\partial t_k} = \{H_k, z(w)\}, \tag{7–3}$$

$$\frac{\partial S(z(w))}{\partial t_k} = \{H_k, S(z(w))\}, \tag{7–4}$$

and similarly for the flows with respect to $\bar{t}_k$. Now the Hamiltonian functions H_k and $\bar{H}_k$ are degree k polynomials of w and w^{-1} respectively.

The consistency conditions (7–1) now take the form of the zero-curvature conditions

$$\partial_{t_j} H_i - \partial_{t_i} H_j + \{H_i,\, H_j\} = 0, \tag{7–5}$$

$$\partial_{t_j} \bar{H}_i + \partial_{\bar{t}_i} H_j + \{\bar{H}_i,\, H_j\} = 0. \tag{7–6}$$

The infinite set of Poisson-commuting flows forms a *Whitham integrable hierarchy* [10]. Equations (7–3) and (7–4) are the Lax–Sato equations for the hierarchy. They generate an infinite set of differential equations for the coefficients (potentials) u_j of the inverse conformal map (5–1). The first equation of the hierarchy is

$$\partial^2_{t_1\bar{t}_1}\phi = \partial_{t_0} \exp(\partial_{t_0}\phi), \quad \partial_{t_0}\phi = \log r^2.$$

The integrable hierarchy describing conformal maps is also known in the soliton literature as the *dispersionless Toda lattice hierarchy*, or SDiff(2) Toda hierarchy [11]; see the next section. The algebra Sdiff(2) of area-preserving diffeomorphisms is the symmetry algebra of this hierarchy [11]. Equations (7–3)–(7–6) describe infinitesimal deformations of the curve such that the area t_0 is kept fixed.

(A relation between conformal maps of slit domains and special solutions to equations of hydrodynamic type, namely the Benney equations, was first observed by Gibbons and Tsarev [12].)

The integrable hierarchy possesses many solutions. The particular solution relevant to conformal maps is selected by the subsidiary condition (7–2). This condition, known as *dispersionless string equation*, has already appeared in the study of the $c = 1$ topological gravity [11; 13; 14] and in the large N limit of a model of normal random matrices [15]. The latter is discussed in Section 9.

8. Toda Lattice Hierarchy and its Dispersionless Limit

We review the two-dimensional Toda lattice hierarchy and show that its dispersionless limit gives the equations describing the conformal maps (5–6), (5–7), (7–3), (7–4).

The two-dimensional Toda hierarchy is defined by two Lax operators

$$L = r(t_0)\; e^{\hbar\partial/\partial t_0} + \sum_{k=0}^{\infty} u_k(t_0)\; e^{-k\hbar\partial/\partial t_0}, \tag{8–1}$$

$$\bar{L} = e^{-\hbar\partial/\partial t_0}\, r(t_0) + \sum_{k=0}^{\infty} e^{k\hbar\partial/\partial t_0} \bar{u}_k(t_0), \tag{8–2}$$

acting in the space of functions of t_0 where the coefficients u_j and $\bar{u}_j$ are functions of t_0 and also of two independent sets of parameters ("times") t_k and $\bar{t}_k$. Note that u_k and $\bar{u}_k$ as well as t_k and $\bar{t}_k$ in (8–1), (8–2) are not necessarily complex conjugate to each other, although we choose them to be so.

The dependence of the coefficient u_k and $\bar{u}_k$ on t_k and $\bar{t}_k$ are given by the Lax–Sato equations:

$$\hbar \frac{\partial L}{\partial t_k} = [H_k, L], \tag{8–3}$$

$$\hbar \frac{\partial L}{\partial \bar{t}_k} = [L, \bar{H}_k], \tag{8–4}$$

and similar equations for $\bar{L}$. The flows are generated by

$$H_k = \left(L^k\right)_+ + \tfrac{1}{2}\left(L^k\right)_0 \tag{8–5}$$

and

$$\bar{H}_k = \left(\bar{L}^k\right)_- + \tfrac{1}{2}\left(\bar{L}^k\right)_0,$$

where the symbol $\left(L^k\right)_\pm$ means positive (negative) parts of the series in the shift operator $e^{\hbar\,\partial/\partial t_0}$. The first equation of the hierarchy is the Toda lattice equation

$$\partial^2_{t_1\bar{t}_1}\phi(t_0) = e^{\phi(t_0+\hbar)-\phi(t_0)} - e^{\phi(t_0)-\phi(t_0-\hbar)},$$

where $r^2 = e^{\phi(t_0+\hbar)-\phi(t_0)}$.

The spectrum of the Lax operator is determined by the linear problem $L\Psi = z\Psi$. The wave function Ψ is expressed through the τ-function $\tau_\hbar$ of the dispersionful hierarchy (8–3), (8–4) by the formula

$$\Psi(z; t_0, t_1, t_2, \cdots) \\ = \tau_\hbar^{-1}(t_0, t_1, t_2, \ldots) z^{t_0/\hbar} e^{(1/\hbar)\sum_{k>0} t_k z^k} e^{\hbar \sum_{k>0}(z^{-k}/k)\,\partial/\partial t_k} \tau_\hbar(t_0, t_1, t_2, \ldots).$$

Among many solutions of the hierarchy, one is of particular interest. It is selected by the *string equation* [16]

$$[L, \bar{L}] = \hbar. \tag{8–6}$$

This solution is known to describe the normal matrix model at finite size of matrices [15].

The dispersionless limit of the Toda hierarchy is a formal semi-classical limit $\hbar \to 0$. To proceed we notice that the shift operator $W = e^{\hbar\partial/\partial t_0}$ obeys the commutation relation $[W, t_0] = \hbar W$. In the semiclassical limit it is supposed to be replaced by the canonical variable w with the Poisson bracket $\{\log w, t_0\} = 1$. The Lax operator then becomes a c-valued function which is identified with the inverse conformal map $z(w)$ (5–1). Similarly, $\bar{L}$ is identified with $S(z(w))$. In their turn, the Lax–Sato equations (8–3) and (8–4) are identified with equations

(7–3) and (7–4) for the conformal map. In the same fashion the dispersionless limit of the string equation (8–6) is identified with (7–2). The semiclassical limits of the wave function and the τ-function give the generating function Ω and the dispersionless τ-function: $\Psi \to e^{\Omega/\hbar}$, $\tau_\hbar \to e^{(\log \tau)/\hbar^2}$. Similarly, equation (6–7) is a semiclassical limit of the Hirota equation for the τ-function of the two-dimensional Toda hierarchy.

9. The τ-Function of the Conformal Map as a Large N Matrix Integral

The integrable structure of conformal maps is identical to the one observed in a class of random matrix models related to noncritical string theories. Moreover, there exists a random matrix model whose large N limit reproduces *exactly* the τ-function for analytic curves.

Consider the partition function of the ensemble of normal random $N \times N$ matrices [15], with the potential (3–2):

$$\tau_\hbar[t, \bar{t}] = \int dM\, dM^\dagger e^{-(1/\hbar)\operatorname{Tr} V(M,M^\dagger)}.$$

(V. Kazakov pointed out to us that the Lax equations (8–3) and (8–4) are generated by the Hermitian 2-matrix model [17] with complex conjugated potentials. The latter and the normal matrix model have an identical $1/N$-expansion.)

A matrix is called normal if it commutes with its Hermitian conjugated $[M, M^\dagger] = 0$. Passing to the eigenvalues $\operatorname{diag}(z_1, \dots, z_N)$ of the matrix M, one obtains the measure of the integral in a factorized form

$$dM\, dM^\dagger \sim \prod_{i=1}^{N} dz_i\, d\bar{z}_i \prod_{k<j} (z_k - z_j)(\bar{z}_k - \bar{z}_j).$$

Then the partition function is represents a two-dimensional Coulomb gas in the potential (3–2)

$$\tau_\hbar[t, \bar{t}] = \int \prod_{k=1}^{N} dz_k\, d\bar{z}_k\; e^{-(1/\hbar)V(z_k,\bar{z}_k)} \; \prod_{i<j} e^{2\log|z_i - z_j|}.$$

To proceed to the large N limit one introduces a parameter $t_0 = \hbar N$ and expresses the integrand in terms of density of eigenvalues as $e^{-\hbar^{-2}\mathcal{E}\{\rho,V\}}$, where $\mathcal{E}\{\rho, V\}$ is given by (3–3). Then the limit for large N (or $\hbar \to 0$) yields to the variational principle of Section 3. In the large N limit the eigenvalues of the matrix homogeneously fill the domain D_+ bound by the curve, characterized by the harmonic moments t_k and the area t_0 and leads to the τ-function defined by (4–1). Other objects introduced in Sections 3 to 7 can also be identified with expectation values of the matrix model. In particular the moments v_k of (2–4) are

$$v_k = \hbar \langle \operatorname{Tr} M^k \rangle$$

and $\Omega^- - \frac{1}{2}v_0 = \hbar\big\langle \mathrm{Tr}\,\log(z-M)\big\rangle$.

In order to identify the Lax operator, we follow [18; 15; 17]. Introduce the basis of orthogonal polynomials $P_n(z) = h_n z^n + \cdots$ $(n \geq 0)$, by the orthonormality relations

$$\langle m|n\rangle \equiv \int d^2z\, \overline{P_n(z)}\, e^{-\frac{1}{\hbar}V(z,\bar z)}\; P_m(z) = \delta_{m,n}.$$

The polynomials are uniquely defined by the potential V up to phase factors. It is easy to see that the τ-function is given by the product of the coefficients $N!\,|h_n h_{n-1}\dots h_0|^2$ of the highest powers of the polynomials $P_n(z) = h_n z^n + \cdots$. Then Lax operators L and $\bar L$ appear as the operators $\langle m|z|n\rangle$ and $\langle m|\bar z|n\rangle$. Since $zP_n(z)$ can be expressed through polynomials of the degree not grater than n, one may represent $\langle m|z|n\rangle$ and $\langle m|\bar z|n\rangle$ in terms of shifts operators $W = e^{\hbar\frac{\partial}{\partial t_0}}$ in the form of (8–1), (8–2), where $r(t_0 = \hbar n) = h_n/h_{n+1}$.

Similar arguments allow one to identify the flows. Consider a variation of some operator $\langle m|O|n\rangle$ under a variation of t_k. We have $\hbar\partial_{t_k}\langle m|O|n\rangle = \langle m|[H_k, O]|n\rangle$, where $H_k = A_k - A^{\dagger}{}_k$ and $\langle m|A_k|n\rangle = \langle m|\partial_{t_k}|n\rangle$. Obviously $H_k = -L^k(W)$ + negative powers of W. Choosing O to be $\bar L$ (see (8–2)) which consists on W^{-1} and positive powers of W, one concludes that H_k does not consists of negative powers of W. This brings us to (8–5).

Finally, the operator $D = \langle m|\hbar\partial_z|n\rangle$ is equal to

$$D = \bar L - \sum_{k\geq 1} k t_k L^{k-1}.$$

The Heisenberg relation $[D, L] = \hbar$ prompts the string equation (8–6).

The matrix model also offers an effective method to derive equations (6–1)–(6–7); see [17], for example.

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