

Vector Bundles and Lax Equations on Algebraic Curves

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Abstract: The Hamiltonian theory of zero-curvature equations with spectral parameter on an arbitrary compact Riemann surface is constructed. It is shown that the equations can be seen as commuting flows of an infinite-dimensional field generalization of the Hitchin system. The field analog of the elliptic Calogero-Moser system is proposed. An explicit parameterization of Hitchin system based on the Tyurin parameters for stable holomorphic vector bundles on algebraic curves is obtained.

1. Introduction

The main goal of this paper is to construct a Hamiltonian theory of zero curvature equations on an algebraic curve introduced in [1], and identify them as infinite-dimensional field analogs of the Hitchin system [2].

The zero curvature equation

$$\partial_t L - \partial_x M + [L, M] = 0, \quad (1.1)$$

where $L(x, t, \lambda)$ and $M(x, t, \lambda)$ are *rational* matrix functions of a *spectral* parameter λ

$$L = u_0(x, t) + \sum_{i,s} u_{is}(x, t)(\lambda - \lambda_i)^{-s}, \quad M = v_0(x, t) + \sum_{j,k} v_{jk}(x, t)(\lambda - \mu_j)^{-k}, \quad (1.2)$$

of degree n and m , respectively, was proposed in [4] as one of the most general type of representation for integrable systems. Equation (1.1), which has to be valid identically in λ , is equivalent to a system of $(n + m + 1)$ matrix equations for the unknown functions

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u_0, v_0, u_{is}, v_{jk} . The number of the equations is less than the number of unknown functions. That is due to a gauge symmetry of (1.1). If $g(x, t)$ is an arbitrary matrix function then the transformation

$$L \mapsto g_x g^{-1} + g L g^{-1}, \quad M \mapsto g_t g^{-1} + g M g^{-1} \tag{1.3}$$

maps solutions of (1.1) into solutions of the same equations. The gauge transformation can be used to normalize L and M . For example, in the gauge $u_0 = v_0 = 0$ the numbers of equations and unknown functions are equal. Hence, Eq. (1.1) is well-defined.

The Riemann-Roch theorem shows that the naive direct generalization of the zero curvature equation for matrix functions that are meromorphic on an algebraic curve of genus $g > 0$ leads to an over-determined system of equations. Indeed, the dimension of $(r \times r)$ matrix functions with fixed degree d divisor of poles in general position equals $r^2(d - g + 1)$. If divisors of L and M have degrees n and m , then the commutator $[L, M]$ is of degree $n + m$. Therefore, the number of equations $r^2(n + m - g + 1)$ is bigger than the number $r^2(n + m - 2g + 1)$ of unknown functions modulo gauge equivalence.

There are two ways to overcome the difficulty in defining the zero curvature equations on algebraic curves. The first one is based on a choice of special ansatz for L and M . In this way a few integrable systems were found with Lax matrices that are elliptic functions of the spectral parameter. The second possibility, based on a theory of *high rank* solutions of the KP equation [3], was discovered in [1]. It was shown that if in addition to fixed poles the matrix functions L and M have moving rg poles with special dependence on x and t , then Eq. (1.1) is a well-defined system on the space of singular parts of L and M at fixed poles. Recently, an algebraic construction of the zero curvature equations on an algebraic curve was proposed in [5].

If matrix functions L and M do not depend on x , then (1.1) reduces to the Lax equation

$$\partial_t L = [M, L]. \tag{1.4}$$

A theory of the Lax equations on an algebraic curve was briefly outlined in [1]. In the next section for each effective degree $N > g$ divisor D on a smooth genus g algebraic curve Γ we introduce a space \mathcal{L}^D of the Lax matrices, and define a hierarchy of commuting flows on it. The spaces of the Lax matrices associated to equivalent divisors are isomorphic. If $D = \mathcal{K}$ is the divisor of zeros of a holomorphic differential, then the space $\mathcal{L}^{\mathcal{K}}$ is identified with an open set of the cotangent bundle $T^*(\widehat{\mathcal{M}})$ of the moduli space of semistable holomorphic vector bundles on Γ , i.e. with an open set of the phase space of the Hitchin system. The commuting hierarchy of the Lax equations on $\mathcal{L}^{\mathcal{K}}$ are commuting flows of the Hitchin system.

The conventional approach to a theory of the Hitchin system is based on a representation of $T^*(\widehat{\mathcal{M}})$ as the Hamiltonian reduction of free infinite-dimensional system modulo infinite-dimensional gauge group. In the finite-gap or algebro-geometric theory of soliton equations involutivity of the integrals of motion does not come for granted, as in the case of the Hamiltonian reduction. Instead, the commutativity of the hierarchy of the Lax equations is a starting point. It implies involutivity of the integrals, whenever the equations are Hamiltonian.

The Lax matrices provide an explicit parameterization of the Hitchin system based on *Tyurin parameters* for framed stable holomorphic bundles on an algebraic curve [6]. Let V be a stable, rank r , and degree rg holomorphic vector bundle on Γ . Then the dimension of the space of its holomorphic sections is $r = \dim H^0(\Gamma, V)$. Let $\sigma_1, \dots, \sigma_r$ be a basis

of this space. The vectors $\sigma_i(\gamma)$ are linear independent at the fiber of V over a generic point $\gamma \in \Gamma$, and are linearly dependent

$$\sum_{i=1}^r \alpha_s^i \sigma_i(\gamma_s) = 0 \tag{1.5}$$

at zeros γ_s of the corresponding section of the determinant bundle associated to V . For a generic V these zeros are simple, i.e. the number of distinct points γ_s is equal to $rg = \deg V$, and the vectors $\alpha_s = (\alpha_s^i)$ of linear dependence (1.5) are uniquely defined up to a multiplication. A change of the basis $\tilde{\sigma}_i = \sum_j g_{ij} \sigma_j$ corresponds to the linear transformation of the vectors $\alpha_s, \tilde{\alpha}_s = g^T \alpha_s$. Hence, an open set $\mathcal{M} \subset \widehat{\mathcal{M}}$ of the moduli space of vector bundles is parameterized by points of the factor-space

$$\mathcal{M} = \mathcal{M}_0 / SL_r, \quad \mathcal{M}_0 \subset S^{rg} \left(\Gamma \times CP^{r-1} \right), \tag{1.6}$$

where SL_r acts diagonally on the symmetric power of CP^{r-1} . In [1, 7] the parameters (γ_s, α_s) were called Tyurin parameters. Recently, the Tyurin parameterization of the Hitchin system for $r = 2$ was found [8].

In Sect. 3 we show that the standard scheme to solve conventional Lax equations using the concept of the Baker-Akhiezer function is evenly applicable to the case of Lax equations on algebraic curves. We would like to emphasize that solution of the Lax equations via the spectral transform of the phase space to algebraic-geometric data does not use a Hamiltonian description of the system. Moreover, *a priori* it's not clear, why the Lax equations are Hamiltonian. In Sect. 4 we clarify this problem using the approach to the Hamiltonian theory of soliton equations proposed in [9–11]. It turns out that for $D = \mathcal{K}$ the universal two-form which is expressed in terms of the Lax matrix and its eigenvectors coincides with canonical symplectic structure on the cotangent bundle $T^*(\mathcal{M})$. If the divisor $D_{\mathcal{K}} = D - \mathcal{K}$ is effective, then the form is non-degenerate on symplectic leaves defined by a choice of the orbits of the adjoint action of SL_r on the singular parts of $L \in \mathcal{L}^D$ at the punctures $P_m \in D_{\mathcal{K}}$.

In Sect. 5 for each degree $N > g$ divisor D on Γ a commuting hierarchy of zero curvature equations is defined. The infinite-dimensional phase space \mathcal{A}^D of the hierarchy can be seen as a space of connections $\partial_x - L(x, q)$ along loops in \mathcal{M}_0 . We would like to emphasize that \mathcal{A}^D does depend on the divisor D and not simply on its equivalence class, as in the case of the Lax equations. If $D_{\mathcal{K}}$ is effective, then the equations of the hierarchy are Hamiltonian after restriction on symplectic leaves.

The Riemann surface of the Bloch solutions of the equation

$$(\partial_x - L(x, q))\psi(x, q) = 0, \quad x \in S^1, \quad q \in \Gamma \tag{1.7}$$

is an analog of the spectral curves in the x -independent case. Algebro-geometric solutions of the hierarchy are constructed in the last section. Note that they can be constructed in all the cases independently of whether the equations are Hamiltonian or not.

It is instructive to present two examples of the zero curvature equations. The first one is a field analog of the elliptic Calogero–Moser system. The elliptic CM system is a system of r particles with coordinates q_i on an elliptic curve with the Hamiltonian

$$H = \frac{1}{2} \left(\sum_i p_i^2 + \sum_{i \neq j} \wp(q_i - q_j) \right), \tag{1.8}$$

where $\wp(q)$ is the Weierstrass function. In [12] the elliptic CM system was identified with a particular case of the Hitchin system on an elliptic curve with a puncture. In Sect. 5 we show that the zero curvature equation on an elliptic curve with a puncture is equivalent to the Hamiltonian system which can be seen as the field analog of the elliptic CM system. For $r = 2$ this system is equivalent to the system on a space of periodic functions $p(x), q(x)$ with canonical Poisson brackets

$$\{p(x), q(y)\} = \delta(x - y). \tag{1.9}$$

The Hamiltonian is

$$H = \int \left(p^2 (1 - q_x^2) - \frac{q_{xx}^2}{2(1 - q_x^2)} + 2(1 - 3q_x^2)\wp(2q) \right) dx. \tag{1.10}$$

The second example is the Krichever-Novikov equation [3]

$$q_t = \frac{1}{4}q_{xxx} + \frac{3}{8q_x}(1 - q_{xx}^2) - \frac{1}{2}Q(q)q_x^2, \tag{1.11}$$

where

$$Q(q) = \partial_q \Phi + \Phi^2, \quad \Phi = \Phi(q, y) = \zeta(q - y) + \zeta(q + y) - \zeta(2q). \tag{1.12}$$

Note that $Q(q)$ does not depend on y . Each solution $q = q(x, t)$ of (1.11) defines a rank 2, genus 1 solution of the KP equation by the formula

$$8u(x, y, t) = \left(q_{xx}^2 - 1 \right) q_x^{-2} - 2q_{xxx}q_x^{-1} + 8q_{xx}\Phi + 4q_x^2 (\partial_q \Phi - \Phi). \tag{1.13}$$

Equation (1.12) has zero curvature representation on the elliptic curve with puncture with $r = 2$. The difference between the two examples is in the choice of orbits at the puncture. In the first example the orbit is that of the diagonal matrix $\text{diag}(1, -1)$, while the second example corresponds to the orbit of the Jordan cell.

2. The Lax Equations

We define first the space of Lax matrices associated with a generic effective divisor D on Γ , and a point $(\gamma, \alpha) = \{\gamma_s, \alpha_s\}$ of the symmetric product $X = S^r g (\Gamma \times CP^{r-1})$. Throughout the paper it is assumed that the points $\gamma_s \in \Gamma$ are distinct, $\gamma_s \neq \gamma_k$.

Let $\mathcal{F}_{\gamma, \alpha}$ be the space of meromorphic vector functions f on Γ , that are holomorphic except at the points γ_s , at which they have a simple pole of the form

$$f(z) = \frac{\lambda_s \alpha_s}{z - z(\gamma_s)} + O(1), \quad \lambda_s \in C. \tag{2.1}$$

The Riemann-Roch theorem implies that

$$\dim \mathcal{F}_{\gamma, \alpha} \geq r(rg - g + 1) - rg(r - 1) = r. \tag{2.2}$$

The first term in (2.2) is the dimension of the space of meromorphic vector-functions with simple poles at γ_s . The second term is the number of equations equivalent to the constraint that poles of f are proportional to the vectors α_s .

The space \mathcal{F}_s of meromorphic functions in the neighborhood of γ_s that have a simple pole at γ_s of the form (2.1) is the space of local sections of the vector bundle $V_{\gamma, \alpha}$

corresponding to (γ, α) under the inverse to the Tyurin map described in terms of Hecke modification of the trivial bundle. The space of global holomorphic sections of $V_{\gamma, \alpha}$ is just the space $\mathcal{F}_{\gamma, \alpha}$. Let \mathcal{M}'_0 be an open set of the parameters (γ, α) such that $\dim \mathcal{F}_{\gamma, \alpha} = r$.

Let $D = \sum_i m_i P_i$ be an effective divisor on Γ that does not intersect with γ . Then we define a space $\mathcal{N}_{\gamma, \alpha}^D$ of meromorphic matrix functions $M = M(q)$, $q \in \Gamma$, such that:

1⁰. M is holomorphic except at the points γ_s , where it has at most simple poles, and at the points P_i of D , where it has poles of degree not greater than m_i ;

2⁰. The coefficient M_{s0} of the Laurent expansion of M at γ_s

$$M = \frac{M_{s0}}{z - z_s} + M_{s1} + M_{s2}(z - z_s) + O((z - z_s)^2), \quad z_s = z(\gamma_s), \quad (2.3)$$

is a rank 1 matrix of the form

$$M_{s0} = \mu_s \alpha_s^T \iff M_{s0}^{ij} = \mu_s^i \alpha_s^j, \quad (2.4)$$

where μ_s is a vector. The constraint (2.4) does not depend on a choice of local coordinate z in the neighborhood of γ_s .

If $(\gamma, \alpha) \in \mathcal{M}'_0$, then the constraints (2.4) are linear independent and

$$\dim \mathcal{N}_{\gamma, \alpha}^D = r^2(N + rg - g + 1) - r^2g(r - 1) = r^2(N + 1), \quad N = \deg D. \quad (2.5)$$

Central to all our further constructions is a map

$$\mathcal{D} : \mathcal{N}_{\gamma, \alpha}^D \mapsto T_{\gamma, \alpha}(\mathcal{M}'_0) \quad (2.6)$$

from $\mathcal{N}_{\gamma, \alpha}^D$ to the tangent space to \mathcal{M}'_0 at the point (γ, α) . The tangent vector $\partial_m = \mathcal{D}(M)$ is defined by derivatives of the coordinates

$$\partial_m z_s = -\text{tr } M_{s0} = -\alpha_s^T \mu_s, \quad z_s = z(\gamma_s), \quad (2.7)$$

$$\partial_m \alpha_s^T = -\alpha_s^T M_{s1} + \kappa_s \alpha_s^T, \quad (2.8)$$

where κ_s is a scalar. The tangent space to CP^{r-1} at a point represented by the vector α_s is a space of r -dimensional vectors v modulo equivalence $v' = v + \kappa_s \alpha_s$. Therefore, the right hand side of (2.8) is a well-defined tangent vector to CP^{r-1} .

Simple dimension counting shows that on an open set of \mathcal{M}'_0 the linear map \mathcal{D} is an injection for $N < g - 1$, and is an isomorphism for $N = g - 1$. Let us define the space $\mathcal{L}_{\gamma, \alpha}^D$ of the Lax matrices as the kernel of \mathcal{D} . In other words: a matrix function $L(q) \in \mathcal{N}_{\gamma, \alpha}^D$ is a Lax matrix if

(i) the singular term of the expansion

$$L = \frac{L_{s0}}{z - z_s} + L_{s1} + L_{s2}(z - z_s) + O((z - z_s)^2), \quad L_{s0} = \beta_s \alpha_s^T, \quad z_s = z(\gamma_s), \quad (2.9)$$

is traceless

$$\alpha_s^T \beta_s = \text{tr } L_{s0} = 0; \quad (2.10)$$

(ii) α_s^T is a left eigenvector of the matrix L_{s1}

$$\alpha_s^T L_{s1} = \alpha_s^T \kappa_s. \quad (2.11)$$

For a non-special degree $N \geq g$ divisor D and a generic set of the parameters (γ, α) , the space $\mathcal{L}_{\gamma, \alpha}^D$ is of dimension

$$\dim \mathcal{L}_{\gamma, \alpha}^D = r^2(N + 1) - rg - rg(r - 1) = r^2(N - g + 1). \tag{2.12}$$

A key characterization of constraints (2.9)–(2.11) is as follows.

Lemma 2.1. *A meromorphic matrix-function L in the neighborhood U of γ_s with a pole at γ_s satisfies the constraints (2.10) and (2.11) if and only if it has the form*

$$L = \Phi_s(z) \widehat{L}_s(z) \Phi_s^{-1}(z), \tag{2.13}$$

where \widehat{L}_s and Φ_s are holomorphic in U , and $\det \Phi_s$ has at most simple zero at γ_s .

Proof. Let g_s be a constant non-degenerate matrix such that

$$\alpha_s^T g_s = e_1^T, \quad e_1^T = (1, 0, 0, \dots, 0). \tag{2.14}$$

If L satisfies (2.9, 2.10), then, the coefficient L'_{s0} of the Laurent expansion at γ_s of the gauge equivalent Lax matrix

$$L'_s = g_s^{-1} L g_s = \frac{L'_{s0}}{z - z_s} + L'_{s1} + O(z - z_s), \quad z_s = z(\gamma_s), \tag{2.15}$$

equals $f e_1^T$, where $f = g_s^{-1} \beta_s$. Therefore, it has non-zero entries at the first column, only,

$$(L'_{s0})^{i,j} = 0, \quad j = 2, \dots, r. \tag{2.16}$$

Further, the vector e_1^T is a left eigenvector for L'_{s1} corresponding to the eigenvalue κ_s . Hence, the first row of L'_{s1} equals

$$(L'_{s1})^{1j} = \kappa_s, \quad (L'_{s1})^{1j} = 0, \quad j = 2, \dots, r. \tag{2.17}$$

From (2.16), (2.17) it follows that the matrix $\widehat{L}_s = f_s^{-1} L'_s f_s$, where f_s is the diagonal matrix

$$f_s(z) = \text{diag}\{(z - z_s), 1, 1, \dots, 1\}, \tag{2.18}$$

is regular at γ_s . Hence, the Lax matrix L has the form (2.13), where

$$\Phi_s = g_s f_s(z). \tag{2.19}$$

Conversely suppose L has the form (2.13), and let α_s be the unique (up to multiplication) vector such that $\alpha_s^T \Phi_s(z_s) = 0$. Then the Laurent expansion of L at γ_s has the form (2.9). The trace of L is holomorphic, which implies (2.10). Using the equality $\alpha_s^T \Phi_s(z_s) = 0$ we obtain that $\alpha_s^T L$ is holomorphic at γ_s and its evaluation at this point is proportional to α_s^T . This implies (2.11) and the lemma is proved.

Let $[D]$ be the equivalence class of a degree $N > g$ divisor D . Then for any set (γ, α) there is a divisor D' equivalent to D that does not intersect with γ . Constraints (2.10) and (2.11) are invariant under the transformation $L \rightarrow hL$, where h is a function holomorphic in the neighborhood of γ_s . Therefore, the spaces $\mathcal{L}_{\gamma,\alpha}^D$ and $\mathcal{L}_{\gamma,\alpha}^{D'}$ of Lax matrices corresponding to equivalent divisors D and D' are isomorphic. They can be regarded as charts of a total space $\mathcal{L}^{[D]}$, the Lax matrices corresponding to $[D]$.

Let us consider in greater detail the case $D = \mathcal{K}$, where \mathcal{K} is the zero divisor of a holomorphic differential dz . Then Ldz , where $L \in \mathcal{L}_{\gamma,\alpha}^{\mathcal{K}}$, is a matrix valued one-form that is holomorphic everywhere except at the points γ_s . The constraints (2.10, 2.11) imply that the space \mathcal{F}_s of local sections of $V_{\gamma,\alpha}$ is invariant under the adjoint action of L ,

$$f \in \mathcal{F}_s \longmapsto L^T(z)f(z) \in \mathcal{F}_s. \tag{2.20}$$

Therefore, the gauge equivalence class of the matrix valued differential Ldz can be seen as a global section of the bundle $End(V_{\gamma,\alpha}) \otimes \Omega^{1,0}(\Gamma)$. It is basic in the Hitchin system theory, that the space of such sections, called Higgs fields, is identified with the cotangent bundle $T^*(\widehat{\mathcal{M}})$.

It is instructive to establish directly the equivalence

$$\mathcal{L}^{\mathcal{K}}/SL_r = T^*(\mathcal{M}), \tag{2.21}$$

using the map (2.6). The formula

$$\langle L, M \rangle = - \sum_s \text{res}_{\gamma_s} \text{Tr} (LM) dz \tag{2.22}$$

defines a natural pairing between $\mathcal{L}_{\gamma,\alpha}^{\mathcal{K}}$ and $\mathcal{N}_{\gamma,\alpha}^D$. For a generic degree $(g - 1)$ divisor D the map (2.6) is an isomorphism. Therefore each tangent vector $w = (\dot{z}_s, \dot{\alpha}_s)$ to \mathcal{M}'_0 at the point $(z_s = z(\gamma_s), \alpha_s)$ can be represented in the form $\mathcal{D}(M)$. From (2.7, 2.8) it follows that (2.22) actually defines a pairing between $\mathcal{L}_{\gamma,\alpha}^{\mathcal{K}}$ and the tangent space $T_{\gamma,\alpha}(\mathcal{M}_0)$,

$$\langle L, w \rangle = \sum_s (\kappa_s \dot{z}_s + \dot{\alpha}_s^T \beta_s). \tag{2.23}$$

This formula shows that the vector β_s and the eigenvalue κ_s in (2.10, 2.11) can be regarded as coordinates of a cotangent vector to \mathcal{M}'_0 . Note that κ_s under the change of dz to another holomorphic differential dz_1 get transformed to $\kappa'_s = \kappa_s dz/dz_1$. Therefore, the pair (γ_s, κ_s) can be seen as a point of the cotangent bundle $T^*(\Gamma)$ to the curve Γ .

The pairing (2.23) descends to pairing of $\mathcal{L}^{\mathcal{K}}/SL_r$ with tangent vectors to \mathcal{M} . Indeed, tangent vectors to \mathcal{M} at a point represented by gauge equivalence class of α are identified with vectors $\dot{\alpha}_s$ modulo transformation $\dot{\alpha}_s^T \rightarrow \dot{\alpha}_s^T + \alpha_s^T W$, where W is a matrix. Under this transformation the right hand side of (2.23) does not change due to the equation

$$\sum_{s=1}^{rg} \beta_s \alpha_s^T = \sum_s \text{res}_{\gamma_s} Ldz = 0, \tag{2.24}$$

which is valid, because Ldz is holomorphic except at γ .

The induced pairing of $\mathcal{L}^{\mathcal{K}}/SL_r$ with $T(\mathcal{M})$ is non-degenerate. Indeed, if $w = \mathcal{D}(M)$, then (2.22) implies that

$$\langle L, w \rangle = \langle L, M \rangle = \sum_i \operatorname{res}_{P_i} \operatorname{Tr}(LM) dz. \tag{2.25}$$

Therefore, if (2.23) is degenerate then there is a nontrivial L which has zero of order m_i at all the points P_i of D . That is impossible because D is a generic degree $(g - 1)$ divisor.

Our next goal is to introduce an explicit parameterization of $\mathcal{L}^{\mathcal{K}}$. Recall that we always assume $(\gamma, \alpha) \in \mathcal{M}'_0$.

Lemma 2.2. *The map*

$$L \in \mathcal{L}^{\mathcal{K}} \longmapsto \{\alpha_s, \beta_s, \gamma_s, \kappa_s\}, \tag{2.26}$$

where pairs of orthogonal vectors $(\alpha_s^T \beta_s) = 0$ are considered modulo gauge transformations

$$\alpha_s \rightarrow \lambda_s \alpha_s, \quad \beta_s \rightarrow \lambda_s^{-1} \beta_s, \tag{2.27}$$

and satisfy Eq. (2.24), is one-to-one correspondence.

Proof. Suppose that images of L and L' under (2.26) coincide, then $(L - L')dz$ is a holomorphic matrix valued differential φ such that

$$\alpha_s^T \varphi(\gamma_s) = 0. \tag{2.28}$$

Let $\mathcal{F}_{\gamma, \alpha}^P$ be the space of meromorphic vector functions with poles at γ_s of the form (2.1) and with simple pole at a point $P \in \Gamma$. By the definition of \mathcal{M}'_0 , the constraints (2.1) are linearly independent. Therefore, $\mathcal{F}_{\gamma, \alpha}^P$ has dimension $2r$, and the vectors of singular part of $f \in \mathcal{F}_{\gamma, \alpha}^P$ at P span the whole space \mathbf{C}^r . From (2.28) it follows that if $f \in \mathcal{F}_{\gamma, \alpha}^P$, then the differential $f^T \varphi$ has no poles at γ_s . As the sum of all the residues of a meromorphic differential equals zero, then the $f^T \varphi$ is regular at P . That implies $\varphi(P) = 0$. The point P is arbitrary, therefore (2.26) is an injection.

The map (2.26) is linear on fibers over (γ, α) . Therefore, in order to complete a proof of the lemma, it is enough to show that dimension of $\mathcal{L}_{\gamma, \alpha}^{\mathcal{K}}$ is greater than or equal to the dimension d of the corresponding data (β_s, κ_s) . The vectors β_s are orthogonal to α_s . Therefore, d equals r^2g minus the rank of the system of equations (2.24).

Let us show that if $(\gamma, \alpha) \in \mathcal{M}'_0$, then the vectors α_s span \mathbf{C}^r . Suppose that they span an l -dimensional subspace, then by a gauge transformation we can reduce the problem to the case when the vectors α_s have the $(r - l)$ vanishing coordinates, $\alpha_s^i = 0, \quad i > l$. The Riemann–Roch theorem then implies that the dimension of the corresponding space $\mathcal{F}_{\gamma, \alpha}$ is not less than $l(rg - g + 1) - rg(l - 1) + (r - l) = (r - l)g + r$.

If the rank of α_s^i is r , then Eqs. (2.24) are linearly independent by themselves, but one of them is already satisfied due to the orthogonality condition for β_s , which implies $\operatorname{Tr}(\beta_s \alpha_s^T) = 0$. Therefore the dimension of the fiber of data (2.26) over $(\gamma, \alpha) \in \mathcal{M}'_0$ equals $r^2(g - 1) + 1$.

On the other hand, for $L \in \mathcal{L}_{\gamma, \alpha}^{\mathcal{K}}$ among constraints (2.10) there are at most $(rg - 1)$ linearly independent, because a meromorphic differential can not have a single simple pole. Hence, dimension counting as in (2.5) implies $\dim \mathcal{L}_{\gamma, \alpha}^{\mathcal{K}} \geq r^2(g - 1) + 1$ and the lemma is proved.

Example. Let Γ be a hyperelliptic curve defined by the equation

$$y^2 = R(x) = x^{2g+1} + \sum_{i=0}^{2g} u_i x^i. \tag{2.29}$$

A set of points γ_s on Γ is a set of pairs (y_s, x_s) , such that

$$y_s^2 = R(x_s). \tag{2.30}$$

A meromorphic differential on Γ with residues $(\beta_s \alpha_s^T)$ at γ_s has the form

$$L \frac{dx}{2y} = \left(\sum_{i=0}^{g-1} L_i x^i + \sum_{s=1}^{rg} (\beta_s \alpha_s^T) \frac{y + y_s}{x - x_s} \right) \frac{dx}{2y}, \tag{2.31}$$

where L_i is a set of arbitrary matrices. The constraints (2.11) are a system of linear equations defining L_i :

$$\sum_{i=0}^g \alpha_n^T L_i x_k^i + \sum_{s \neq n} (\alpha_n^T \beta_s) \alpha_s^T \frac{y_n + y_s}{x_n - x_s} = \kappa_n \alpha_n^T, \quad n = 1, \dots, rg \tag{2.32}$$

in terms of data (2.26). In a similar way the Lax matrices can be explicitly written for any algebraic curve using the Riemann theta-functions.

For $g > 1$, the correspondence (2.26) descends to a system of local coordinates on \mathcal{L}^K/SL_r over an open set \mathcal{M}_0 of \mathcal{M}'_0 , which we define as follows.

As shown above, for $(\gamma, \alpha) \in \mathcal{M}'_0$ the matrix α_s^i is of rank r . We call (γ, α) a non-special set of the Tyurin parameters if additionally they satisfy the constraint: there is a subset of $(r + 1)$ indices s_1, \dots, s_{r+1} such that all minors of $(r + 1) \times r$ matrix $\alpha_{s_j}^i$ are non-degenerate. The action of the gauge group on the space of non-special sets of the Tyurin parameters \mathcal{M}_0 is free.

Let us define charts of coordinates on a smooth bundle of equivalence classes of Lax matrices over \mathcal{M}_0 . Consider the open set of \mathcal{M}_0 such that the vectors $\alpha_j, j = 1, \dots, r$, are linearly independent and all the coefficients of an expansion of α_{r+1} in this basis do not vanish

$$\alpha_{r+1} = \sum_{s=1}^r c_s \alpha_s, \quad c_s \neq 0. \tag{2.33}$$

Then for each point of this open set there exists a unique matrix $W \in GL_r$, such that $\alpha_j^T W$ is proportional to the basis vector e_j with the coordinates $e_j^i = \delta_j^i$, and $\alpha_{r+1}^T W$ is proportional to the vector $e_0 = \sum_j e_j$. Using the global gauge transformation defined by W ,

$$B_s = W^{-1} \beta_s, \quad A_s = W^T \alpha_s, \tag{2.34}$$

and the part of local transformations

$$A_s \rightarrow \lambda_s A_s; \quad B_s \rightarrow \lambda_s^{-1} B_s, \tag{2.35}$$

for $s = 1, \dots, r + 1$, we obtain that on the open set of \mathcal{M}_0 each equivalence class has representation of the form (A_s, B_s) such that

$$A_i = e_i, \quad i = 1, \dots, r; \quad A_{r+1} = e_0. \tag{2.36}$$

This representation is unique up to local transformations (2.35) for $s = r + 2, \dots, rg$.

In the gauge (2.36) Eq. (2.24) can be easily solved for B_1, \dots, B_{r+1} . Using (2.36), we get

$$B_j^i + B_{r+1}^i = - \sum_{s=r+2}^{rg} B_s^i A_s^j. \tag{2.37}$$

The orthogonality condition of B_j to $A_j = e_j$ implies that $B_j^j = 0$. Hence,

$$B_{r+1}^i = - \sum_{s=r+2}^{rg} B_s^i A_s^i. \tag{2.38}$$

The sets of $r(g - 1) + 1$ pairs of orthogonal vectors A_s, B_s modulo the transformations (2.35), and points $\{\gamma_s, \kappa_s\} \in S^{rg}(T^*(\Gamma))$ provide a parameterization of an open set of $T^*(\mathcal{M})$. Here and below $\mathcal{M} = \mathcal{M}_0/SL_r$.

In the same way, taking various subsets of $(r + 1)$ indices we obtain charts of local coordinates which cover $T^*(\mathcal{M})$. In Sect. 4 we provide a similar explicit parameterization of \mathcal{L}^D for divisors D such that $D_{\mathcal{K}} = D - \mathcal{K}$ is an effective divisor.

Our next goal is to construct a hierarchy of commuting flows on a total space \mathcal{L}^D of a vector bundle over an open set of \mathcal{M}_0 . Let us identify the tangent space $T_L(\mathcal{L}^D)$ to \mathcal{L}^D at the point L with the space of meromorphic matrix functions spanned by derivatives $\partial_\tau L|_{\tau=0}$ of all one-parametric deformations $L(q, \tau) \in \mathcal{L}^D$ of L .

Lemma 2.3. *The commutator $[M, L]$ of matrix functions $L \in \mathcal{L}_{\gamma, \alpha}^D$ and $M \in \mathcal{N}_{\gamma, \alpha}^{D'}$ is a tangent vector to \mathcal{L}^D at L if and only if its divisor of poles outside the points γ_s is not greater than D .*

Proof. First of all, let us show that the tangent space $T_L(\mathcal{L}^D)$ can be identified with a space of matrix functions T on Γ with poles of order not greater than m_i at P_i , and double poles at the points γ_s , where they have an expansion of the form

$$T = \dot{z}_s \frac{\beta_s \alpha_s^T}{(z - z_s)^2} + \frac{\dot{\beta}_s \alpha_s^T + \beta_s \dot{\alpha}_s^T}{z - z_s} + T_{s1} + O(z - z_s). \tag{2.39}$$

Here \dot{z}_s is a constant, and $\dot{\alpha}_s, \dot{\beta}_s$ are vectors that satisfy the constraint

$$\alpha_s^T \dot{\beta}_s + \dot{\alpha}_s^T \beta_s = 0. \tag{2.40}$$

The vectors α_s, β_s are defined by L . In addition it is required that the following equation holds:

$$\alpha_s^T T_{s1} = \dot{\alpha}_s \kappa_s + \alpha_s \dot{\kappa}_s - \dot{\alpha}_s^T L_{s1} - \dot{z}_s \alpha_s^T L_{s2}, \tag{2.41}$$

where L_{s1}, L_{s2} and κ_s are defined by (2.9, 2.11), and $\dot{\kappa}_s$ is a constant.

Equations (2.40) and (2.41) can be easily checked for a tangent vector $\partial_\tau L|_{\tau=0}$, if we identify $(\dot{z}_s, \dot{\alpha}_s, \dot{\beta}_s)$ with

$$\dot{z}_s = \partial_\tau z(\gamma_s(\tau))|_{\tau=0}, \quad \dot{\alpha}_s = \partial_\tau \alpha_s(\tau)|_{\tau=0}, \quad \dot{\beta}_s = \partial_\tau \beta_s(\tau)|_{\tau=0} \tag{2.42}$$

and T_{s1} with

$$T_{s1} = (\partial_\tau L_{s1} - \dot{z}_s L_{s2})|_{\tau=0}. \tag{2.43}$$

Direct counting of a number of the constraints shows that the space of matrix functions that have poles of order m_i at P_i , and satisfy (2.39-2.41) equals $r^2(N + 1)$, which is the dimension of \mathcal{L}^D . Therefore, these relations are necessary and sufficient conditions for T to be a tangent vector.

From (2.10, 2.11) it follows that, if we define \dot{z}_s and $\dot{\alpha}_s$ with the help of formulae (2.7, 2.8), then the expansion of $[M, L]$ at γ_s satisfies the constraints (2.39-2.41). The lemma is thus proved.

The lemma directly implies that the Lax equation $L_t = [M, L]$ is a well-defined system on an open set of \mathcal{L}^D , whenever we can define $M = M(L)$ as a function of L that outside of the points γ_s commutes with L up to a meromorphic function with poles at the points P_i of order not greater than m_i .

Let us fix a point $P_0 \in \Gamma$ and local coordinates w in the neighborhoods of the punctures $P_0, P_i \in D$. Our next goal is to define gauge invariant functions $M_a(L)$ that satisfy the conditions of Lemma 2.3. They are parameterized by sets

$$a = (P_i, n, m), \quad \text{where } n > 0, m > -m_i \text{ are integers.} \tag{2.44}$$

As follows from (2.5), for generic $L \in \mathcal{L}_{\gamma, \alpha}^D$ there is a unique matrix function $M_a(q)$ such that:

- (i) it has the form (2.3,2.4) at the points γ_s ;
- (ii) outside of the divisor γ it has pole at the point P_i , only, where the singular part at M_a coincides with the singular part of $w^{-m}L^n$, i.e.

$$M_a^- = M_a(q) - w^{-m}L^n(q) = O(1) \text{ is regular at } P_i, \tag{2.45}$$

- (iii) M_a is normalized by the condition $M_a(P_0) = 0$.

Theorem 2.1. *The equations*

$$\partial_a L = [M_a, L], \quad \partial_a = \partial/\partial t_a \tag{2.46}$$

define a hierarchy of commuting flows on an open set of \mathcal{L}^D , which descends to the commuting hierarchy on an open set of \mathcal{L}^D/SL_r .

By definition, M_a only depends on L , i.e. $M_a = M_a(L)$. Equation (2.45) implies that $[M_a, L]$ satisfies the conditions of Lemma 2.3. Therefore, the right-hand side of (2.46) is a tangent vector to \mathcal{L}^D at the point L . Hence, (2.46) is a well-defined dynamical system on an open set of \mathcal{L}^D .

The Laurent expansion of (2.46) at γ_s shows that the projection $\pi_*(\partial_a) \in T(\mathcal{M}_0)$ of the vector $\partial_a \in T(\mathcal{L}^D)$ equals

$$\pi_*(\partial_a) = \mathcal{D}(M_a). \tag{2.47}$$

Now let us prove the second statement of the theorem. Commutativity of flows (2.46) is equivalent to the equation

$$\partial_a M_b - \partial_b M_a - [M_a, M_b] = 0. \tag{2.48}$$

The left-hand side of (2.48) equals zero at P_0 , and, as follows from (2.47) its expansion at γ_s satisfies (2.39-2.41). Therefore, it equals zero identically, if it is regular at D . This easily follows from standard arguments used in KP theory. If indices a and b correspond to the same point P_i , i.e. $a = (P_i, n, m)$, $b = (P_i, n_1, m_1)$, then in the neighborhood of P_i we have

$$\begin{aligned} \partial_a M_b &= w^{-m_1} \partial_a L^{n_1} + \partial_a M_b^- \\ &= w^{-m_1} [M_a, L^{n_1}] + \partial_a M_b^- = w^{-m_1} [M_a^-, L^{n_1}] + \partial_a M_b^-, \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} [M_a, M_b] &= [w^{-m} L^n + M_a^-, w^{-m_1} L^{n_1} + M_b^-] \\ &= w^{-m} [L^n, M_b^-] - w^{-m_1} [L^{n_1}, M_a^-] + 0(1) \end{aligned} \tag{2.50}$$

From (2.49, 2.50) it follows that the left-hand side of (2.48) is regular at P_i . From the definition of M_a , it is regular at all the other points of D as well. In a similar way we prove (2.48) for indices $a = (P_i, n, m)$, $b = (P_j, n', m')$ for $P_i \neq P_j$.

Let us now define an extended hierarchy of commuting flows on generic fibers of the evaluation map $\mathcal{L}_{\gamma, \alpha}^D \rightarrow L(P_0) = L_0$. Note that these fibers are invariant with respect to (2.46). Additional flows are parameterized by indices

$$a = (P_0, m; l), \quad m > 0, \quad l = 1, \dots, r. \tag{2.51}$$

Let L_0 be a matrix with distinct eigenvalues, and let us fix a representation of L_0 in the form $\Psi_0 K_0 \Psi_0^{-1}$, where K_0 is a diagonal matrix. Then for each $L \in \mathcal{L}_{\gamma, \alpha}^D$, such that $L(q) = L_0$, there exists a unique holomorphic matrix function Ψ , $\Psi(q) = \Psi_0$, which diagonalizes L in the neighborhood of q , i.e. $L = \Psi K \Psi^{-1}$. For each index a of the form (2.51) we define M_a as the unique matrix $M_a \in \mathcal{N}_{\gamma, \alpha}^{n, P_0}$ that in the neighborhood of P_0 has the form

$$M_a = w^{-m} \Psi(w) E_l \Psi^{-1}(w) + O(w), \tag{2.52}$$

where E_l is the diagonal matrix $E_l^{ij} = \delta^{il} \delta^{jl}$.

Theorem 2.2. *The equations*

$$\partial_a L = [M_a, L], \quad a = (P_0, m; l) \tag{2.53}$$

defines commuting flows on the fiber of the evaluation map $\mathcal{L}^D \rightarrow L_0$. The flows (2.53) commute with flows (2.46).

The proof is almost identical to that of the previous theorem.

3. The Baker-Akhiezer Functions

In this section we show that standard procedure in the algebro-geometric theory of soliton equations to solve conventional Lax equations using the concept of the Baker–Akhiezer functions ([13, 14]) is evenly applicable to the case of Lax equations on algebraic curves.

Let $L \in \mathcal{L}^D$ be a Lax matrix. The characteristic equation

$$R(k, q) \equiv \det(k - L(q)) = k^r + \sum_{j=1}^r r_j(q)k^{r-j} = 0 \tag{3.1}$$

defines a *time-independent* algebraic curve $\widehat{\Gamma}$, which is an r -fold branch cover of Γ . The following statement is a direct corollary of Lemma 2.1.

Lemma 3.1. *The coefficients $r_j(q)$ of the characteristic equation (3.1) are holomorphic functions on Γ except at the points P_i of the divisor D , where they have poles of order jm_i , respectively.*

For a non-special divisor D the dimension of the space S^D of sets of meromorphic functions $\{r_j(Q), j = 1, \dots, r\}$ with the divisor of poles jD equals

$$\dim S^D = \frac{Nr(r + 1)}{2} - r(g - 1). \tag{3.2}$$

Note that dimension counting in the case of the special divisor \mathcal{K} gives

$$\dim S^{\mathcal{K}} = r^2(g - 1) + 1. \tag{3.3}$$

Equation (3.1) defines a map $\mathcal{L}^D \mapsto S^D$. The coefficients of an expansion of r_j in some basis of S^D can be seen as functions on \mathcal{L}^D . The Lax equation implies that these functions are integrals of motion. Usual arguments show that they are independent. These arguments are based on the solution of the *inverse spectral problem*, which reconstruct L , modulo gauge equivalence, from a generic set of spectral data: a smooth curve $\widehat{\Gamma}$ defined by $\{r_j\} \in S^D$, and a point of the Jacobian $J(\widehat{\Gamma})$, i.e. an equivalence class $[\widehat{\gamma}]$ of degree $\widehat{g} + r - 1$ divisor $\widehat{\gamma}$ on $\widehat{\Gamma}$. Here \widehat{g} is the genus of $\widehat{\Gamma}$.

For a generic point of S the corresponding spectral curve $\widehat{\Gamma}$ is smooth. Its genus \widehat{g} can be found with the help of the Riemann–Hurwitz formula $2\widehat{g} - 2 = 2r(g - 1) + \deg \nu$, where ν is the divisor on Γ , which is the projection of the branch points of $\widehat{\Gamma}$ over Γ . The branch points are zeros on $\widehat{\Gamma}$ of the function $\partial_k R(k, Q)$. This function has poles on all the sheets of $\widehat{\Gamma}$ over P_i of order $(r - 1)m_i$. Because the numbers of poles and zeros of a meromorphic function are equal then $\deg \nu = Nr(r - 1)$ and we obtain that

$$\widehat{g} = \frac{Nr(r - 1)}{2} + r(g - 1) + 1. \tag{3.4}$$

Moreover, a product of $\partial_k R$ on all the sheets of $\widehat{\Gamma}$ is a well-defined meromorphic function on Γ . Its divisor of zeros coincides with ν and the divisor of poles is $r(r - 1)D$. Therefore, these divisors are equivalent, i.e. in the Jacobian $J(\Gamma)$ of Γ we have the equality

$$[\nu] = r(r - 1)[D] \in J(\Gamma). \tag{3.5}$$

For a generic point $Q = (q, k)$ of $\widehat{\Gamma}$ there is a unique eigenvector $\psi = \psi(Q)$ of L ,

$$L(q)\psi(Q) = k\psi(Q), \tag{3.6}$$

normalized by the condition that a sum of its components ψ_i equals 1,

$$\sum_{i=1}^r \psi_i = 1. \tag{3.7}$$

The coordinates of ψ are rational expressions in k and the entries of L . Therefore, they define $\psi(Q)$ as a meromorphic vector-function on $\widehat{\Gamma}$. The degree of the divisor $\widehat{\gamma}$ of its poles can be found in the usual way. Let $\Psi(q), q \in \Gamma$, be a matrix with columns $\psi(Q^i)$, where $Q^i = (q, k_i(q))$ are preimages of q on $\widehat{\Gamma}$,

$$\Psi(q) = \{\psi(Q^1), \dots, \psi(Q^r)\}. \tag{3.8}$$

This matrix depends on an ordering of the roots $k_i(q)$ of (3.1), but the function $F(q) = \det^2 \Psi(q)$ is independent of this. Therefore, F is a meromorphic function on Γ . Its divisor of poles equals $2\pi_*(\widehat{\gamma})$, where $\pi : \widehat{\Gamma} \rightarrow \Gamma$ is the projection. In general position, when the branch points of $\widehat{\Gamma}$ over Γ are simple, the function F has simple zeros at the images of the branch points, and double zeros at the points γ_s , because evaluations of ψ at preimages of γ_s span the subspace orthogonal to α_s . Therefore, the zero divisor of F is $v + 2\gamma$, where $\gamma = \gamma_1 + \dots + \gamma_{rg}$, and we obtain the equality for equivalence classes of the divisors

$$2[\pi_*(\widehat{\gamma})] = [v] + 2[\gamma] = 2[\gamma] + r(r - 1)D, \tag{3.9}$$

which implies

$$\deg \widehat{\gamma} = \deg v/2 + rg = \widehat{g} + r - 1. \tag{3.10}$$

Let Ψ_0 be the matrix defined by (3.8) for $q = P_0$. Normalization (3.7) implies that Ψ_0 leaves the co-vector $e_0 = (1, \dots, 1)$ invariant, i.e.

$$e_0 \Psi_0 = e_0. \tag{3.11}$$

The spectral curve $\widehat{\Gamma}$ and the pole divisor $\widehat{\gamma}$ are invariant under the gauge transformation $L \rightarrow \Psi_0^{-1} L \Psi_0, \psi \rightarrow \Psi_0^{-1} \psi$, but the matrix Ψ_0 gets transformed to the identity $\Psi_0 = I$. Let $F = \text{diag}(f_1, \dots, f_r)$ be a diagonal matrix, then the gauge transformation

$$L \rightarrow F L F^{-1}, \psi(Q) \rightarrow f^{-1}(Q) F \psi, \text{ where } f(Q) = \sum_{i=1}^r f_i \psi_i(Q), \tag{3.12}$$

which preserves the normalization (3.7) and the equality $\Psi_0 = I$, changes $\widehat{\gamma}$ to an equivalent divisor $\widehat{\gamma}'$ of zeros of the meromorphic function $f(Q)$. The gauge transformation of L by a permutation matrix corresponds to a permutation of preimages $P_0^i \in \widehat{\Gamma}$ of $P_0 \in \Gamma$, which was used to define Ψ_0 .

A matrix g with different eigenvalues has representation of the form $g = \Psi_0 F$, where Ψ_0 satisfy (3.11) and F is a diagonal matrix. That representation is unique up to conjugation by a permutation matrix. Therefore, the correspondence described above $L \rightarrow \{\widehat{\Gamma}, \widehat{\gamma}, \Psi_0\}$ descends to a map

$$\mathcal{L}^D / SL_r \mapsto \{\widehat{\Gamma}, [\widehat{\gamma}]\}, \tag{3.13}$$

which is well-defined on an open set of \mathcal{L}^D / SL_r .

According to the Riemann–Roch theorem for each smooth genus \widehat{g} algebraic curve $\widehat{\Gamma}$ with fixed points q^1, \dots, q^r , and for each nonspecial degree $\widehat{g} + r - 1$ effective divisor $\widehat{\gamma}$ there is a unique meromorphic function $\psi_i(Q)$, $Q \in \widehat{\Gamma}$ with divisor of poles in $\widehat{\gamma}$, which is normalized by the conditions $\psi_i(q^j) = \delta_i^j$. Let $\psi(Q)$ be a meromorphic vector-function with the coordinates $\psi_i(Q)$. Note that it satisfies (3.7).

Let $\widehat{\Gamma}$ be a curve defined by Eq. (3.1), where r_j is a generic set of meromorphic functions on Γ with divisor of poles in jD . Then for each point $q \in \Gamma$ we define a matrix $\Psi(q)$ with the help of (3.8). It depends on a choice of order of the roots $k_i(q)$ of Eq. (3.1) but the matrix function

$$L(q) = \Psi(q)K(q)\Psi^{-1}(q), \quad K(q) = \text{diag}(k_1(q), \dots, k_r(q)), \tag{3.14}$$

is independent of the choice, and therefore, is a meromorphic matrix function on Γ . It has poles of degree m_i at $P_i \in D$ and is holomorphic at the points of the branch divisor ν . By reversing the arguments used for the proof of (3.10), we get that the degree of the zero divisor γ of $\det \Psi$ equals rg . In general position the zeros γ_s are simple. From Lemma 2.1 it follows that an expansion of L at γ_s satisfies constraints (2.10,2.11), where α_s is a unique up to multiplication vector orthogonal to the vector-columns of $\Psi(\gamma_s)$. Hence, L is a Lax matrix-function.

If the points P_0^i used for normalization of ψ_j are preimages of $P_0 \in \Gamma$, then L , given by (3.14), is diagonal at $q = P_0$, and the correspondence $\{\widehat{\Gamma}, \widehat{\gamma}\} \rightarrow L$ descends to a map

$$\{\widehat{\Gamma}, [\widehat{\gamma}]\} \rightarrow \mathcal{L}^D/SL_r, \tag{3.15}$$

which is well-defined on an open set of the Jacobian bundle over S , where it is inverse to (3.13).

Now, let $L = L(q, t)$ be a solution of the Lax equations (2.46,2.53). Then the spectral curve $\widehat{\Gamma}$ of $L(q, t)$ is time-independent and can be regarded as a generating form of the integrals of the Lax equations. The divisor $\widehat{\gamma}$ of poles of the eigenvector ψ , defined by (3.6, 3.7) does depend on t_a .

It is now standard procedure to show that $[\widehat{\gamma}]$ evolves linearly on $J(\widehat{\Gamma})$. From the Lax equation $\partial_a L = [M_a, L]$ it follows that, if ψ is an eigenvector of L , then $(\partial_a - M_a)\psi$ is also an eigenvector. Therefore,

$$(\partial_a - M_a)\psi(Q, t) = f_a(Q, t)\psi(Q, t), \tag{3.16}$$

where $f_a(Q, t)$ is a scalar meromorphic function on $\widehat{\Gamma}$. The vector-function

$$\widehat{\psi}(Q, t) = \varphi(Q, t)\psi(Q, t), \quad \varphi(Q, t) = \exp\left(-\int_0^{t_a} f_m(Q, \tau)d\tau\right) \tag{3.17}$$

satisfies the equations

$$L(q, t)\widehat{\psi}(Q, t) = k\widehat{\psi}(Q, t), \quad (\partial_a - M_a(q, t))\widehat{\psi}(q, t) = 0. \tag{3.18}$$

It turns out that the pole divisor $\widehat{\gamma}(t)$ of ψ under the gauge transform (3.17) gets transformed to a *time-independent* divisor $\widehat{\gamma} = \widehat{\gamma}(0)$ of poles of $\widehat{\psi}$. All the time dependence of $\widehat{\psi}(Q, t)$ is encoded in the form of its essential singularities, which it acquires at the constant poles of f_a .

Let $L(q, t)$ be a solution of the hierarchy of Eqs. (2.46), (2.53). Here and below we assume that only a finite number of "times" t_a are not equal to zero. For brevity we

denote the variables t_a corresponding to indices (2.44) and (2.51) by $t_{(i,n,m)}$ and $t_{(0,m;l)}$, respectively. Commutativity of the hierarchy implies that there is a unique common gauge transform $\widehat{\psi}(Q, t) = \varphi(Q, t)\psi(Q, t)$ such the $\widehat{\psi}$ solves all the auxiliary linear equations (3.18).

Lemma 3.2. *Let $\widehat{\psi}(Q, t)$, $\widehat{\psi}(Q, 0) = \psi(Q, 0)$ be the common solution of equations (3.18). Then*

1⁰. $\widehat{\psi}$ is a meromorphic function on $\widehat{\Gamma}$ except at the points P_i^l and P_0^l , which are preimages on $\widehat{\Gamma}$ of the points $P_i \in D$ and P_0 on Γ , respectively. Its divisor of poles on $\widehat{\Gamma}$ outside of P_i^l , P_0^l is not greater than $\widehat{\gamma}$;

2⁰. In the neighborhood of P_i^l the function $\widehat{\psi}$ has the form

$$\widehat{\psi} = \xi_{i,l}(w, t) \exp \left(\sum_n t_{(i,n,m)} w^{-m} k^n \right), \tag{3.19}$$

where $\xi_{i,l}(w, t)$ is a holomorphic vector-function, and $k = k_l(q)$ is the corresponding root of Eq. (3.1);

3⁰. In the neighborhood of P_0^l the function $\widehat{\psi}$ has the form

$$\widehat{\psi} = \chi_l(w, t) \exp \left(\sum_n t_{(0,m;l)} w^{-m} \right), \tag{3.20}$$

where χ_l is a holomorphic vector-function such that evaluation of its coordinates at P_0^l equals $\chi_i^j(P_0^l) = \delta^{ij}$.

The function $\widehat{\psi}(Q, t)$ is a particular case of the conventional Baker–Akhiezer functions. As shown in [14], for any generic divisor $\widehat{\gamma}$ of degree $\widehat{g} + r - 1$ there is a unique vector function $\widehat{\psi}(Q, t)$ which satisfy all the properties 1⁰ – 3⁰. It can be written explicitly in terms of the Riemann theta-function of the curve $\widehat{\Gamma}$.

Theorem 3.1. *Let $\widehat{\psi}(Q, t)$ be the Baker–Akhiezer vector function associated with a non-special divisor $\widehat{\gamma}$ on $\widehat{\Gamma}$. Then there exist unique matrix functions $L(q, t)$, $M_a(q, t)$ such that Eqs. (3.18) hold.*

As a corollary we get that the Lax operator $L(q, t) \in \mathcal{L}^D$ constructed with the help of $\widehat{\psi}$ solves the whole hierarchy of the Lax equations (2.46,2.53).

4. Hamiltonian Approach

As we have seen, the spectral transform which identifies the space of gauge equivalent Lax matrices with a total space of a Jacobian bundle over the moduli space of the spectral curves does not involve a Hamiltonian description of the Lax equations. Moreover, *a priori* it is not clear, why all the systems constructed above are Hamiltonian. In this section we show that the general algebraic approach to the Hamiltonian theory of the Lax equations proposed in [9, 10] and developed in [11] is evenly applicable to the Lax equations on the Riemann surfaces.

The entries of $L(q) \in \mathcal{L}^D$ can be regarded as functions on \mathcal{L}^D . Therefore, L by itself can be seen as a matrix-valued function and its external derivative δL as a matrix-valued one-form on \mathcal{L}^D . The matrix Ψ (3.8) with columns formed by the canonically normalized eigenvectors $\psi(Q^i)$ of L can also be regarded as a matrix function on \mathcal{L}^D

defined modulo permutation of the columns. Hence, its differential $\delta\Psi$ is a matrix-valued one-form on \mathcal{L}^D . In the same way we consider the differential δK of the diagonal matrix K (3.14). Let us define a two-form $\Omega(q)$ on \mathcal{L}^D with values in a space of meromorphic functions on Γ by the formula

$$\Omega(q) = \text{Tr} \left(\Psi^{-1} \delta L \wedge \delta \Psi - \Psi^{-1} \delta \Psi \wedge \delta K \right). \tag{4.1}$$

This form does not depend on an order of the eigenvalues of L , and therefore, is well defined on \mathcal{L}^D . Fix a holomorphic differential dz on Γ . Then the formula

$$\omega = -\frac{1}{2} \left(\sum_{s=1}^{rg} \text{res}_{\gamma_s} \Omega dz + \sum_{P_i \in D} \text{res}_{P_i} \Omega dz \right), \tag{4.2}$$

defines a scalar-valued two-form on \mathcal{L}^D .

The equation

$$\delta L = \Psi \delta K \Psi^{-1} + \delta \Psi K \Psi^{-1} + \Psi K \delta \Psi^{-1} \tag{4.3}$$

implies

$$\Omega = 2 \delta \left(\text{Tr} \left(K \Psi^{-1} \delta \Psi \right) \right) = 2 \delta \left(\text{Tr} \left(\Psi^{-1} L \delta \Psi \right) \right). \tag{4.4}$$

We would like to emphasize that though the last formula looks simpler than (4.1) and directly shows that ω is a *closed* two-form, the original definition is more universal. As shown in [9, 10], it provides symplectic structure for general soliton equations.

Lemma 4.1. *The two-form ω defined by (4.2) is invariant under gauge transformations defined by matrices g that preserve the co-vector $e_0 = (1, \dots, 1)$, $e_0 g = e_0$.*

Proof. If g preserves e_0 , then the gauge transformation

$$L' = g^{-1} L g, \quad \Psi' = g^{-1} \Psi \tag{4.5}$$

preserves normalization (3.7) of the eigenvectors. If $h = (\delta g) g^{-1}$, then from (4.4) it follows that under (4.5) Ω gets transformed to $\Omega' = \Omega + F$, where

$$F = -2 \delta \left(\text{Tr} (L h) \right) = -2 \text{Tr} (\delta L \wedge h + L h \wedge h). \tag{4.6}$$

The additional term F is a meromorphic function on Γ with poles at the points γ_s and P_i . Therefore, the sum of residues at these points of the differential $F dz$ equals zero and the lemma is proved.

It is necessary to emphasize that in the generic case the form ω is not gauge invariant with respect to the whole group SL_r , because it does depend on a choice of the normalization of the eigenvectors. A change of normalization corresponds to the transformation $\Psi' = \Psi V, L' = L$, where $V = V(Q)$ is a diagonal matrix, which might depend on Q . The corresponding transformation of Ω has the form:

$$\Omega' = \Omega + 2 \delta \left(\text{Tr} (K v) \right) = \Omega + 2 \text{Tr} (\delta K \wedge v), \quad v = \delta V V^{-1}. \tag{4.7}$$

Here we use the equation $\delta v = v \wedge v = 0$ which is valid because v is diagonal.

Let $\mathcal{P}_0^D \subset \mathcal{L}^D$ be a subspace of the Lax matrices such that restriction of $\delta k dz$ to \mathcal{P}_0^D is a holomorphic differential. This subspace is a leaf of foliation on \mathcal{L}^D defined by the common level sets of the functions defined on \mathcal{L}^D by the formulae

$$T_{i,j,l} = \text{res}_{P_i^l} \left((z - z(P_i))^j k dz \right), \quad j = 0, \dots, (m_i - d_i), \quad (4.8)$$

where d_i is the order of zero dz at P_i (compare with the definition of the universal configuration space in [9]). Note that although the functions (4.8) are multivalued, their common level sets are leaves of a well-defined foliation on \mathcal{L}^D .

Lemma 4.2. *The two-form ω defined by (4.2) restricted to $\mathcal{P}_0^D \subset \mathcal{L}^D$ is gauge invariant, i.e. it descends to a form on $\mathcal{P}^D = \mathcal{P}_0^D / SL_r$.*

Let $L \in \mathcal{L}^K$ be a Lax matrix corresponding to the zero divisor \mathcal{K} of a holomorphic differential dz , then $L dz$ has poles at the points γ_s , only. Therefore, $\mathcal{P}_0^K = \mathcal{L}^K$.

Lemma 4.3. *The two-form ω on \mathcal{L}^K defined by the formula (4.2) descends to a form on \mathcal{L}^K / SL_r , which under the isomorphism (2.21) coincides with the canonical symplectic structure on the cotangent bundle $T^*(\mathcal{M})$.*

Proof. The first statement is a direct corollary of the previous lemma. The second one follows from the equality

$$\text{res}_{\gamma_s} \Omega dz = -2 \left(\delta \kappa_s \wedge \delta z_s + \sum_{i=1}^r \delta \beta_s^i \wedge \delta \alpha_s^i \right), \quad (4.9)$$

which can be proved as follows. Let L'_s be the matrix defined by the gauge transformation (2.15), and let Ω'_s be the function defined by (4.1) for $L = L'$. Then as shown above,

$$\text{res}_{\gamma_s} \Omega'_s dz = \text{res}_{\gamma_s} \Omega dz + 2 \text{res}_{\gamma_s} \text{Tr} \left(\delta g_s g_s^{-1} \wedge \delta L - L \delta g_s g_s^{-1} \wedge \delta g_s g_s^{-1} \right). \quad (4.10)$$

From (2.9), (2.10) it follows that the second term in (4.10) equals

$$II = -2 \text{Tr} \left((\beta_s \delta \alpha_s^T + \delta \beta_s \alpha_s^T) \wedge \delta g_s g_s^{-1} + \beta_s \alpha_s^T \delta g_s g_s^{-1} \wedge \delta g_s g_s^{-1} \right). \quad (4.11)$$

Using the equality $\delta \alpha_s^T g_s + \alpha_s^T \delta g_s = 0$, which follows from (2.14), we get

$$II = 2 \text{Tr} \left(\delta \beta_s \wedge \delta \alpha_s^T \right) = -2 \left(\delta \alpha_s^T \wedge \delta \beta_s \right). \quad (4.12)$$

The matrix L'_s under the gauge transformation $\widehat{L} = f_s^{-1} L'_s f_s$, where $f_s = f_s(z)$ is the diagonal matrix (2.18), gets transformed to a holomorphic matrix. Therefore,

$$0 = \text{res}_{\gamma_s} \Omega'_s dz + 2 \text{res}_{\gamma_s} \text{Tr} \left(\delta f_s f_s^{-1} \wedge \delta L'_s - L'_s \delta f_s f_s^{-1} \wedge \delta f_s f_s^{-1} \right). \quad (4.13)$$

The last term in (4.13) equals zero because f_s is diagonal. From (2.14)–(2.18) it follows that

$$\text{res}_{\gamma_s} \Omega'_s dz = -2 \text{res}_{\gamma_s} \text{Tr} \left(\delta f_s f_s^{-1} \wedge \delta L'_s \right) = 2 \delta z_s \wedge \delta \kappa_s. \quad (4.14)$$

Equations (4.10)–(4.14) imply (4.9). In the coordinates A_s and B_s (2.34)–(2.38) on an open set of $T^*(\mathcal{M})$ the form ω due to (2.36) equals

$$\omega = \sum_{s=1}^{rg} \delta\kappa_s \wedge \delta z_s + \sum_{s=r+1}^{rg} \delta B_s^T \wedge \delta A_s, \quad g > 1 \tag{4.15}$$

and the Lemma is proved.

Let us now consider the contribution to ω from poles of Ldz at the points P_m of the divisor $D_{\mathcal{K}} = D - \mathcal{K}$. The residue of the last term in (4.1) restricted to \mathcal{P}_0^D vanishes. Therefore,

$$\omega_m = -\frac{1}{2} \text{res}_{P_m} \Omega dz = \text{res}_{P_m} \text{Tr} \left(L \delta \Psi \Psi^{-1} \wedge \delta \Psi \Psi^{-1} \right) dz. \tag{4.16}$$

If Ldz has a simple pole at P_m , then its residue L_m is a point of the orbit \mathcal{O}_m of the adjoint action of GL_r , corresponding to the fixed singular part of kdz , which defines the leaf \mathcal{P}_0^D . Let ξ be a matrix, which we regard as a point of the Lie algebra $\xi \in \mathfrak{sl}_r$. The formula

$$\partial_{\xi} L_m = [L_m, \xi], \tag{4.17}$$

defines a tangent vector $\partial_{\xi} \in T_{L_m}(\mathcal{O}_m)$ to the orbit at L_m . The correspondence $\xi \rightarrow \partial_{\xi}$ is an isomorphism between $\mathfrak{sl}_r/\mathfrak{sl}_r(L_m)$, and $T_{L_m}(\mathcal{O}_m)$. Here $\mathfrak{sl}_r(L_m)$ is a subalgebra of the matrices, that commute with L_m . Evaluation of the form $(\delta \Psi \Psi^{-1})$ at ∂_{ξ} is equal to ξ . Hence, (4.16) restricted to \mathcal{P}_0^D coincides with the canonical symplectic structure on the orbit \mathcal{O}_m . Its evaluation on a pair of vectors ξ, η is equal to

$$\omega_m(\xi, \eta) = \text{Tr} (L_m [\xi, \eta]). \tag{4.18}$$

If Ldz has a multiple pole at P_m , then we define \tilde{L}_m as the equivalence class of the singular part of Ldz . By definition two matrix differentials \tilde{L} and \tilde{L}' meromorphic in the neighborhood of P_m are equivalent if $\tilde{L} - \tilde{L}'$ is a holomorphic differential. Let \mathcal{G}_- be a group of the invertible holomorphic matrix functions in the neighborhood of P_m . The transformation $\tilde{L} \rightarrow g \tilde{L} g^{-1}$, $g \in \mathcal{G}_-$ defines a representation of \mathcal{G}_- on the finite-dimensional space of singular parts of meromorphic differentials. Let $\tilde{\mathcal{O}}_m$ be an orbit of this representation.

If \mathcal{H}_- is the Lie algebra of \mathcal{G}_- , then the equivalence class of the right-hand side of (4.17) for $\xi \in \mathcal{H}_-$ depends only on the equivalence class of \tilde{L}_m . Therefore, (4.17) defines an isomorphism between the tangent space to $\tilde{\mathcal{O}}_m$ at \tilde{L}_m and $\mathcal{H}_-/\mathcal{H}_-(\tilde{L}_m)$, where $\mathcal{H}_-(\tilde{L}_m)$ is the subalgebra of holomorphic matrix functions ξ such that $[L_m, \xi]$ is holomorphic at P_m . The formula

$$\omega_m = \text{res}_{P_m} \text{Tr} \left(\tilde{L}_m [\xi, \eta] \right) \tag{4.19}$$

defines a symplectic structure on $\tilde{\mathcal{O}}_m$.

Lemma 4.4. *If $D_{\mathcal{K}} = D - \mathcal{K} > 0$ is an effective divisor, then the map*

$$L \mapsto \{z_s, \kappa_s, \alpha_s, \beta_s, \tilde{L}_m, \}, \tag{4.20}$$

is a bijective correspondence between points of the bundle \mathcal{L}^D over \mathcal{M}_0 and sets of the data (4.20) subject to the constraints $(\alpha_s^T \beta_s) = 0$, and

$$\sum_{s=1}^{rg} \beta_s \alpha_s^T + \sum_{P_m \in D'} \operatorname{res}_{P_m} \tilde{L}_m = 0, \tag{4.21}$$

modulo gauge transformations (2.27).

If we fix a gauge on a open set of \mathcal{L}^D by (2.36), then the reconstruction formulae for B_1, \dots, B_{r+1} become

$$B_{r+1}^i = - \sum_{s=r+2}^{rg} B_s^i A_s^i - \sum_m \operatorname{res}_{P_m} \tilde{L}_m^{ii}, \tag{4.22}$$

and

$$B_j^i = -B_{r+1}^i - \sum_{s=r+2}^{rg} B_s^i A_s^j - \sum_m \operatorname{res}_{P_m} \tilde{L}_m^{ij}. \tag{4.23}$$

If $g > 1$, then for $D_{\mathcal{K}} > 0$ the data $\{z_s, \kappa_s, A_s, B_s, \tilde{L}_m \in \tilde{\mathcal{O}}_m\}$ provide explicit coordinates on an open set of \mathcal{P}^D .

Theorem 4.1. *Let D be a divisor such that $D_{\mathcal{K}} \geq 0$, where \mathcal{K} is the zero divisor of a holomorphic differential dz . Then the form ω defined by (4.2), restricted to \mathcal{P}_0^D descends to a non-degenerate closed two-form on \mathcal{P}^D :*

$$\omega = \omega_0 + \sum_{P_m \in D_{\mathcal{K}}} \omega_m, \tag{4.24}$$

where ω_0 and ω_m are given by (4.15), and (4.19), respectively.

The representation of the form ω in terms of the Lax operator and its eigenvectors provide a straightforward and universal way to show that the Lax equations are Hamiltonian, and to construct the action-angle variables.

By definition a vector field ∂_t on a symplectic manifold is Hamiltonian, if the contraction $i_{\partial_t} \omega(X) = \omega(\partial_t, X)$ of the symplectic form is an exact one-form $dH(X)$. The function H is the Hamiltonian corresponding to the vector field ∂_t .

Theorem 4.2. *Let ∂_a be the vector fields corresponding to the Lax equations (2.46, 2.53). Then the contraction of ω defined by (4.2) restricted to \mathcal{P}^D equals*

$$i_{\partial_a} \omega = \delta H_a, \tag{4.25}$$

where

$$H_a = -\frac{1}{n+1} \operatorname{res}_{P_i} \operatorname{Tr} \left(w^{-m} L^{n+1} \right) dz, \quad a = (P_i, n, m), \tag{4.26}$$

$$H_a = -\operatorname{res}_{P_0} \left(w^{-m} k_l \right) dz, \quad a = (P_0, m; l). \tag{4.27}$$

Here $k_l = k_l(q)$ is the l^{th} eigenvalue of L in the neighborhood of the puncture P_0 .

Proof. The Lax equation $\partial_a L = [M_a, L]$, $\partial_a k = 0$, and Eq. (3.16)

$$\partial_a \Psi = M_a \Psi + \Psi F_a, \tag{4.28}$$

where Ψ is the matrix of eigenvectors (3.8), and $F_a = \text{diag}(f_a(Q^1), \dots, f_a(Q^r))$, imply

$$i_{\partial_a} \omega = -\frac{1}{2} \left(\sum_{s=1}^{rg} \text{res}_{\gamma_s} \Lambda dz + \sum_{P_i \in D} \text{res}_{P_i} \Lambda dz \right), \tag{4.29}$$

where $\Lambda = \Lambda(q)$ equals

$$\Lambda = \text{Tr} \left(\Psi^{-1} [M_a, L] \delta \Psi - \Psi^{-1} \delta L (M_a \Psi + \Psi F_a) - \Psi^{-1} (M_a \Psi + \Psi F_a) \delta K \right). \tag{4.30}$$

Using, as before, the equality $L \delta \Psi - \delta \Psi K = \Psi \delta K - \delta L \Psi$, we get that

$$\text{Tr} \left(\Psi^{-1} [M_a, L] \delta \Psi \right) = \text{Tr} \left(\Psi^{-1} M_a \Psi \delta K - M_a \delta L \right). \tag{4.31}$$

Using the fact that K and F are diagonal, we also obtain the equation

$$\text{Tr} \left(\Psi^{-1} \delta L \Psi F_a \right) = \text{Tr} (\delta K F_a). \tag{4.32}$$

From (4.31), (4.32) it follows that

$$i_{\partial_a} \omega = \sum_{P_i \in D} \text{res}_{P_i} \text{Tr} (\delta K F_a) dz + R_a, \tag{4.33}$$

where

$$R_a = \sum_{s=1}^{rg} \text{res}_{\gamma_s} \text{Tr} (\delta L M_a) dz + \sum_{P_i \in D} \text{res}_{P_i} \text{Tr} (\delta L M_a) dz. \tag{4.34}$$

Note that in the first term of (4.33) a sum of residues at γ_s has been dropped because K and F_a are holomorphic at these points.

Consider first the case of the Lax equations (2.46). The matrix M_a for $a = (P_i, n, m)$ is holomorphic everywhere except at the points γ_s and P_i . Therefore, $R_{i,n,m} = 0$. The corresponding diagonal matrix $F_{i,n,m}$ is holomorphic at the points $P_j \in D, j \neq i$. From (2.45) it follows that $F_{i,n,m}$ in the neighborhood of P_i has the form

$$F_{i,n,m} = -w^{-m} K^n + O(1). \tag{4.35}$$

The form $\delta K dz$ restricted to \mathcal{P}^D is holomorphic in the neighborhood of P_i . Therefore,

$$-\text{res}_{P_i} \text{Tr} (\delta K F_{i,n,m}) dz = \text{res}_{P_i} \text{Tr} (w^{-m} K^n \delta K) dz = \frac{1}{n+1} \text{res}_{P_i} \text{Tr} (w^{-m} L^{n+1}) dz. \tag{4.36}$$

The matrix F_a corresponding to $a = (P_0, m; l)$ is holomorphic at the points of D . Therefore, the right-hand side of (4.33) reduces just to R_a . Because, $M_{0,m;l}$ is holomorphic except at the points γ_s and P_0 , we have in this case the equation

$$R_{0,m;l} = -\text{res}_{P_0} \text{Tr} (\delta L M_{0,m;l}) dz, \tag{4.37}$$

which, with the help of (2.52), implies (4.27). The theorem is therefore proved. It shows that the Lax equations restricted to \mathcal{P}^D are Hamiltonian whenever the restriction of ω is non-degenerate.

Corollary 4.1. *If $D_{\mathcal{K}}$ is an effective divisor, then the Lax equations (2.46), (2.53) restricted to \mathcal{P}^D are Hamiltonian. The corresponding Hamiltonians (4.26), (4.27) are in involution*

$$\{H_a, H_b\} = 0. \tag{4.38}$$

The basic relation which implies all Eqs. (4.38) is involutivity of all the eigenvalues of the Lax matrices at different points of Γ , i.e.

$$\{k_l(q), k_{l_1}(q_1)\} = 0. \tag{4.39}$$

Example. Let us consider the Lax matrices on an elliptic curve $\Gamma = C/\{2n\omega_1, 2m\omega_2\}$ with one puncture, which without loss of generality we put at $z = 0$. In this example we denote the parameters γ_s and κ_s by q_s and p_s , respectively.

In the gauge $\alpha_s = e_s$, $e_s^j = \delta_s^j$ the j^{th} column of the Lax matrix L^{ij} has poles only at the points q_j and $z = 0$. From (2.10) it follows that L^{jj} is regular everywhere, i.e. it is a constant. Equation (2.11) implies that $L^{ji}(q_j) = 0$, $i \neq j$ and $L^{jj} = p_j$. An elliptic function with two poles and one zero fixed is uniquely defined up to a constant. It can be written in terms of the Weierstrass σ -function as follows:

$$L^{ij}(z) = f^{ij} \frac{\sigma(z + q_i - q_j) \sigma(z - q_i) \sigma(q_j)}{\sigma(z) \sigma(z - q_j) \sigma(q_i - q_j) \sigma(q_i)}, \quad i \neq j; \quad L^{ii} = p_i. \tag{4.40}$$

Let f^{ij} be a rank 1 matrix $f^{ij} = a^i b^j$. As it was mentioned above, the equations $\alpha_i = e_i$ fix the gauge up to transformation by diagonal matrices. We can use these transformation to make $a^i = b^i$. The corresponding momentum is given then by the collection $(a^i)^2$ and we fix it to the values $(a^i)^2 = 1$. The matrix L given by (4.40) with $f^{ij} = 1$ is gauge equivalent to the Lax matrix \tilde{L} with a spectral parameter for the elliptic Calogero–Moser system found in [15]:

$$\tilde{L}^{ii} = p_i, \quad \tilde{L}^{ij} = \Phi(q_i - q_j, z), \quad i \neq j, \tag{4.41}$$

where

$$\Phi(q, z) = \frac{\sigma(z - q)}{\sigma(z) \sigma(q)} e^{\zeta(z)q}. \tag{4.42}$$

Note that \tilde{L} has essential singularity at $z = 0$, which is due to the gauge transformation by the diagonal matrix $\hat{\Phi} = \text{diag}(\Phi(q_i, z))$, which removes poles of L at the points q_i .

The Hamiltonian of the elliptic CM system (1.8) is equal to

$$H_{\text{CM}} = \frac{1}{2} \text{res}_0 \text{Tr} \left(z^{-1} L^2 \right) dz. \tag{4.43}$$

For the sequel, we would like to express H_{CM} in terms of the first two coefficients of the Laurent expansion of the marked branch of the eigenvalue of L at $z = 0$. Indeed, expansions of the eigenvalues of L at $z = 0$ have the form

$$\begin{aligned} k_1(z) &= (r - 1)z^{-1} + k_{11} + k_{12}z + O(z^2), \\ k_l(z) &= -z^{-1} + k_{l1} + k_{l2}z + O(z^2), \quad l > 1. \end{aligned} \tag{4.44}$$

The equation

$$H_1 = \sum_{i=1}^r p_i = \text{Tr } L = \sum_{l=1}^r k_l(z). \tag{4.45}$$

implies

$$H_1 = \sum_{l=1}^r k_{l1}, \quad \sum_{l=1}^r k_{l2} = 0. \tag{4.46}$$

From (4.44) and (4.46) it follows that

$$2H_{\text{CM}} = 2rk_{12} + \sum_{l=1}^r k_{l1}^2. \tag{4.47}$$

Trace of L^m has the only pole at $z = 0$. Hence, we have the equations

$$\text{res}_0 \text{Tr}(L^2) = 2 \left((r-1)k_{11} - \sum_{l=2}^r k_{l1} \right) = 0, \tag{4.48}$$

$$\text{res}_0 \text{Tr}(L^3) = 3 \left((r-1)^2 k_{12} + (r-1)k_{11}^2 + \sum_{l=2}^r (k_{l2} - k_{l1}^2) \right) = 0. \tag{4.49}$$

Equations (4.48) and (4.49) imply

$$H_1 = rk_{11}, \quad 2H_{\text{CM}} = r^2 k_{12} + rk_{11}^2. \tag{4.50}$$

Our next goal is to construct the action-angle variables for ω .

Theorem 4.3. *Let $L \in \mathcal{L}^D$ be a Lax matrix, and let $\widehat{\gamma}_s$ be the poles of the normalized (3.7) eigenvector ψ . Then the two-form ω defined by (4.2) is equal to*

$$\omega = \sum_{s=1}^{\widehat{g}+r-1} \delta k(\widehat{\gamma}_s) \wedge \delta z(\widehat{\gamma}_s). \tag{4.51}$$

The meaning of the right-hand side of this formula is as follows. The spectral curve is equipped by definition with the meromorphic function $k(Q)$. The pull back to $\widehat{\Gamma}$ of the abelian integral $z(Q) = \int^Q dz$ on Γ is a multi-valued holomorphic function on $\widehat{\Gamma}$. The evaluations $k(\widehat{\gamma}_s)$, $z(\widehat{\gamma}_s)$ at the points $\widehat{\gamma}_s$ define functions on the space \mathcal{L}^D , and the wedge product of their external differentials is a two-form on \mathcal{L}^D . (Note that the differential $\delta z(\widehat{\gamma}_s)$ of the multi-valued function $z(\widehat{g}_s)$ is single-valued, because the periods of dz are constants).

Proof. The proof of formula (4.51) is very general and does not rely on any specific form of L . Let us present it briefly following the proof of Lemma 5.1 in [11] (more details can be found in [16]).

Let γ_s^j, P_i^j be preimages on $\widehat{\Gamma}$ of the points $\gamma_s \in \Gamma$ and $P_i \in D$. Then the form ω is equal to

$$\omega = -\frac{1}{2} \sum_{j=1}^r \left(\sum_{s=1}^{rg} \text{res}_{\gamma_s^j} \widetilde{\Omega} dz + \sum_i \text{res}_{P_i^j} \widetilde{\Omega} dz \right), \tag{4.52}$$

where $\widetilde{\Omega}$ is a meromorphic function on $\widehat{\Gamma}$ defined by the formula

$$\widetilde{\Omega}(Q) = \psi^*(Q) \delta L(q) \wedge \delta \psi(Q) - \psi^*(Q) \delta \psi(Q) \wedge \delta k, \quad Q = (k, q) \in \widehat{\Gamma}. \tag{4.53}$$

The expression $\psi_n^*(Q)$ is the dual eigenvector, which is the row-vector solution of the equation

$$\psi^*(Q) L(q) = k \psi^*(Q), \tag{4.54}$$

normalized by the condition

$$\psi^*(Q) \psi(Q) = 1. \tag{4.55}$$

Note that $\psi^*(Q)$ can be identified with the only row of the matrix $\Psi^{-1}(q)$ which is not orthogonal to the column $\psi(Q)$ of $\Psi(q)$. That implies that $\psi^*(Q)$ as a function on the spectral curve has poles at the points γ_s^j , and at the branching points of the spectral curve. Equation (4.55) implies that it has zeroes at the poles $\widehat{\gamma}_s$ of $\psi_n(Q)$. These analytical properties will be crucial in the sequel.

The differential $\widetilde{\Omega} dz$ is a meromorphic differential on the spectral curve $\widehat{\Gamma}$. Therefore, the sum of its residues at the punctures P_i^j, γ_s^j is equal to the negative of the sum of the other residues on $\widehat{\Gamma}$. There are poles of two types. First of all, $\widetilde{\Omega}$ has poles at the poles $\widehat{\gamma}_s$ of ψ . Note that $\delta \psi$ has a pole of the second order at $\widehat{\gamma}_s$. Taking into account that ψ^* has zero at $\widehat{\gamma}_s$ we obtain

$$\text{res}_{\widehat{\gamma}_s} \widetilde{\Omega} = (\psi^* \delta L \psi)(\widehat{\gamma}_s) \wedge \delta z(\widehat{\gamma}_s) + \delta k(\widehat{\gamma}_s) \wedge \delta z(\widehat{\gamma}_s) = 2\delta k(\widehat{\gamma}_s) \wedge \delta z(\widehat{\gamma}_s). \tag{4.56}$$

The last equality follows from the standard formula for variation of the eigenvalue of an operator, $\psi^* \delta L \psi = \delta k$.

The second set of poles of $\widetilde{\Omega}$ is the set of branch points q_i of the cover. The pole of ψ^* at q_i cancels with the zero of the differential $dz, dz(q_i) = 0$, considered as a differential on $\widehat{\Gamma}$. The vector-function ψ is holomorphic at q_i . If we take an expansion of ψ in the local coordinate $(z - z(q_i))^{1/2}$ (in general position when the branch point is simple) and consider its variation we get that

$$\delta \psi = -\frac{d\psi}{dz} \delta z(q_i) + O(1). \tag{4.57}$$

Therefore, $\delta \psi$ has simple pole at q_i . In the similar way we have

$$\delta k = -\frac{dk}{dz} \delta z(q_i). \tag{4.58}$$

Equalities (4.57) and (4.58) imply that

$$\text{res}_{q_i} (\psi^* \delta L \wedge \delta \psi) dz = \text{res}_{q_i} \left[(\psi^* \delta L d\psi) \wedge \frac{\delta k dz}{dk} \right]. \tag{4.59}$$

Due to skew-symmetry of the wedge product we may replace δL in (4.59) by $(\delta L - \delta k)$. Then, using the identities $\psi^*(\delta L - \delta k) = \delta\psi^*(k - L)$ and $(k - L)d\psi = (dL - dk)\psi$, we obtain

$$\text{res}_{q_i} (\psi^* \delta L \wedge \delta \psi) dz = -\text{res}_{q_i} (\delta \psi^* \psi) \wedge \delta k dz = \text{res}_{q_i} (\psi^* \delta \psi) \wedge \delta k dz. \tag{4.60}$$

Note that the term with dL does not contribute to the residue, because $dL(q_i) = 0$. The right-hand side of (4.60) cancels with a residue of the second term in the sum (4.53) and the theorem is proved.

Remark. The right-hand side of (4.51) can be identified with a particular case of universal algebraic-geometric symplectic form proposed in [9]. It is defined on the generalized Jacobian bundles over a proper subspace of the moduli spaces of Riemann surfaces with punctures. In the case of families of hyperelliptic curves that form was pioneered by Novikov and Veselov [17].

Let ϕ_k be coordinates on the Jacobian $J(\widehat{\Gamma})$ of the spectral form. The isomorphism of the symmetric power of the spectral curve and the Jacobian is defined by the Abel map

$$\phi_i(\widehat{\gamma}) = \sum_s \int^{\widehat{\gamma}_s} d\omega_i, \tag{4.61}$$

where $d\omega_i$ is the basis of normalized holomorphic differentials on $\widehat{\Gamma}$, corresponding to a choice of a basis of a - and b -cycles on $\widehat{\Gamma}$ with the canonical matrix of intersections. Restricted to \mathcal{P}^D , the differential $\delta k dz$ is holomorphic. Therefore, it can be represented as a sum of the basis differentials

$$\delta k dz = \sum_i \delta I_i d\omega_i. \tag{4.62}$$

The coefficients of the sum are differentials on \mathcal{P}^D of the functions

$$I_i = \oint_{a_i} k dz. \tag{4.63}$$

From (4.51) it follows that $\omega = \delta\alpha$, where

$$\alpha = \sum_{s=1}^{\widehat{g}+r-1} \int^{\widehat{\gamma}_s} \delta k dz = \sum_{i=1}^{\widehat{g}} \delta I_i \wedge \phi_i. \tag{4.64}$$

Corollary 4.2. *The form ω restricted to \mathcal{P}^D equals*

$$\omega = \sum_{i=1}^{\widehat{g}} \delta I_i \wedge \delta \phi_i. \tag{4.65}$$

For the case when $D_{\mathcal{K}} \geq 0$, this result was obtained first in [18].

It is instructive to show that (4.65) directly implies that ω is non-degenerate for $D_{\mathcal{K}} \geq 0$. First of all, (4.65) implies that the forms δI_i are linear independent. Indeed, if they are linear dependent at $s \in S^D$, then there is a vector v tangent to S^D at s , such that

$\delta I_i(v) = 0$. Due to (4.62) we conclude $\partial_v k \equiv 0$. It is impossible for generic s , because the equation

$$\partial_v k = \frac{\sum_{j=1}^r \partial_v r_j k^{r-j}}{R_k(k, Q)} \equiv 0, \tag{4.66}$$

implies, then, that $k(Q)$ satisfies an algebraic equation of degree less than r , i.e. the spectral curve $\widehat{\Gamma}$ can not be an r -sheeted branch cover of Γ .

The second argument needed in order to complete the proof is that the dimension of the space $S_{\mathcal{P}}^D \subset S^D$ of the spectral curves corresponding to \mathcal{P}^D equals \widehat{g} . The number of conditions that singular parts of eigenvalues of L at the points $P_m \in D_{\mathcal{K}}$ are constant along \mathcal{P}^D equals $(r \deg D_{\mathcal{K}})$ minus 1, due to the relation

$$\sum_{P_m \in D_{\mathcal{K}}} \text{res}_{P_m} (\text{Tr } L) dz = 0, \tag{4.67}$$

which is valid, because the singular parts of L at γ_s are traceless. From (3.2) we get

$$2 \dim S_{\mathcal{P}}^D = Nr(r - 1) - 2r(g - 1) + 2 = 2 \widehat{g} = \dim \mathcal{P}^D. \tag{4.68}$$

5. The Zero-Curvature Equations

The main goal of this section is to present the non-stationary analog of the Lax equations on an algebraic curve as an infinite-dimensional Hamiltonian system.

Let \mathcal{A}^D be a space of the $(r \times r)$ matrix function $L(x, q) = L(x + T, q)$ of the real variable x such that:

1⁰. $L(x, q)$ is a meromorphic function of the variable $q \in \Gamma$ with poles at D and at the points $\gamma_s(x)$, where it has the form (2.3), i.e. $L(x, q) \in \mathcal{N}_{\gamma(x), \alpha(x)}^D$,

$$L(x, z) = \frac{\beta_s(x) \alpha^T(x)}{z - z_s(x)} + L_{s1}(x) + O((z - z_s(x))), \quad z_s(x) = z(\gamma_s(x)). \tag{5.1}$$

2⁰. The vector $\mathcal{D}(L(x, q))$ defined by the map (2.6) is *tangent* to the loop $\{\gamma(x), \alpha(x)\}$, i.e.

$$\partial_x z_s(x) = -\alpha_s^T(x) \beta_s(x), \quad \partial_x \alpha_s^T(x) = -\alpha_s^T(x) L_{s1}(x) + \kappa_s(x) \alpha_s^T(x), \tag{5.2}$$

where $\kappa_s(x)$ is a scalar function.

Remark. It is necessary to emphasize, that although the loops $S^1 \mapsto \mathcal{N}^D/SL_r$ are lifted to matrix functions $L'(x, q) \in \mathcal{N}^D$, $x \in \mathbb{R}$, such that

$$L'(x + T, q) = gL'(x, q)g^{-1} + \partial_x g g^{-1}, \quad \gamma = g(x) \in GL_r, \tag{5.3}$$

without loss of generality we may consider functions periodic in x , because L' with the monodromy property (5.3) is gauge equivalent to a periodic matrix function L .

The space \mathcal{A}_{σ}^D of the matrix functions, corresponding to a loop $\sigma = \{\gamma(x), \alpha(x)\}$ in \mathcal{M}_0 , is the space of sections of finite-dimensional affine bundle over the loop, because for any two functions $L_1, L_2 \in \mathcal{A}_{\sigma}^D$ their difference is the Lax matrix, $L_1 - L_2 \in \mathcal{L}^D$. Therefore, for a generic divisor D the space \mathcal{A}_{σ}^D is non-trivial only if $\deg D = N \geq g$. The functional dimension of \mathcal{A}_{σ}^D is equal to $r^2(N - g + 1)$, while the functional dimension of \mathcal{A}^D equals $r^2(N + 1)$.

Lemma 5.1. *If $D = \mathcal{K}$ is the zero divisor of a holomorphic differential dz , then the map*

$$L \in \mathcal{A}^{\mathcal{K}} \longmapsto \{\alpha_s(x), \beta_s(x), \gamma_s(x), \kappa_s(x)\} \tag{5.4}$$

is a bijective correspondence of $\mathcal{A}^{\mathcal{K}}$ and the space of functions periodic in x such that

$$\partial_x z(\gamma_s(x)) = -\alpha_s^T(x)\beta_s(x), \quad \sum_{s=1}^{rg} \beta_s(x)\alpha_s(x)^T = 0, \tag{5.5}$$

modulo the gauge transformations

$$\alpha_s(x) \longmapsto \lambda_s(x)\alpha_s(x), \quad \beta_s \longmapsto \lambda_s^{-1}(x)\beta_s(x), \quad \kappa_s(x) \longmapsto \kappa_s(x) + \partial_x \ln \lambda_s(x), \tag{5.6}$$

$$\alpha_s(x) \longmapsto W(x)^T \alpha_s(x), \quad \beta_s(x) \longmapsto W^{-1}(x)\beta_s(x), \tag{5.7}$$

where $\lambda_s(x)$ is a non-vanishing function periodic in x and $W(x) \in \widehat{GL}_r$ a periodic non-degenerate matrix function.

Note that from (5.2) it follows that locally in the neighborhood of $\gamma_s(x)$ the matrix function $L(x, Q) \in \mathcal{A}_\sigma^D$ can be regarded as a connection of the bundle $\widehat{\mathcal{V}}$ over $S^1 \times \Gamma$ along the loop $\{\gamma(x), \alpha(x)\}$. Indeed, if \mathcal{F} is a space of local sections of this bundle, which can be identified with the space of meromorphic vector functions $f(x, z)$ that have the form (2.1) in the neighborhood of γ_s , then

$$\left(\partial_x + L^T(x, z) \right) f(x, z) \in \mathcal{F}_s. \tag{5.8}$$

Another characterization of the constraints (5.2) is as follows.

Lemma 5.2. *A meromorphic matrix-function L in the neighborhood of $\gamma_s(x)$ with a pole at $\gamma_s(x)$ satisfies the constraints (5.2) if and only if there exists a holomorphic matrix function $\Phi_s(x, z)$ with at most a simple zero of $\det \Phi_s$ at γ_s such that L is gauge equivalent*

$$L = \Phi_s \widehat{L} \Phi_s^{-1} + \partial_x \Phi_s \Phi_s^{-1} \tag{5.9}$$

to a holomorphic matrix function \widehat{L} .

The tangent space to \mathcal{A}^D is the space of functions of x with values in the tangent space to the space of Lax matrices $T(\mathcal{L}^D)$.

Lemma 5.3. *Let $L \in \mathcal{A}_\sigma^D$ and $M \in \mathcal{N}_{\gamma(x), \alpha(x)}^{D'}$, then the commutator $[\partial_x - L, M] = M_x + [M, L]$ is a tangent vector to \mathcal{A}^D at L if and only if its divisor of poles outside of $\gamma_s(x)$ is not greater than D .*

From Eqs. (5.2) it follows that the Laurent expansion of the matrix function $T = M_x + [L, M]$ at the point $\gamma_s(x)$ has the form (2.39), where \dot{z}_s and $\dot{\alpha}_s$ are given by formulae (2.7, 2.8). That proves that T is a tangent vector to \mathcal{L}^D .

Lemma 5.3 shows that the zero-curvature equation

$$L_t = M_x + [M, L] \tag{5.10}$$

is a well-defined system, whenever we can define $M(L)$, such that the conditions of the lemma are satisfied. Our goal is to construct the zero-curvature equations that are equivalent to *differential equations*. That requires $M(L)$ to be expressed in terms of L and its derivatives in x .

It is instructive enough to consider the case when all the multiplicities of the points $P_i \in D$ equal $m_i = 1$. Let \mathcal{A}_0^D be an open set in \mathcal{A}^D such that the singular part of $L \in \mathcal{A}_0^D$ at P_i has different eigenvalues

$$L(x, q) = w_i^{-1} C_i(x) u^{(i)}(x) C_i^{-1}(x) + O(1), \quad w_i = w_i(q), \quad w_i(P_i) = 0, \\ u^{(i)} = \text{diag} \left(u_1^{(i)}(x), \dots, u_r^{(i)}(x) \right), \quad u_k^{(i)}(x) \neq u_l^{(i)}(x), \quad k \neq l. \quad (5.11)$$

Lemma 5.4. *Let $L(x, w)$ be a formal Laurent series*

$$L = \sum_{j=-1}^{\infty} l_j(x) w^j \quad (5.12)$$

such that $l_{-1}(x) = C(x)u(x)C^{-1}(x)$, where u is a diagonal matrix, with distinct diagonal elements. Then there is a unique formal solution $\Psi_0 = \Psi_0(x, w)$ of the equation

$$(\partial_x - L(x, w)) \Psi(x, w) = 0, \quad (5.13)$$

which has the form

$$\Psi_0(x, w) = C(x) \left(\sum_{s=0}^{\infty} \xi_s(x) w^s \right) e^{\int_{x_0}^x h(x', w) dx'}, \quad h = \text{diag}(h_1, \dots, h_r), \quad (5.14)$$

normalized by the conditions

$$\xi_0^{ij} = \delta^{ij}, \quad \xi_s^{ii}(x) = 0. \quad (5.15)$$

The coefficients $\xi_s(x)$ of (5.14) and the coefficients $h_s(x)$ of the Laurent series

$$h(x, w) = \sum_{s=-1}^{\infty} h_s(x) w^s, \quad h_{-1} = u, \quad (5.16)$$

are differential polynomials of the matrix elements of L .

Substitution of (5.14) into (5.13) gives a system of the equations, which have the form

$$h_s - [u, \xi_{s+1}] = R(\xi_0, \dots, \xi_s; h_0, \dots, h_{s-1}), \quad s = -1, 0, 1, \dots \quad (5.17)$$

They recursively determine the off-diagonal part of ξ_{s+1} , and the diagonal matrix h_s as polynomial functions of matrix elements of $l_i(x)$, $i \leq s$.

Corollary 5.1. *Let Ψ_0 be the formal solution (5.14) of Eq. (5.13). Then for any diagonal matrix E the expression $w^{-m} \Psi_0 E \Psi_0^{-1}$ does not depend on x_0 , and is formally meromorphic, i.e. it has the form*

$$w^{-m} \Psi_0 E \Psi_0^{-1} = \sum_{s=-m}^{\infty} m_s(x) w^{-s}. \quad (5.18)$$

The coefficients $m_s(x)$ are differential polynomials on the matrix elements of the coefficients $l_i(x)$.

Expression (5.18) is meromorphic and does not depend on x_0 , because the essential singularities of the factors commute with E and so cancel each other.

We are now in position to define matrices M_a ,

$$a = (P_i, m; l), \quad m \geq 1, \quad l = 1, \dots, r, \tag{5.19}$$

which are differential polynomials on entries of L , and satisfy the conditions of Lemma 5.3. Let $\Psi_0(x, q) = \Psi_0(x, w(q))$ be the formal solution of Eq. (5.13) constructed above for the expansion (5.11) of $L \in \mathcal{A}_0^D$ at P_i . Then, we define $M_{(i,m;l)}(x, q)$ as the unique meromorphic matrix function, which has the form (2.3), (2.4) at the points $\gamma_s(x)$, and is holomorphic everywhere else except at the point P_i , where

$$M_{(i,m;l)}(x, q) = w^{-m}(q)\Psi_0(x, q)E_l\Psi_0^{-1}(x, q) + O(1), \quad E_l^{ij} = \delta_l^i\delta^{jl}. \tag{5.20}$$

As before, we normalize $M_{(i,m;l)}$ by the condition $M_{(i,m;l)}(x, P_0) = 0$.

It is necessary to mention, that M_a , as a function of L , is defined only locally, because it depends on a representation of the singular part of L at P_i in the form (5.11).

Theorem 5.1. *The equations*

$$\partial_a L = \partial_x M_a + [M_a, L], \quad a = (P_i, m; l) \tag{5.21}$$

define a hierarchy of commuting flows on \mathcal{A}_0^D .

Let the coefficients of (5.12) be periodic functions of x . Then, Lemma 5.4 implies that

$$\Psi_0(x + T, w) = \Psi_0(x, w)e^{p(w)}, \quad p = \int_0^T h(x, w)dx. \tag{5.22}$$

Therefore, the columns of Ψ_0 are Bloch solutions of Eq. (5.13), i.e. the solutions that are eigenvectors of the monodromy operator. The diagonal elements of the matrix $p(w)$ are the formal quasimomentum of the operator (5.13).

Our next goal is to show that for $D_{\mathcal{K}} \geq 0$ the zero curvature equations are Hamiltonian on suitable symplectic leaves, and identify their Hamiltonians with coefficients of the quasimomentum matrices p_i corresponding to the expansion (5.11) of L at the punctures P_i ,

$$p_i(w) = \sum_{s=-1}^{\infty} H_{(i,s)} w^s, \quad H_{(i,s)} = \text{diag} \{H_{(i,s;l)}\}. \tag{5.23}$$

Let us fix a holomorphic differential dz with simple zeros, and a set of diagonal matrix functions $v^{(i)}(x)$. Then for a divisor D , such $D_{\mathcal{K}}$ is effective, we define first a subspace \mathcal{B}^D of \mathcal{A}_0^D by the constraints

$$\partial_x \left(u^{(i)}(x) - v^{(i)}(x) \right) = 0, \tag{5.24}$$

where $u^{(i)}$ are the matrices of eigenvalues (5.11) of the singular parts of $L \in \mathcal{A}_0^D$. Next we define a foliation of \mathcal{B}^D . The leaves $\widehat{\mathcal{P}}_0^D$ of the foliation are parameterized by sets of constant diagonal matrices $c^{(m)}$ with distinct diagonal elements, and are defined by the equations

$$u^{(m)}(x) - v^{(m)}(x) = c^{(m)}, \quad \text{if } dz(P_m) \neq 0. \tag{5.25}$$

We would like to stress the difference between the constraints (5.24) and (5.25). Equations (5.24) imply that for all the points of the divisor D the differences $(u^{(i)}(x) - v^{(i)}(x))$ are x -independent matrices. For $P_m \in D_{\mathcal{K}}$ we require additionally that the difference equals the fixed matrix.

As before, we define a two-form on $\widehat{\mathcal{P}}_0^D$ by formula (4.2), where now

$$\Omega(q) = \text{Tr} \left(\int_{x_0}^{x_0+T} (\Psi^{-1} \delta L \wedge \delta \Psi) dx - (\Psi^{-1} \delta \Psi)(x_0) \wedge \delta p \right) \tag{5.26}$$

and Ψ is the matrix of the Bloch solutions of (5.13), i.e.

$$(\partial_x - L(x, q))\Psi(x, q) = 0, \quad \Psi(x + T, q) = \Psi(x, q)e^{P(q)}. \tag{5.27}$$

We would like to emphasize that this definition is a slight modification of the formula for symplectic structure for soliton equations, proposed in [9]. The second term in (5.26) gives zero contribution in the conventional theory. It is here to remove the dependence on the choice of x_0 in the definition as may be seen as follows. The monodromy property (5.27) implies

$$\text{Tr} \left(\Psi^{-1} \delta L \wedge \delta \Psi \right) (x + T) - \text{Tr} \left(\Psi^{-1} \delta L \wedge \delta \Psi \right) (x) = \text{Tr} \left((\Psi^{-1} \delta L \Psi)(x) \wedge \delta p \right). \tag{5.28}$$

Using the equations $\delta L \Psi = \delta \Psi_x - L \delta \Psi$, we obtain

$$\text{Tr} \left(\Psi^{-1} \delta L \Psi \right) = \text{Tr} \left(\partial_x \left(\Psi^{-1} \delta \Psi \right) \right). \tag{5.29}$$

Hence, the form Ω does not depend on a choice of the initial point $x = x_0$.

The same arguments as before show that ω when restricted to $\widehat{\mathcal{P}}_0^D$ does not depend on the normalization of the Bloch solutions.

Theorem 5.2. *The formula (4.2) with Ω given by (5.26) defines a closed two-form on $\widehat{\mathcal{P}}_0^D$. This is gauge invariant with respect to the affine gauge group GL_r .*

If $D \geq \mathcal{K}$, then the contraction of ω by the vector field ∂_a defined by (5.21) equals

$$i_{\partial_a} \omega = \delta H_a, \tag{5.30}$$

where for $a = (P_l, m; l)$,

$$H_{(i,m;l)} = -\text{res}_{P_l} \text{Tr} \left(w^{-m} E_l p \right) dz, \tag{5.31}$$

and p is the quasi-momentum matrix.

The proof of this theorem proceeds along identical lines to the proof of the stationary analogs of these results presented above. First, we show that under the gauge transformation $L' = g^{-1} L g - g^{-1} \partial_x g$, $\Psi' = g^{-1} \Psi$ the form Ω gets transformed to

$$\Omega' = \Omega + \text{Tr} \int_{x_0}^{x_0+T} (2\delta h \wedge \delta L - 2L \delta h \wedge \delta h + \delta h_x \wedge \delta h) dx, \tag{5.32}$$

where $\delta h = \delta g g^{-1}$. Note that the last term does not contribute to the residues. The first two terms are meromorphic on Γ with poles at γ_s and $P_l \in D$, only. Therefore, a sum of their contributions to residues of $\Omega' dz$ equals zero. Hence, ω does descend to a form on

$$\widehat{\mathcal{P}}^D = \widehat{\mathcal{P}}_0^D / \widehat{GL}_r. \tag{5.33}$$

Using (5.32) for the gauge transformation (5.9), where Φ_s depends on a point z in the neighborhood of γ_s , we obtain

$$\text{res}_{\gamma_s} \Omega dz = -2 \int_{x_0}^{x_0+T} \left(\delta \kappa_s(x) \wedge \delta z_s(x) + \sum_{i=1}^r \delta \beta_s^i(x) \wedge \delta \alpha_s^i(x) \right) dx. \tag{5.34}$$

From (5.14) we obtain that if $dz(P_i) = 0$, then

$$\begin{aligned} & \text{res}_{P_i} \Omega dz \\ &= \text{Tr} \left(\int_{x_0}^{x_0+T} \left(\delta u^{(i)}(x) \wedge \int_{x_0}^x \delta u^{(i)}(y) dy \right) dx - \delta u^{(i)}(x_0) \wedge \int_{x_0}^{x_0+T} \delta u^{(i)}(x) dx \right). \end{aligned} \tag{5.35}$$

Equations (5.24) imply that the restriction of $\delta u^{(i)}$ to $\widehat{\mathcal{P}}_0^D$ is x -independent. Then, from (5.35) it follows that the points $P_i \in \mathcal{K}$ give zero contribution to ω . From (5.14) and (5.25) it follows that the form $\delta \Psi \Psi^{-1}$ when restricted to $\widehat{\mathcal{P}}_0^D$ is holomorphic in the neighborhood of $P_m \in D_{\mathcal{K}}$. Therefore, in this neighborhood

$$\left(\delta \Psi_x \Psi^{-1} + \delta \Psi \Psi_x^{-1} \right) \Big|_{\widehat{\mathcal{P}}_0^D} = 0(1). \tag{5.36}$$

Using this equality we obtain that on \mathcal{P}_0^D the following equation holds:

$$\text{res}_{P_m} \Omega dz = -2 \text{res}_{P_m} \text{Tr} \left(\int_{x_0}^{x_0+T} \left(L \delta \Psi \Psi^{-1} \wedge \delta \Psi \Psi^{-1} dx \right) \right) dz. \tag{5.37}$$

Therefore, restricted to $\widehat{\mathcal{P}}_0^D$ the form ω is equal to the integral over the period of (4.24). The proof of Eq. (5.30), where H_a is given by (5.31) is almost identical to the proof of (4.26).

Important remark. The formulae (5.34) and (5.37) do not directly imply that ω restricted to $\widehat{\mathcal{P}}$ is non-degenerate, because of the constraints (5.24). The conventional theory of the soliton equations, and results of the next section provide some evidence that it is non-degenerate for $D_{\mathcal{K}} \geq 0$, although at this moment the author does not know a direct proof of that. Anyway, Eq. (5.30) shows that Eqs. (5.21) are Hamiltonian on suitable subspaces of \mathcal{P}^D . Then, commutativity of flows implies

$$\{H_a, H_b\} = 0. \tag{5.38}$$

The previous results can be easily extended for the case when the leading coefficient of the singular part of L at the puncture P_i has multiple eigenvalues.

Lemma 5.5. *Let $L(x, w)$ be a formal Laurent series (5.12) such that $l_{-1} = C(x)uC^{-1}(x)$, and $u = u_i \delta^{ij}$ is a constant diagonal matrix. Then there is a unique formal solution $\Psi_0 = \Psi_0(x, w)$ of Eq. (5.13), which has the form*

$$\Psi_0(x, w) = C(x) \left(\sum_{s=0}^{\infty} \xi_s(x) w^{-s} \right) \mathcal{T}(x, w), \quad \mathcal{T}(x_0, w) = 1, \tag{5.39}$$

where

$$\xi_0^{ij} = \delta^{ij}; \quad \xi_s^{ij}(x) = 0, \text{ if } u_i = u_j, s \geq 1, \tag{5.40}$$

and the logarithmic derivative $h(x, w)$ of \mathcal{T} is a formal series with non vanishing entries only for indices (i, j) , such that $u_i = u_j$, i.e.

$$h = \partial_x \mathcal{T} \mathcal{T}^{-1} = u w^{-1} + \sum_{s=0}^{\infty} h_s(x) w^s, \quad h^{ij} = 0, \text{ if } u_i \neq u_j. \tag{5.41}$$

The coefficients $\xi_s(x)$ of (5.39) and the coefficients $h_s(x)$ of (5.41) are differential polynomials of the matrix elements of L .

Substitution of (5.39) in (5.13) gives a system of the equations which have the form (5.17) They recursively determine ξ_{s+1}^{ij} for indices (i, j) such that $u_i \neq u_j$ and the matrix h_s , as polynomial functions of the matrix elements of $l_i(x)$, $i \leq s$.

Corollary 5.2. *Let Ψ_0 be the formal solution (5.39) of Eq. (5.13). Then for any diagonal matrix $E = E_i \delta^{ij}$ such that $E_i = E_j$, if $u_i = u_j$, the expression $w^{-m} \Psi_0 E \Psi_0^{-1}$ does not depend on x_0 , and is formally meromorphic. The coefficients $m_s(x)$ of its Laurent expansion (5.18) are differential polynomials of the entries of the coefficients $l_i(x)$.*

The expression $w^{-m} \Psi_0 E \Psi_0^{-1}$ is meromorphic and does not depend on x_0 because $[\mathcal{T}, E] = 0$.

The corollary implies that if singular parts of L at the punctures P_i have multiple eigenvalues, then the commuting flows are parameterized by sets

$$a = (P_i, m; E_\lambda), \tag{5.42}$$

where E_λ is a diagonal matrix that satisfies the condition of Corollary 5.2. The Hamiltonians of the corresponding equations are equal to

$$H_a = -\text{res}_{P_i} \text{Tr} \left(w^{-m} E_\lambda \int_0^T h(x) dx \right) dz. \tag{5.43}$$

Example. Field analog of the elliptic CM system. Let us consider the zero curvature equation on the elliptic curve with one puncture. We use the same notation as in Sect. 4. In the gauge $\alpha_s = e_s$, $e_s^j = \delta_s^j$, the phase space can be identified with the space of elliptic matrix functions such that L^{ij} has a pole at the point $q_j(x)$ and $z = 0$, only. From (5.2) it follows that the residue of L^{jj} at q_j equals $-q_{jx}$. Therefore, $L^{jj} = p_j + q_{jx}(\zeta(z) - \zeta(z - q_j) - \zeta(q_j))$. Equation (5.2) implies also that $L^{ji}(q_j) = 0$, $i \neq j$.

Let us assume, as in the case of the elliptic CM system, that the singular part of L at the puncture $z = 0$ is a point of the orbit of the adjoint action corresponding to the diagonal matrix $\text{diag}(r - 1, -1, \dots, -1)$. Then, taking into account the momentum map corresponding to the gauge transformation by diagonal matrices, we get that the non-stationary analog of the Lax matrix for the CM system has the form

$$L^{ii} = p_i + q_{ix} (\zeta(z) - \zeta(z - q_i) - \zeta(q_i)), \tag{5.44}$$

$$L^{ij} = f_i f_j \frac{\sigma(z + q_i - q_j) \sigma(z - q_i) \sigma(q_j)}{\sigma(z) \sigma(z - q_j) \sigma(q_i - q_j) \sigma(q_i)}, \quad i \neq j. \tag{5.45}$$

The values f_i^2 are fixed to

$$f_i^2 = 1 + q_{ix}, \quad \sum_{i=1}^r q_{ix} = 0. \tag{5.46}$$

According to (5.34), the symplectic form equals

$$\omega = \int_0^T \left(\sum_{i=1}^r \delta p_i(x) \wedge \delta q_i(x) \right) dx \mapsto \{p_i(x), q_j(y)\} = \delta_{ij} \delta(x - y). \quad (5.47)$$

The commuting Hamiltonians are coefficients of the Laurent expansion at $z = 0$ of the quasimomentum, corresponding to the only simple eigenvalue of the singular part of L at $z = 0$. To find them we look for the solution of (5.13) in the form

$$\psi = C(x, z) e^{\int_0^x h(x', z) dx'}, \quad (5.48)$$

$$C = \left(\sum_{s=0}^{\infty} C^{(s)}(x) z^s \right), \quad h = \sum_{s=-1}^{\infty} h_s(x) z^s, \quad (5.49)$$

where $C^{(0)}$ is the eigenvector of the singular part of L , corresponding to the eigenvalue $(r - 1)$, i.e.

$$C_i^{(0)} = f_i, \quad (5.50)$$

and the coefficients $C^{(s)}$ for $s > 0$ are vectors, normalized by the condition

$$\sum_{i=1}^r f_i C_i^{(s)} = 0, \quad s > 0. \quad (5.51)$$

Substitution of (5.44,5.45) into (5.13) gives a system of the equations for the coordinates C_i of the vector C :

$$\begin{aligned} \partial_x C_i + h C_i &= q_{ix} C_i [\zeta(z) - \zeta(z - q_i) - \zeta(q_i)] \\ &+ f_i \sum_{j \neq i} f_j C_j [\zeta(z) - \zeta(z - q_j) + \zeta(q_i - q_j) - \zeta(q_i)], \end{aligned} \quad (5.52)$$

where we use the identity

$$\frac{\sigma(z + q_i - q_j) \sigma(z - q_i) \sigma(q_j)}{\sigma(z) \sigma(z - q_j) \sigma(q_i - q_j) \sigma(q_i)} = \zeta(z) - \zeta(z - q_j) + \zeta(q_i - q_j) - \zeta(q_i). \quad (5.53)$$

Taking the expansion of (5.52) at $z = 0$, we find recursively the coefficients of $C_i^{(s)}$ and densities h_s of the Hamiltonians. The first two steps are as follows:

The coefficients at z^{-1} of the right- and left-hand sides of (5.52) give

$$h_{-1} = q_{ix} + \sum_{j \neq i} f_j^2 = q_{ix} + (r - f_i^2) = r - 1. \quad (5.54)$$

The next system of equations is

$$f_{ix} + f_i h_0 + (r - 1) C_i^{(1)} = p_i f_i + q_{ix} C_i^{(1)} + f_i \sum_{j \neq i} \left(f_j C_j^{(1)} + f_j^2 V_{ij} \right), \quad (5.55)$$

where

$$V_{ij} = \zeta(q_j) + \zeta(q_{ij}) - \zeta(q_i), \quad q_{ij} = q_i - q_j. \tag{5.56}$$

Using (5.51), we get

$$rC_i^{(1)} + f_i h_0 = p_i f_i - f_{ix} + f_i \sum_{j \neq i} f_j^2 V_{ij}. \tag{5.57}$$

Multiplying (5.57) by f_i and taking a sum over i , we find upon using (5.51) and skew-symmetry of V_{ij} ,

$$r h_0 = \sum_{i=1}^r p_i f_i^2 = \sum_{i=1}^r p_i (1 + q_{ix}). \tag{5.58}$$

In the same way we get the system of equations for $C_i^{(2)}$,

$$\begin{aligned} rC_i^{(2)} + \partial_x C_i^{(1)} + h_0 C_i^{(1)} + h_1 f_i \\ = p_i C_i^{(1)} + q_{ix} f_i \wp(q_i) + f_i \sum_{j \neq i} f_j \left(C_j^{(1)} V_{ij} + f_j \wp(q_j) \right). \end{aligned} \tag{5.59}$$

Consequently the expression for the density of the second Hamiltonian is

$$\begin{aligned} r^2 h_1 \\ = r \sum_i \left[q_{ix} f_i^2 \wp(q_i) + C_i^{(1)} (f_{ix} + p_i f_i) + \sum_{j \neq i} \left(f_i^2 f_j C_j^{(1)} V_{ij} + f_i^2 f_j^2 \wp(q_j) \right) \right] \\ = r \sum_i \left[(r - 1) f_i^2 \wp(q_i) + C_i^{(1)} \left(f_{ix} + p_i f_i + \sum_{j \neq i} f_i f_j^2 V_{ji} \right) \right]. \end{aligned} \tag{5.60}$$

For the first line we have used the equation $\sum_i (f_i C_{ix}^{(1)} + f_{ix} C_{ix}^{(1)}) = 0$. From (5.57) it follows that the second term in (5.60) equals

$$II = -r h_0^2 + \sum_i \left(p_i^2 f_i^2 - f_{ix}^2 + \sum_{j \neq i} (f_i^2)_x f_j^2 V_{ij} + \sum_{j, k \neq i} f_i^2 f_j^2 f_k^2 V_{ij} V_{ki} \right). \tag{5.61}$$

For any triple of distinct integers $i \neq j \neq k \neq i$ the following equation holds:

$$V_{ij} V_{ki} + V_{jk} V_{ij} + V_{ki} V_{jk} = -\wp(q_i) - \wp(q_j) - \wp(q_k). \tag{5.62}$$

In order to prove (5.62), it is enough to check that the left-hand side, which is a symmetric function of all the variables q_i, q_j, q_k , as a function of the variable q_i , has double pole at $q_i = 0$, and is regular at $q_i = q_j$. In the same way one can obtain the well-known relation

$$V_{ij} V_{ji} = -\wp(q_i) - \wp(q_j) - \wp(q_{ij}). \tag{5.63}$$

Equations (5.62, 5.63) imply

$$\sum_i \sum_{j,k \neq i} f_i^2 f_j^2 f_k^2 V_{ij} V_{ki} = - \sum_i \left(r f_i^2 (r - f_i^2) \wp(q_i) + \sum_{j \neq i} f_i^2 f_j^4 \wp(q_{ij}) \right). \quad (5.64)$$

From (5.56) it follows

$$\begin{aligned} \sum_{j \neq i} (f_i^2)_x f_j^2 V_{ij} &= \frac{1}{2} \sum_{j \neq i} \left((f_i^2)_x f_j^2 - f_i^2 (f_j^2)_x \right) (\zeta(q_{ij}) - 2\zeta(q_i)) \\ &= -\frac{1}{2} \sum_i \left(2r q_{ixx} \zeta(q_i) - \sum_{j \neq i} q_{ijxx} \zeta(q_{ij}) \right) \\ &\quad + \frac{1}{2} \sum_{j \neq i} (q_{ixx} q_{jx} - q_{jxx} q_{ix}) \zeta(q_{ij}). \end{aligned} \quad (5.65)$$

The first sum is equal to

$$\begin{aligned} \frac{1}{2} \sum_i \left(2r q_{ixx} \zeta(q_i) - \sum_{j \neq i} q_{ijxx} \zeta(q_{ij}) \right) \\ = \frac{1}{2} \sum_i \left(2r q_{ix} f_i^2 \wp(q_i) - \sum_{j \neq i} q_{ijx}^2 \wp(q_{ij}) \right) + \partial_x F, \end{aligned} \quad (5.66)$$

where

$$F = \frac{1}{2} \sum_i \left(2r f_i^2 \zeta(q_i) - \sum_{j \neq i} q_{ijx} \zeta(q_{ij}) \right). \quad (5.67)$$

The first terms in (5.60), (5.64), (5.65) cancel each other. The function F , as a function of the variable q_i , has poles at the points $0, q_j, j \neq i$ and the sum of its residues at these points equals

$$r f_i^2 - \sum_{j \neq i} q_{ijx} = r. \quad (5.68)$$

Therefore, it has the same monodromy properties with respect to all the variables. The functions $q_i(x)$ represent loops on the elliptic curve. Therefore, $q_i(x + T) = q_i(x) + b_i$, where b_i is a period of the elliptic curve. The constraint (5.46) implies $\sum_i b_i = 0$. Then, from (5.68) it follows that F is a *periodic* function of x . The densities of the Hamiltonians are defined up to a total derivative of periodic functions in x . Hence, a density of the

second Hamiltonian of the hierarchy equals

$$r^2 \tilde{h}_1 = -\frac{1}{r} \left(\sum_i (p_i(1 + q_{ix})) \right)^2 + \sum_i \left(p_i^2(1 + q_{ix}) - \frac{q_{ixx}^2}{4(1 + q_{ix})} \right) \tag{5.69}$$

$$-\frac{1}{2} \sum_{j \neq i} ((q_{ixx}q_{jx} - q_{jxx}q_{ix}) \zeta(q_{ij})) \tag{5.70}$$

$$+\frac{1}{2} \sum_{j \neq i} \left((1 + q_{ix})(1 + q_{jx})^2 + (1 + q_{jx})(1 + q_{ix})^2 - q_{ijx}^2 \right) \wp(q_{ij}). \tag{5.71}$$

The transformation $p_i \rightarrow p_i + f(x)$ does not change h_s for $s > 0$. In particular, the first two terms in (5.69) can be rewritten as

$$-\frac{1}{r} \left(\sum_i (p_i(1 + q_{ix})) \right)^2 + \sum_i p_i^2(1 + q_{ix}) = \frac{1}{2r} \sum_{i,j} (p_i - p_j)^2(1 + q_{ix})(1 + q_{jx}). \tag{5.72}$$

The symplectic form (5.47) restricted to the subspace

$$\sum_i q_i = 0, \quad \sum_i p_i = 0, \tag{5.73}$$

is non-degenerate. The Hamiltonians $H_s = \int_0^T h_s(x) dx$ restricted to this space generate a hierarchy of commuting flows, which we regard as field analog of the elliptic CM system. For $r = 2$ the Hamiltonian $2H_1$ has the form (1.10), where $q = q_1 = -q_2$, $p = p_1 = -p_2$.

6. The Algebraic-Geometric Solutions

So far, our consideration of the Bloch solutions (5.27) has been purely local and formal. For generic $L \in \mathcal{A}_0^D$ the series (5.14, 5.16) for the formal solutions $\Psi(x, q)$, and quasimomentum have zero radius of convergence. The main goal of this section is to construct algebraic-geometric solutions of the zero curvature equations, for which these series do converge and, moreover, have meromorphic continuations on a compact Riemann surface.

Let $\widehat{T}(q)$ be a restriction of the monodromy operator $f(x) \rightarrow f(x + T)$ to the space of solutions of the equation $(\partial_x - L(x, q))f = 0$, where f is a vector function. Then, we define the Riemann surface $\widehat{\Gamma}$ of the Bloch solutions by the characteristic equation

$$R(\mu, q) \equiv \det(\mu - \widehat{T}(q)) = \mu^r + \sum_{j=1}^r R_j(q)\mu^{r-j} = 0. \tag{6.1}$$

Lemma 6.1. *The coefficients $R_j(q)$ of the characteristic equation (6.1) are holomorphic functions on Γ except at the points P_i of the divisor D .*

Proof. In the basis defined by columns of the fundamental matrix of solutions to the equation $(\partial_x - L)F(x, q; x_0) = 0$, $F(x_0, q, x_0) = 1$, the operator $\widehat{T}(q)$ can be identified with the matrix

$$\widehat{T}(q) = F(x_0 + T, q; x_0). \tag{6.2}$$

A priori this matrix is holomorphic on Γ except at the points of the divisor D and at points of the loops $\gamma_s(x)$, where L has singularities. From Lemma 5.2 it follows that in the neighborhood of the loop we have

$$F = \Phi_s(x, q)\widetilde{F}(x, q)\Phi_s^{-1}(x, q), \quad \widetilde{F}(x_0, q) = 1, \tag{6.3}$$

where \widetilde{F} is a holomorphic matrix function, and Φ_s is defined by (2.14), (2.18), (2.19). The function Φ_s is periodic, because γ_s, α_s are periodic. In the neighborhood of the loop γ_s , the functions $R_j(q)$ coincide with the coefficients of the characteristic equation for \widetilde{F} . Therefore, they are holomorphic in that neighborhood. The lemma is then proved.

It is standard in the conventional spectral theory of periodic linear operators that for a generic operator the Riemann surface of the Bloch functions is smooth and has infinite genus. For algebro-geometric or finite-gap operators the corresponding Riemann surface is singular, and is birational equivalent to a *smooth* algebraic curve.

It is instructive to consider first, as an example of such operators, the case, when L does not depend on x , i.e. $L \in \mathcal{L}^D$. In this case the equation $(\partial_x - L)\psi = 0$ can be easily solved. The Bloch solutions have the form

$$\psi = \psi_0 e^{kx}, \tag{6.4}$$

where ψ_0 is an eigenvector of L , and k is the corresponding eigenvalue. These solutions are parameterized by points Q of the spectral curve $\widehat{\Gamma}_0$ of L . The image of $\widehat{\Gamma}_0$ under the map into $\mathbf{C}^1 \times \Gamma$ defined by formula

$$(k, q) \in \widehat{\Gamma}_0 \mapsto (\mu = e^{kT}, q) \in \mathbf{C}^1 \times \Gamma \tag{6.5}$$

is the Riemann surface $\widehat{\Gamma}$ defined by (6.1), where the coefficients are symmetric polynomials of $e^{k_i(q)T}$.

For example, if $\widehat{\Gamma}_0$ is defined by the equation

$$k^2 + u(q) = 0, \tag{6.6}$$

where $u(q)$ is a meromorphic function with double poles at the points of D , then $\widehat{\Gamma}$ is defined by the equation

$$\mu^2 + 2R_1\mu + 1 = 0, \quad R_1(q) = \cosh(\sqrt{u(q)}). \tag{6.7}$$

The Riemann surface defined by (6.7) is singular. Projections onto Γ of the points of self-intersection of $\widehat{\Gamma}$ are roots of the equation

$$u(q) = \left(\frac{\pi N}{2T}\right)^2, \tag{6.8}$$

where N is an integer. The coefficient $u(q)$ has poles of the second order at D , $u = a_i^2 w^{-2} + O(w^{-1})$, where w is a local coordinate at $P_i \in D$. Therefore, as $|N| \rightarrow \infty$,

the roots of (6.7) tend to the points of D . The coordinates of the singular points $q_{i,N}$ that tend to P_i equal

$$w(q_{i,N}) = 2T a_i (\pi N)^{-1} + O(N^{-2}). \tag{6.9}$$

As usual in perturbation theory, for generic L each double eigenvalue $q_{i,n}$ splits into two smooth branch points $q_{i,n}^\pm$. By analogy with the conventional theory we expect, that if L is an analytic function of x , then the differences $|w(q_{i,N}) - w(q_{i,N}^\pm)| < O(N^{-k})$ will decay faster than any power of N^{-1} .

Localization of the branch points is a key element in the construction [19] of a theory of theta-functions for infinite genus hyperelliptic curves of the Bloch solutions for periodic Sturm–Liouville operators. In [20] a general approach for the construction of Riemann surfaces of the Bloch functions was proposed. The model of the spectral curves developed in [20] was chosen in [21] as a starting point of the theory of general (non-hyperelliptic) infinite-genus Riemann surfaces. It was shown that for such surfaces many classical theorems of algebraic geometry take place.

Algebro-geometric or finite-gap operators can be seen, as operators for which there are only a finite number of multiple eigenvalues that split into smooth branch points. Let $\widehat{\Gamma}$ be a smooth genus \widehat{g} algebraic curve that is an r -branch cover of Γ . Note that unlike the stationary case, for given a rank r there is no relation between \widehat{g} , and the genus g of Γ . As \widehat{g} increases the dimension of the space of r -sheeted cover increases. It equals $2(\widehat{g} - rg + r - 1)$.

Assume that the preimages P_i^l, P_0^l on $\widehat{\Gamma}$ of the points of a divisor D , and a point P_0 on Γ are not branch points. The definition of the Baker-Akhiezer function corresponding to this data and to a non-special degree $\widehat{g} + r - 1$ divisor $\widehat{\gamma}$ on $\widehat{\Gamma}$ is as follows:

1⁰. ψ is a meromorphic vector function on $\widehat{\Gamma}$ except at the points P_i^l . Its divisor of poles on $\widehat{\Gamma}$ outside of P_i^l is not greater than $\widehat{\gamma}$.

2⁰. In the neighborhood of P_i^l the vector function ψ has the form

$$\psi = \xi_{i,l}(q, t) \exp \left(\sum_m t_{(i,m;l)} w^{-m} \right), \tag{6.10}$$

where $\xi_{i,l}(q, t)$ is a holomorphic vector-function.

3⁰. Evaluation of ψ at the punctures P_0^l are vectors with coordinates $(\psi(P_0^l))^{(i)} = \delta^{il}$.

Theorem 6.1. *Let $\psi(q, t)$ be the Baker–Akhiezer vector function associated with a non-special divisor $\widehat{\gamma}$ on $\widehat{\Gamma}$. Then, there exist unique matrix functions $M_{(i,m;l)}(q, t) \in \mathcal{N}_{\gamma(t), \alpha(t)}^D$ such that the equations*

$$(\partial_{(i,m;l)} - M_{(i,m;l)}) \psi(q, t) = 0 \tag{6.11}$$

hold.

Now, let $v_l^{(i)}(x)$ be a set of periodic functions, $\int_0^T v_l^{(i)} dx = 0$, and $u_l^{(i)}$ be a set of constants. Then the change of the independent variables

$$t_{(i,1;l)} = x u_l^{(i)} + v_l^{(i)}(x) + t'_{(i,1;l)} \tag{6.12}$$

define the Baker–Akhiezer function ψ , as a function of (q, t) and the variable x . From (6.11) it follows that

$$(\partial_x - L)\psi = 0, \quad L = \sum_{i,l} \left(u_l^{(i)} + \partial_x v_l^{(i)} \right) M_{(i,1;l)}. \tag{6.13}$$

As follows from Lemma 5.2, the vector $\mathcal{D}(M_a)$, $a = (i, m; l)$, corresponding to M_a under (2.6), is tangent to $(\gamma_s(t_a), \alpha_s(t_a))$. Therefore, $\mathcal{D}(L)$ is tangent to $(\gamma_s(x), \alpha_s(a))$.

In general, L constructed above is not a periodic function of x . It is periodic, if we impose additional constraints on the set of data that are the curve $\widehat{\Gamma}$ and the constants $u_l^{(i)}$. We call the set $\{\widehat{\Gamma}, u_l^{(i)}\}$ admissible if there exists a meromorphic differential dp on $\widehat{\Gamma}$ which has second order poles at P_i^l ,

$$dp = -u_l^{(i)} dw \left(w^{-2} + O(1) \right), \tag{6.14}$$

and such that all periods of dp are multiples of $2\pi i/T$,

$$\oint_c dp = \frac{2\pi i m_c}{T}, \quad m_c \in \mathbb{Z}, \quad c \in H_1(\Gamma, \mathbb{Z}). \tag{6.15}$$

Lemma 6.2. *The Baker–Akhiezer function ψ , associated with an admissible set of data $\{\widehat{\Gamma}, u_l^{(i)}\}$ satisfies the equation*

$$\psi(x + T, q) = g\psi(x, q)\mu(q), \quad \mu = e^{p(q)T}, \tag{6.16}$$

where g is the diagonal matrix $g = \text{diag} (\mu(P_0^1), \dots, \mu(P_0^r))$.

From (6.15) it follows that the function μ defined by the multi-valued abelian integral p is single-valued. Equation (6.16) follows from the uniqueness of the Baker-Akhiezer function, because the left- and the right-hand sides have the same analytic properties.

The matrix function L constructed with the help of ψ satisfies the monodromy property

$$L(x + T, q) = gL(x, q)g^{-1}. \tag{6.17}$$

Let $\mathcal{S} = \mathcal{S}(T, p)$ be a space of curves $\widehat{\Gamma}$ with meromorphic differential dp satisfying (6.15). We would like to mention that the closure of \mathcal{S} , as $T \rightarrow \infty$, coincides with the space of all genus \widehat{g} branching covers of Γ .

Corollary 6.1. *A set of data $\widehat{\Gamma} \in \mathcal{S}$, $[\widehat{\gamma}] \in J(\widehat{\Gamma})$, and a set of periodic functions $v_l^{(i)}(x)$ define with the help of the corresponding Baker-Akhiezer function a solution of the hierarchy (5.21) on $\mathcal{B}^D/\widehat{\mathcal{G}}L_r$.*

The finite-gap or algebro-geometric solutions are singled out by the constraint that there is a Lax matrix $L_1 \in \mathcal{L}^{nD}$ such that

$$[\partial_x - L, L_1] = 0. \tag{6.18}$$

Indeed, let k be a function on $\widehat{\Gamma}$ with divisor of poles $n\widehat{D}$, where \widehat{D} is the preimage of D . If n is big enough this exists. Let ψ be the Baker–Akhiezer function on $\widehat{\Gamma}$, then as it was shown above there is a unique Lax matrix L_1 such that

$$L_1(t, q)\psi(t, q) = k(q)\psi(t, q). \tag{6.19}$$

Equation (6.19) implies that the spectral curve of L_1 is birationally equivalent to the Riemann surface $\widehat{\Gamma}$ of Bloch solutions for L .

Theorem 6.2. *The form ω defined by (4.2) and (5.26) restricted to the space of algebro-geometric solutions, corresponding to a set of function $v_l^{(i)}(x)$ equals*

$$\omega = \sum_{s=1}^{\widehat{g}+r-1} \delta p(\widehat{\gamma}_s) \wedge \delta z(\widehat{\gamma}_s). \quad (6.20)$$

The meaning of the right-hand side of this formula is analogous to that of formula (4.51). It shows that the form ω restricted to the space of algebro-geometric solutions is non-degenerate.

It is well-known that the finite-gap solutions of the KdV hierarchy are dense in the space of all periodic solutions ([22]). As shown in [20] the finite-gap solutions are dense for the KP-2 equation as well. It seems quite natural to expect that the similar result is valid for the zero-curvature equations on an arbitrary algebraic curve, as well. In the conjectured scenario the infinite dimensional space \mathcal{B}^D can be identified with a direct limit of finite-dimensional spaces \mathcal{L}^{nD} , as $n \rightarrow \infty$. We are going to address that problem in the near future.

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