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Two-dimensionalized Toda lattice, commuting difference operators, and holomorphic bundles

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Abstract. Higher-rank solutions of the equations of the two-dimensionalized Toda lattice are constructed. The construction of these solutions is based on the theory of commuting difference operators, which is developed in the first part of the paper. It is shown that the problem of recovering the coefficients of commuting operators can be effectively solved by means of the equations of the discrete dynamics of the Tyurin parameters characterizing the stable holomorphic vector bundles over an algebraic curve.

Contents

§ 1. Introduction

The present paper is devoted to the circle of problems related to the construction of higher-rank algebro-geometric solutions of the two-dimensionalized Toda lattice

$$\partial^2_{\xi\eta}\varphi_n = e^{\varphi_n-\varphi_{n-1}} - e^{\varphi_{n+1}-\varphi_n}. \tag{1.1}$$

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Solutions of this kind for the Kadomtsev–Petviashvili (KP) equation

$$3u_{yy} = (4u_t - 6uu_x + u_{xxx})_x \tag{1.2}$$

were constructed earlier in the authors' paper [1]. The construction, as well as the very notion of *rank* of solutions, is based on the theory of commuting one-dimensional differential operators [2]. In modern mathematical physics this theory appeared as an auxiliary algebraic aspect of the integration theory for non-linear soliton systems and the spectral theory of periodic finite-zone operators [3]–[6].

The KP equation, like any other soliton equation, is only one of the equations of the corresponding hierarchy. A solution of the full KP hierarchy is a function $u = u(t)$ depending on an infinite set of times $t = (t_1 = x,\ t_2 = y,\ t_3 = t,\ t_4, \dots)$. In contrast to the algebro-geometric solutions of $(1+1)$-systems of KdV-type equations, which were singled out [7] by the stationary condition for one of the flows of the KdV-hierarchy, the algebro-geometric solutions of the KP equation are singled out by the stationary condition with respect to *two* times of the hierarchy, $\partial_{t_n} u = \partial_{t_m} u = 0$. This condition is equivalent to the existence of a pair of commuting ordinary differential operators

$$[L_n, L_m] = 0, \qquad L_n = \sum_{i=0}^{n} u_i \partial_x^i, \qquad L_m = \sum_{j=0}^{m} v_j \partial_x^j, \tag{1.3}$$

which commute with the auxiliary Lax operators $\partial_y - L_2$ and $\partial_t - L_3$ for the KP equation which were first found in [8], [9],

$$L_2 = \partial_x^2 + u, \qquad L_3 = \partial_x^3 + \frac{3}{2}\, u\partial_x + w. \tag{1.4}$$

Thus, the pairs of commuting ordinary scalar linear differential operators determine the invariant spaces for the entire KP hierarchy. By the rank r of such a pair we mean the number of linearly independent joint eigenfunctions, that is, the number of solutions of the equations

$$L_n \psi^i = E\psi^i, \qquad L_m \psi^i = w\,\psi^i, \qquad i = 1, \dots, r, \tag{1.5}$$

for the pairs of complex numbers (E, w) for which the space of solutions is non-empty. The rank of a pair of commuting differential operators is a *common divisor* of their orders.

As a purely algebraic problem, the classification problem for commuting ordinary scalar differential operators was posed already in the 1920s by Burchnall and Chaundy [10], [11] who had made great progress in the solution for operators of coprime orders (in which case the rank is always equal to 1). The effective classification of the commuting operators of rank one was completed in [4]. Burchnall and Chaundy noted (see [11]) that the general problem for rank $r > 1$ seems to be extremely difficult.

The first steps were made in [12] and [13]. A method of effective classification of commuting differential operators of rank $r > 1$ in general position was conceived by the authors in [1] and [2]. The commuting pairs of rank r depend on $(r-1)$

arbitrary functions of one variable, a smooth algebraic curve Γ with a *single* marked point P, and a set of Tyurin parameters (characterizing a stable framed holomorphic bundle). We say that these constructions are *one-point* constructions.

The conditions distinguishing the algebro-geometric solutions can be formulated for the equations of the $2D$ Toda lattice in the same way as those in the KP theory. Here the stationary condition for solutions with respect to two flows of the corresponding hierarchy is equivalent to the existence of a pair of commuting difference operators

$$L = \sum_{i=-N_-}^{N_+} u_i(n) T^i, \qquad A = \sum_{i=-M_-}^{M_+} v_i(n) T^i, \tag{1.6}$$

which commute with the Lax operators

$$\mathcal{L}_1 = \partial_\xi - T - w(n), \qquad \mathcal{L}_2 = \partial_\eta - c(n) T^{-1}, \tag{1.7}$$

$$w(n) = \varphi_{n\xi}, \qquad c(n) = e^{\varphi_n - \varphi_{n+1}}, \tag{1.8}$$

for the $2D$ Toda lattice. Here and below, T stands for the shift operator with respect to the discrete variable, $Ty_n = y_{n+1}$.

For difference operators the whole (now classical) theory of pairs of commuting operators of rank $r = 1$ was based solely on *two-point constructions* ([14], [15]). The rings of these operators turned out to be isomorphic to the rings $A(\Gamma, P^\pm)$ of meromorphic functions on an algebraic curve Γ with poles at a pair of marked points $P^\pm$. There is a remarkable *almost graded* structure in these algebras. The relation between them and Laurent–Fourier-type bases on Riemann surfaces was developed by the authors in [16] for string theory.

The classification problem for commuting difference operators remains unsolved in full scope. The analysis of the problem presented in § 2 of the present paper enabled the authors to single out some new substantial points. It turned out that commuting difference operators of rank $r = 1$ can be obtained by means of *multi-point* constructions. *A posteriori* this fact does not seem surprising. For commuting differential operators with matrix coefficients, the appearance of multipoint constructions is well known [4]. For difference operators there is apparently no natural way to single out the case of operators with scalar coefficients. It seems reasonable to assume that the difference analogue of commuting linear differential operators with scalar coefficients should be pairs of commuting difference operators arising in the framework of only *one- and two-point* constructions. The most important argument for the authors to support this conjecture is that the corresponding rings of commuting difference operators remain invariant with respect to the hierarchy of the two-dimensionalized Toda lattice only in these cases.

A principal difference between these two cases should be stressed. For the two-point construction the functional parameters do not arise and the coefficients of the operators can be computed in terms of Riemann theta-functions. In the one-point case there are *functional parameters* arising in the construction of commuting operators for the ranks $l > 1$. The coefficients of the corresponding operators depend on l arbitrary functions of a discrete variable, on a smooth algebraic curve Γ with a *single* marked point P, and on the set of Tyurin parameters. As in the continuous case [2], the problem of recovering the joint eigenfunctions of such operators from the algebro-geometric spectral data reduces to the Riemann problem on the corresponding algebraic curve and cannot be solved explicitly. At the same time, as noted in [1], the problem of recovering the coefficients of commuting operators sometimes has an explicit solution. This solution is based on the equations for the Tyurin parameters, equations which describe the dependence of the parameters on the initial point of the normalization. We note that in a recent paper [17] the first author established a relationship between continuous deformations of the Tyurin parameters describing the higher-rank solutions of the KP equation, and the theory of Hitchin systems [18]. A study of the discrete dynamics of the Tyurin parameters enabled the authors to find the explicit form of the commuting operators arising in the framework of the one-point construction of rank two on an elliptic curve. The ring of these operators is generated by a pair of operators L and A of orders 4 and 6, respectively:

$$L = \sum_{i=-2}^{2} u_i(n)T^i, \qquad A = \sum_{i=-3}^{3} v_i(n)T^i.$$

The coefficients of these operators depend on two arbitrary functions γ_n and s_n. In the simplest example the explicit form of the operator L is given by the formulae

$$L = L_2^2 - \wp(\gamma_n) - \wp(\gamma_{n-1}),$$

where L_2 is the Schrödinger difference operator

$$L_2 = T + v_n + c_n T^{-1}$$

with coefficients

$$\begin{aligned} 4c_{n+1} &= (s_n^2 - 1)F(\gamma_{n+1}, \gamma_n)F(\gamma_{n-1}, \gamma_n), \\ 2v_{n+1} &= s_n F(\gamma_{n+1}, \gamma_n) - s_{n+1}F(\gamma_n, \gamma_{n+1}), \end{aligned}$$

where

$$F(u, v) = \zeta(u + v) - \zeta(u - v) - 2\zeta(v).$$

Here and henceforth, $\wp(u) = \wp(u|\omega, \omega')$ and $\zeta(u) = \zeta(u|\omega, \omega')$ are the standard Weierstrass functions.

These formulae are discrete analogues of the formulae for the differential operator L_4 of order 4 and rank 2 obtained by the authors in [19]. In [20] an explicit form was found for the functional parameter in the formulae for L_4, which corresponds to the Dixmier operator [12]. The coefficients of this operator are polynomial functions of a continuous independent variable x. The problem of constructing a discrete analogue of the Dixmier operator seems to be of interest.

The construction of higher-rank algebro-geometric solutions for the equations of the $2D$ Toda lattice, which was briefly presented in [21] and [22], can be found in § 3 below. This result is based on the construction of the Baker–Akhiezer vector functions by means of deformations of eigenfunctions of the commuting difference operators. These deformations are completely determined by the behaviour of the Baker–Akhiezer functions in a neighbourhood of the marked points which is determined by certain *grafting* matrix functions. We must again highlight the principal difference between the two-point and one-point cases.

In the two-point case, in which there are no functional parameters in the construction of the commuting difference operators, the Baker–Akhiezer function is determined by two grafting functions $\Psi_+(\xi, z)$ and $\Psi_-(\eta, z)$, each of which is determined by an ordinary differential equation whose coefficients contain arbitrary functions of one of the continuous variables (ξ or η, respectively). In the one-point case we have the opposite situation. The construction of the commuting difference operators involves arbitrary functions of a discrete variable, but there is no arbitrariness in the definition of the grafting function $\Psi_0(n, \xi, \eta, z)$.

In both cases the multiparameter Baker–Akhiezer functions $\psi_n = (\psi_n^i)$ thus constructed satisfy the equations

$$\mathcal{L}_1\psi_n = 0, \qquad \mathcal{L}_2\psi_n = 0, \tag{1.9}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are operators of the form (1.7). The condition that the equations (1.9) be consistent is equivalent to (1.1). Moreover, it follows from (1.9) that for any n the Baker–Akhiezer function, when regarded as a function of the variables (ξ, η), satisfies a linear Schrödinger equation of the form

$$\bigl(\partial^2_{\xi\eta} + v_n(\xi, \eta)\partial_\xi + u_n(\xi, \eta)\bigr)\psi_n = 0. \tag{1.10}$$

As a consequence, we see that our constructions lead to a broad class of Schrödinger operators in a magnetic field that are integrable on one energy level.

The inverse spectral problem *on one energy level* for Schrödinger operators in a magnetic field was posed and solved in [23] in the case of rank 1. A construction of two-dimensional Schrödinger operators of arbitrary rank that are integrable on one energy level was proposed in [19]. The corresponding constructions were *two-point*. The possibility of constructing integrable Schrödinger operators in the framework of one-point constructions has not been discussed before the present paper.

It should be stressed that, in essence, the problem of constructing solutions of the equations of the $2D$ Toda lattice and the problem of constructing integrable Schrödinger operators in a magnetic field are equivalent. In the modern theory of integrable systems the equation (1.1) and its Lax representation were first obtained by the Zakharov–Shabat scheme as a two-dimensional analogue of the one-dimensional Toda lattice [24]. It turned out later that these equations were well known in classical differential geometry in the equivalent form of chains of Laplace transformations for a two-dimensional Schrödinger equation [25].

Let us consider an arbitrary two-dimensional Schrödinger operator L in a magnetic field:

$$2L = (\partial_{\bar z} + B)(\partial_z + A) + 2V, \tag{1.11}$$

where $\partial_z = \partial_x - i\partial_y$ and $\partial_{\bar z} = \partial_x + i\partial_y$. The functions $H = (B_z - A_{\bar z})/2$ and $U = -H - V$ are usually called the *magnetic field* and the *potential*, respectively. For brevity, the potential is understood to be the function V in what follows.

The operator L is defined up to gauge transformations $L \to e^{-f}Le^{f}$. The only invariants of L are the potential V and the magnetic field H. The *Laplace transformations*, which are two-dimensional analogues of the Bäcklund–Darboux transformations, are defined as follows:

$$(\partial_{\bar z}+B)(\partial_z+A)+2V \longmapsto \widetilde V(\partial_z+A)V^{-1}(\partial_{\bar z}+B)+2V. \tag{1.12}$$

The potential $\widetilde V$ and the magnetic field $\widetilde H$ of the transformed operator are given by the formulae

$$\widetilde V = V+\widetilde H, \qquad 2\widetilde H = 2H+\Delta\log V. \tag{1.13}$$

If a function ψ satisfies the equation $L\psi=0$, then the function $\widetilde\psi=(\partial_z+A)\,\psi$ satisfies the equation $\widetilde L\,\widetilde\psi=0$. Let us consider the lattice of the Laplace transformations

$$V_{n+1}=V_n+H_{n+1}, \qquad 2H_{n+1}=2H_n+\Delta\log V_n. \tag{1.14}$$

The change $V_k=\log(\varphi_n-\varphi_{k-1})$ takes this lattice to the $2D$ Toda equations (1.1).

§ 2. Commuting difference operators

2.1. Setting of the problem. The main objective of the present section is the construction of an effective classification of commuting difference operators, that is, difference operators satisfying the condition

$$[L,A]=0, \tag{2.1}$$

of the form

$$L=\sum_{i=-N_-}^{N_+}u_i(n)T^i, \quad A=\sum_{i=-M_-}^{M_+}v_i(n)T^i, \quad N_\pm=r_\pm n_\pm, \quad M_\pm=r_\pm m_\pm, \tag{2.2}$$

with scalar coefficients, where $r_\pm$ are the greatest common divisors of the highest and lowest orders of the operators, respectively, that is, the numbers $n_\pm$ and $m_\pm$ are non-zero and coprime,

$$(n_+,m_+)=(n_-,m_-)=1. \tag{2.3}$$

Suppose that the highest and lowest coefficients of these operators are non-zero. We denote them for brevity by $u^\pm(n)$ and $v^\pm(n)$,

$$u^\pm(n)=u_{\pm N_\pm}(n)\neq 0, \qquad v^\pm(n)=v_{\pm M_\pm}(n)\neq 0. \tag{2.4}$$

The equation (2.1) is invariant with respect to the gauge transformations of the form

$$L,A \longmapsto \widetilde L=gLg^{-1}, \widetilde A=gAg^{-1}, \qquad \widetilde L=\sum_{i=-N_-}^{N_+}g(n+i)g^{-1}(n)u_i(n)T^i, \tag{2.5}$$

where $g(n) \neq 0$ is an arbitrary everywhere non-zero function of a discrete variable. By using a gauge transformation if necessary, we can always assume that the highest coefficient of L is identically equal to 1,

$$u^+(n) = 1. \tag{2.6}$$

This normalization, which is assumed in what follows, preserves a partial gauge freedom, that is, it is invariant with respect to the gauge transformations corresponding to functions $g(n)$ such that $g(n+N_+) = g(n)$. It follows from the equation (2.1) that the highest coefficient of the operator A satisfies the equality $v^+(n+N_+) = v^+(n)$. Hence, by using gauge transformations if necessary, we can assume in addition that the following conditions hold:

$$v^+(n+r_+) = v^+(n) = v^+_{\bar{n}}. \tag{2.7}$$

Here $\bar{n}$ stands for the residue of n modulo r_+, that is, $n \to \bar{n} = n \pmod{r_+}$, $0 \leqslant \bar{n} < r_+ - 1$. The remaining gauge freedom corresponds to functions $g(n)$ such that $g(n+r_+) = g(n)$.

Fixing the gauge freedom by the conditions (2.6) and (2.7), we see that the lowest coefficients of the operators are of the form

$$u^-(n) = h^{-1}(n-N_-)h(n), \qquad v^-(n) = v^-_{\widetilde{n}} h^{-1}(n-M_-)h(n). \tag{2.8}$$

Here $\widetilde{n} = n \pmod{r_-}$.

For $r_+ = r_- = r > 1$, the commutativity equations have an additional invariance. Namely, for any i, $0 \leqslant i < r$, one can construct from any function y_n of the discrete variable n a new function Y_n having non-zero values only at the points of the corresponding sublattice,

$$Y_n = \begin{cases} 0, & n \neq i \pmod{r}, \\ y_{\bar{n}}, & n = r\bar{n} + i. \end{cases} \tag{2.9}$$

Under this correspondence, every operator of order N determines some operator of order rN. If one takes r pairs of commuting operators $[L_i, A_i] = 0$ (not necessarily different), then the direct sum of these operators (each regarded as an operator acting on the corresponding sublattice) satisfies the non-degeneracy condition (2.4) and the commutativity equation (2.1).

To eliminate the indicated freedom, we consider commuting pairs of *indecomposable* operators, that is, operators that cannot be reduced to a block-diagonal or block-Jordan form by decomposing the lattice n into a direct sum of sublattices of the form $kr + i$, $0 \leqslant i < r - 1$. To satisfy the indecomposability condition, it suffices to assume that for any $i = 1, \dots, r-1$ there is an n_0 such that

$$v_{rm_+ - i}(n_0) \neq 0, \qquad v_{-rm_- + i}(n_0) \neq 0. \tag{2.10}$$

2.2. Spectral curve. Formal infinities. The definition of *affine* spectral curve in the discrete case does not differ from the continuous version. Let us consider the linear space $\mathcal{L}(E)$ of solutions of the equation $Ly = Ey$, that is,

$$y_{N_+} + \sum_{i=-N_-}^{N_+-1} u_i(n) y_{n+i} = E y_n. \tag{2.11}$$

The dimension of this space is equal to the order of the linear operator L, which means that $\dim \mathcal{L}(E) = N_+ + N_-$. The restriction of the operator A to $\mathcal{L}(E)$,

$$A(E) = A\big|_{\mathcal{L}(E)}, \tag{2.12}$$

is a finite-dimensional linear operator. The spectral curve parametrizing the joint eigenfunctions of L and A is defined by the characteristic equation

$$R(w, E) = \det(w - A(E)) = 0. \tag{2.13}$$

The matrix elements of the operator $A(E)$ in the basis c^i of solutions of the equation (2.11) determined by the initial conditions

$$c_n^i = \delta_n^i, \qquad i, n = -N_-, \dots, N_+ - 1\,, \tag{2.14}$$

are polynomial functions of the variable E. Hence, $R(w, E)$ is a polynomial not only in the variable w but also in the variable E.

Compactification of the spectral curve. The construction of eigenfunctions of commuting operators at infinity (as $E \to \infty$) in the discrete case, which repeats the continuous case in general, contains a number of new substantial points.

We denote by $\mathcal{L}_+$ the linear space of solutions of the equation

$$L\Phi = z^{-N_+}\Phi, \qquad \Phi = \{\Phi_n\}, \tag{2.15}$$

that have the form

$$\Phi_n(z) = z^{-n}\left(\sum_{s=0}^{\infty} \xi_s(n) z^s\right). \tag{2.16}$$

We assume here that $\xi_0(n)$ is non-zero for at least one value n.

Lemma 2.1. *The space $\mathcal{L}_+$, regarded as a linear space over the field k_+ of Laurent series in the variable z, is of dimension N_+. It is generated by the solutions Φ^i, $i = 0, \dots, N_+ - 1$, uniquely determined by the conditions*

$$\Phi_n^i(z) = z^{-n}\delta_{ni}, \qquad n, i = 0, \dots, N_+ - 1. \tag{2.17}$$

To prove the lemma, it suffices to note that the substitution of the series (2.16) into the equation (2.15) leads to the following system of recursion equations for determining the coefficients of the expansion (2.16):

$$\begin{aligned} &\xi_0(n + N_+) = \xi_0(n), \\ &\xi_1(n + N_+) = \xi_1(n) - u_{n+N_+-1}(n)\xi_0(n + N_+ - 1), \ \dots . \end{aligned} \tag{2.18}$$

It follows from this system that any solution Φ is uniquely determined by the initial data $\xi_s(i)$, $i=0,\dots,N_+-1$.

The space $\mathcal{L}_+$ is invariant with respect to the action of the operator A. Hence,

$$A\Phi_n^i(z)=\sum_{j=0}^{N_+-1}A_{ij}(z)\Phi_n^j(z),\qquad A_{ij}(z)=\sum_{s=-M_+}^{\infty}A_{ij}^{(s)}z^s. \tag{2.19}$$

The highest coefficient is the matrix

$$A_{ij}^{(M_+)}=v_i^+\delta_{j,\,i+M_+\,(\mathrm{mod}\,N_+)}. \tag{2.20}$$

The change $z\to\varepsilon z$, where $\varepsilon^{N_+}=1$, determines an automorphism of the space $\mathcal{L}_+(z)$. This implies that the coefficients of the characteristic polynomial

$$\det(w-A_{ij}(z))=w^{N_+}+\sum_{i=0}^{N_+-1}a_i(E)w^i,\qquad E=z^{-N_+}, \tag{2.21}$$

are series in the variable $E=z^{-N_+}$. Therefore, the eigenvalues of this matrix are Laurent series in the variable z (rather than in some fractional power $z^{1/k}$, $k>1$).

Suppose that the following conditions hold:

$$v_i^+\neq v_j^+,\qquad i\neq j, \tag{2.22}$$

where the v_i^+ are defined in (2.7). In this case the matrix $A_{ij}(z)$ has a unique eigenvector $\Psi_n^{(i)}(z)$, $i=0,\dots,r_+-1$, of the form (2.16),

$$\Psi_n^{(i)}(z)=z^{-n}\left(\delta_{i,\bar n}+\sum_{s=1}^{\infty}\xi_s(n)z^s\right). \tag{2.23}$$

The leading term of the expansion is an eigenvector of the matrix (2.20) corresponding to the eigenvalue v_i^+. The change $z\to\varepsilon^{n_+}z$ multiplies the leading term by ε^{-in_+}. Hence, the vector $z^i\Psi_n^{(i)}(z)$ can be expanded in powers of

$$z_0=z^{r_+}=E^{1/n_+}. \tag{2.24}$$

Therefore, for $i>0$ and for any $j=0,\dots,r_+-1$ we have the equalities

$$\begin{aligned}\Psi_{kr_++j}^{(i)}(z)&=z^{-i}z_0^{-k}(O(z_0)), && j<i,\\ \Psi_{kr_++j}^{(i)}(z)&=z^{-i}z_0^{-k}(O(1)), && j\geqslant i.\end{aligned} \tag{2.25}$$

It follows from the indecomposability condition (2.10) in which we set $n_0=0$ (without loss of generality) that for $i>0$ the zeroth component of the eigenvector is of the form $\Psi_0^{(i)}=O(z^{r-i})$. Therefore, the eigenvectors (normalized in the standard way)

$$\psi_n^{(i)}=\frac{\Psi_n^{(i)}(z)}{\Psi_0^{(i)}(z)},\qquad \psi_0^{(i)}=1, \tag{2.26}$$

have the following form for $i > 0$:

$$\begin{aligned} \psi^{(i)}_{kr_++j}(z_0) &= O(z_0^{-k-1}), \qquad i \leqslant j, \\ \psi^{(i)}_{kr_++j}(z_0) &= O(z_0^{-k}), \qquad j < i. \end{aligned} \tag{2.27}$$

For $i = 0$ we have

$$\psi^{(0)}_{kr_++j}(z_0) = z_0^{-k}(1 + O(z_0)). \tag{2.28}$$

Thus, assuming that the conditions (2.22) are satisfied, we have constructed a set of N_+ formal eigenvectors of the operator $A(E)$ in a neighbourhood of the point at infinity $E = \infty$. We can construct another set of N_- formal eigenfunctions in a similar way assuming that

$$v_i^- \neq v_j^-, \qquad i \neq j, \tag{2.29}$$

where the elements v_i^- are defined by the equality (2.8). To this end, we consider a linear space $\mathcal{L}_-$ over the field k_- of Laurent series in the variable z_-; this space is generated by the solutions of the equation

$$L\Phi = z^{-N_-}\Phi \tag{2.30}$$

that have the form

$$\Phi_n^-(z) = z_-^n\left(\sum_{s=0}^{\infty} \xi_s^-(n) z_-^s\right). \tag{2.31}$$

Here it is assumed that the value $\xi_0^-(n)$ is non-zero for at least one value of n.

Repeating the above arguments, we see that the characteristic polynomial of the operator $A^-(z_-)$ induced by the action of A on the space $\mathcal{L}_-$ is of the form

$$\det\bigl(w - A^-(z_-)\bigr) = \prod_{i=0}^{r_-}\prod_{k=0}^{n_-}\bigl(w - \widehat{v}_i^-(z_{k,-})\bigr), \qquad z_{k,-} = \varepsilon_1^k z_-^r, \tag{2.32}$$

where $\varepsilon_1^{n_-} = 1$ and the series $\widehat{v}_i^- = \widehat{v}_i^-(z_{0,-})$ has the form

$$\widehat{v}_i^-(z_{0,-}) = v_i^- z_{0,-}^{-m_-}(1 + O(z_{0,-})). \tag{2.33}$$

The normalized eigenfunction of $A^-(z_-)$ corresponding to the eigenvalue $w = \widehat{v}_i^-(z_{0,-})$ has the following form for any $i = 0, \dots, r-1$:

$$\begin{aligned} \psi^{(i)}_{kr_-+j}(z_{0,-}) &= O(z_{0,-}^{k}), \qquad i \geqslant j, \\ \psi^{(i)}_{kr_-+j}(z_{0,-}) &= O(z_{0,-}^{k+1}), \qquad i < j. \end{aligned} \tag{2.34}$$

In the direct sum of the spaces $\mathcal{L}_+$ and $\mathcal{L}_-$ one can choose a basis c_n^i normalized by the conditions

$$c_n^i = \delta_{i,n}, \qquad i, n = N_-, \dots, N_+ - 1. \tag{2.35}$$

In this basis the operator A has matrix elements equal to those obtained when constructing the affine spectral curve. Hence, the characteristic equation (2.13) coincides with the product of the characteristic equations for $A(z)$ and $A^-(z_-)$, that is,

$$\det\bigl(w - A(E)\bigr) = \left[\prod_{i=0}^{r_+}\prod_{k=0}^{n_+-1}\bigl(w - \widehat{v}_i(z_k)\bigr)\right]\left[\prod_{i=0}^{r_-}\prod_{k=0}^{n_--1}\bigl(w - \widehat{v_i^-}(z_{k,-})\bigr)\right], \qquad (2.36)$$

where the product in the first and second groups of factors is taken over all roots in E^{-1} of degrees n_+ and n_-, respectively:

$$E = z_k^{-n_+} = z_{k,-}^{-n_-}. \qquad (2.37)$$

The decomposition (2.36) gives comprehensive information on the compactification of the spectral curve if the operator $A(E)$ has N distinct eigenvalues for almost all points E. This is the case of rank-one commuting operators treated in the next subsection.

2.3. Commuting operators of rank 1. The conditions (2.22) and (2.29) ensure that the eigenvalues of $A(E)$ corresponding to distinct factors in the first and second product of the formula (2.36) are distinct. We require in addition that the factors in different groups also do not coincide. To this end, it suffices to assume one of the conditions

$$\text{(i)}\ \ m_+n_- \neq m_-n_+, \qquad \text{(ii)}\ \ v_i^+ \neq v_j^-. \qquad (2.38)$$

In this case it follows from the equality (2.36) that the affine curve given by the equation (2.13) is compactified in a neighbourhood of the point at infinity by $l = r_+ + r_-$ points $P_{i_\pm}^{\pm}$, $i_\pm = 0,\dots,r_\pm - 1$. In a neighbourhood of the point at infinity, and hence for almost all values of E, the spectral curve Γ has $N = N_+ + N_-$ sheets. Hence, corresponding to every point of the spectral curve is a unique eigenfunction ψ_n of the operators L and A. Let us state the main theorem.

Theorem 2.1. *Let a pair of indecomposable commuting operators satisfy the conditions* (2.22), (2.29), *and* (2.38). *Then*:

(1) *the spectral curve* Γ *given by the characteristic equation* (2.13) *is compactified at infinity by* $l = r_+ + r_-$ *points* $P_{i_\pm}^{\pm}$ *with neighbourhoods in which one can take the local coordinates of the form*

$$z_{k,\pm} = E^{-1/n_\pm};$$

(2) *the joint eigenfunction of the pair of commuting operators*

$$L\psi_n(Q) = E\psi_n(Q), \qquad A\psi_n(Q) = w\psi_n(Q), \qquad Q = (w,E) \in \Gamma,$$

normalized by the condition $\psi_0 = 1$, *is a meromorphic function on* Γ *whose divisor of poles* $D = \{\gamma_s\}$ *away from the marked points* $P_i^{\pm}$ *does not depend on* n. *In a neighbourhood of the marked points the function* ψ_n *is of the form* (2.27), (2.28), (2.34), *that is, if* n *is represented in the form* $n = kr_+ + j_+ = k'r_- + j_-$ *with* $0 \leqslant j_\pm < r_\pm$, *then*:

(2a) ψ_n has poles of order k at the points $P_0^+, P_{j_++1}^+, \dots, P_{r_+-1}^+$ and poles of order $k+1$ at the points $P_1^+, \dots, P_{j_+}^+$;

(2b) ψ_n has zeros of order k' at the points $P^-_{j_-}, \dots, P^-_{r_- -1}$ and zeros of order $k'+1$ at the points $P^-_0, \dots, P^-_{j_- -1}$;

(3) in general position, in which case the spectral curve is smooth and irreducible, the number of poles γ_s (counted according to their multiplicities) of the function $\psi_n(Q)$ away from the marked points is equal to the genus g of the curve Γ.

We have thus constructed a map assigning a curve Γ with l marked points $P^\pm_{i_\pm}$ and a divisor of degree g to a pair of commuting operators satisfying the conditions of the theorem:

$$[L, A] = 0 \mapsto \{\Gamma,\, P^\pm_{i_\pm},\, D = \{\gamma_s\}\}, \qquad 0 \leqslant i_\pm < r_\pm, \quad s = 1, \dots, g. \tag{2.39}$$

Let us now show that one can uniquely recover the commuting operators from these algebro-geometric data.

We consider an arbitrary smooth curve Γ with $l = r_+ + r_-$ marked points $P^\pm_{i_\pm}$. It follows from the Riemann–Roch theorem that for any non-special divisor $D = (\gamma_1, \dots, \gamma_g)$ there is a function $\psi_n(Q)$, unique up to proportionality, whose divisor of poles away from the marked points does not exceed D and which has poles and zeros of multiplicities prescribed by (2a) and (2b) of the above theorem at the points $P^\pm_i$.

Indeed, the conditions (2a) and (2b) mean that the function $\psi_n(Q)$ belongs to the space $\mathcal{L}(D_n)$ of meromorphic functions associated with the divisor D_n,

$$D_n = D + k \sum_{i_+=0}^{r_+-1} P^+_{i_+} + \sum_{i_+=1}^{j_+} P^+_{i_+} - k' \sum_{i_-=0}^{r_- -1} P^-_{i_-} - \sum_{i_-=0}^{j_- -1} P^-_{i_-}\,,$$

where $n = kr_+ + j_+ = k'r_- + j_-$. The degree of this divisor is g, and therefore the space $\mathcal{L}(D_n)$ is one-dimensional by the Riemann–Roch theorem.

We denote by $\mathcal{A}(\Gamma, P^\pm_{i_\pm})$ the ring of meromorphic functions on Γ with poles at $P^\pm_{i_\pm}$.

Theorem 2.2. *Let $\psi(Q) = \{\psi_n(Q)\}$ be a sequence of functions corresponding to the algebro-geometric data* (2.39). *Then for any function $f \in \mathcal{A}(\Gamma, P^\pm_i)$ there exists a unique difference operator L_f (whose coefficients do not depend on Q) such that*

$$L_f \psi(Q) = f(Q)\psi(Q).$$

If the function $f(Q)$ has poles of order n_+ and n_- at the points $P^\pm_{i_\pm}$, respectively, then the operator L_f is of the form (2.2).

The proof follows immediately from the Riemann–Roch theorem. Indeed, if a function f has poles at the points $P^\pm_i$ of orders $n_\pm$, respectively, then the function $f(Q)\psi_n(Q)$ belongs to the linear space of the following form:

$$f(Q)\psi_n(Q) \in \mathcal{L}\biggl(D_n + n_+ \sum_{i_+=0}^{r_+-1} P^+_{i_+} + n_- \sum_{i_-=0}^{r_- -1} P^-_{i_-}\biggr).$$

The dimension of this space is equal to $N_+ + N_- + 1$. It follows from the definition of ψ_n that the functions ψ_{n+i}, $-rn_- \leqslant i \leqslant rn_+$, form a basis of this space. The coefficients $u_i(n)$ of the operator L_f are simply the coefficients of the expansion of $f\psi_n$ in the basis functions ψ_{n+i}.

The function $\psi_n(Q)$ can be explicitly represented by means of the Riemann theta function in the standard way. For simplicity, we present these formulae for $r = r_+ = r_-$. We first define the functions h_j, $j = 0, \dots, r-1$, by the rule

$$h_j(Q) = \frac{f_j(Q)}{f_j(P_j^+)}, \qquad f_j(Q) = \frac{\theta(A(Q)+Z_j)}{\theta(A(Q)+Z_0)} \frac{\prod_{i=0}^{j-1} \theta(A(Q)+S_i^-)}{\prod_{i=1}^{j} \theta(A(Q)+S_i^+)}, \tag{2.40}$$

where

$$S_i^{\pm} = \mathcal{K} - A(P_i^{\pm}) - \sum_{s=1}^{g-1} A(\gamma_s), \quad i = 0, \dots, r-1, \tag{2.41}$$

$$Z_j = Z_0 + \sum_{i=0}^{j-1} A(P_i^-) - \sum_{i=1}^{j} A(P_i^+), \quad Z_0 = \mathcal{K} - \sum_{s=1}^{g} A(\gamma_s). \tag{2.42}$$

We then denote by $d\Omega^{(0)}$ a unique meromorphic differential on Γ that has simple poles at $P_j^{\pm}$ with residues ∓ 1 and is normalized by the conditions

$$\oint_{a_k^0} d\Omega^{(0)} = 0. \tag{2.43}$$

The coordinates of its vector $U^{(0)}$ of b_0-periods are

$$U_k^{(0)} = \frac{1}{2\pi i} \oint_{b_k^0} d\Omega^{(0)} = \sum_{j=0}^{r-1} \bigl(A(P_j^-) - A(P_j^+)\bigr). \tag{2.44}$$

Lemma 2.2. *The functions $\psi_n(Q)$ are equal to*

$$\psi_{rk+j} = h_j(Q) \frac{\theta(A(Q) + kU^{(0)} + Z_j)\theta(A(P_j^+) + Z_j)}{\theta(A(Q)+Z_j)\theta(A(P_j^+) + kU^{(0)} + Z_j)} \exp\left(k \int^Q d\Omega^0\right). \tag{2.45}$$

The proof of the formula (2.45) reduces to a simple check that the function defined by this formula is single-valued on Γ and has all the necessary analytic properties.

Corollary 2.1. *The coefficients of commuting rank-*1 *operators are quasi-periodic meromorphic functions of the variable n.*

2.4. Rank > 1. The case of separated infinities. It follows from the construction of formal joint eigenfunctions of the operators L and A that the operator $A(E)$ is diagonalizable for almost all values of E. We call the k-tuple $\mu = (\mu_1, \dots, \mu_k)$ of multiplicities of the eigenvalues of $A(E)$ the *vector rank* of the commuting operators. In the problem of commuting differential operators with scalar coefficients, the rank is always scalar ($k=1$) and is a divisor of the orders of the operators [2]. The appearance of vector ranks in the problem of commuting differential operators with matrix coefficients was discovered in [26]. For commuting operators of vector rank μ the characteristic equation is

$$\det(w-A(E))=\prod_{i=1}^{k} R_i^{\mu_i}(w,E). \tag{2.46}$$

It is important to note that the conditions (2.22) and (2.29) on the highest coefficients of the commuting operators, which ensured the simplicity of the eigenvalues of the operators $A(z)$ and $A^-(z_-)$ in a formal neighbourhood of the point at infinity, are inconsistent with the auxiliary linear problems for the $2D$ Toda equations. Indeed, it follows from the equations

$$[L,\mathcal{L}_i]=[A,\mathcal{L}_i]=0, \tag{2.47}$$

where L, A, and $\mathcal{L}_i$ are of the form (1.6) and (1.7), that

$$\begin{aligned} u_{N_+}(n+1)&=u_{N_+}(n), \qquad & u_{N_-}(n)c(n-N_-)&=u_{N_+}(n-1)c(n),\\ v_{M_+}(n+1)&=v_{M_+}(n), \qquad & v_{M_-}(n)c(n-N_-)&=v_{M_+}(n-1)c(n). \end{aligned}$$

The last equalities mean that the values $v_i^{\pm}$ defined by (2.7) and (2.8) satisfy the relations

$$v_i^+=v^+, \qquad v_i^-=v^-. \tag{2.48}$$

In this case there are no *a priori* obstacles to the appearance of multiple eigenvalues of one of the operators $A(z)$ or $A^-(z_-)$ separately.

We say the commuting operators for which the eigenvalues of $A(z)$ do not coincide with the eigenvalues of $A^-(z_-)$ are operators with *separated infinities*. For commuting operators to have separated infinities it is sufficient that one of the conditions (2.38) hold, that is, one of the conditions (i) $m_+n_-\neq m_-n_+$ and (ii) $v^+\neq v^-$.

By the *type* of a pair of commuting operators with separated infinities we mean the tuples of multiplicities $\mu_{i_\pm}^{\pm}$ of *distinct* eigenvalues of the operators $A(z)$ and $A^-(z_-)$,

$$\sum_{i_\pm\in I_\pm}\mu_{i_\pm}^{\pm}=r_\pm. \tag{2.49}$$

Here $I_\pm$ are the finite sets parametrizing the different eigenvalues of $A(z)$ and $A^-(z_-)$. Corresponding to distinct eigenvalues of these operators are distinct points 'at infinity' which compactify the components Γ_i of the affine spectral curve that are given by the equations $R_i(w,E)=0$. A simple computation of the degree of the divisor of the joint eigenfunctions (for details see the proof of assertion (1) in Theorem 2.3 below) shows that there is no component Γ_i compactified by points corresponding to eigenvalues of only one of the operators $A(z)$ or $A^-(z_-)$. Since the multiplicity of an eigenvalue of $A(E)$ is constant on each of the components Γ_i, it follows that

(i) *for any value of the index $i_\pm$ there is at least one index $j_\mp$ with $\mu_{i_\pm}^{\pm}=\mu_{j_\mp}^{\mp}$.*

It seems plausible to the authors that there are no other conditions on the types of commuting operators with separated infinities, that is, for any tuple of positive integers $\mu^{\pm}_{i_{\pm}}$ satisfying the equality (2.49) and the above condition (i) there are commuting operators of the form (2.2) with separated infinities such that the given tuple is the type of these commuting operators.

Below we prove this assertion for the types of the form (r, r). This type corresponds to commuting operators of the form (2.2) with separated infinities which have equal greatest common divisors of the highest and lowest coefficients, that is, $r = r_+ = r_-$, and which have the maximal possible scalar rank $\mu = r$.

Lemma 2.3. *Let a commuting pair of indecomposable operators of the form* (2.2), *where* $r = r_+ = r_-$, *be of rank* r. *Then there is a unique Laurent series*

$$v^+(x) = v^+ x^{-m_+} + O(x^{-m_+ + 1}) \tag{2.50}$$

such that there exists a solution $\Psi(z)$ *of the equations*

$$L\Psi(z) = z^{-rn_+}\Psi(z), \qquad A\Psi(z) = v(z^r)\Psi(z) \tag{2.51}$$

which is of the form

$$\Psi_n(z) = z^{-n}\biggl(1 + \sum_{s=1}^{\infty} \xi_s(n) z^s\biggr). \tag{2.52}$$

The space of solutions of the equations (2.51) *in* $\mathcal{L}(z)$ *is generated by the series* $\Psi(\varepsilon^k z)$ *with* $\varepsilon^r = 1$.

A practically identical assertion holds for the eigenvalues and eigenvectors of the restriction of the operator A to the space $\mathcal{L}(z_-)$.

Let us preserve the notation Γ for the curve given by the equation $R(w, E) = 0$, where $R(w, E)$ is a root of degree r of the characteristic polynomial,

$$\det(w - A(E)) = R^r(w, E).$$

It follows from the assertion of the lemma and its analogue for the operator $A^-(z_-)$ that in a neighbourhood of the point at infinity we have the factorization

$$R(w, E) = \biggl[\prod_{k=0}^{n_+ - 1} \bigl(w - v^+(z_k)\bigr)\biggr]\biggl[\prod_{k=0}^{n_- - 1} \bigl(w - v^-(z_{k,-})\bigr)\biggr], \tag{2.53}$$

where the product in the first and second group of factors is taken over all roots of E^{-1} of degrees n_+ and n_-, respectively.

Corresponding to every point of the curve Γ is an r-dimensional space of joint eigenfunctions of the operators L and A. We fix a basis $\psi^i(Q)$ in this space by the conditions

$$\psi^i_n(Q) = \delta_{in}, \qquad i, n = 0, \dots, r - 1. \tag{2.54}$$

For any n the components $\psi^i_n(Q)$ are meromorphic functions. The form of these functions in a neighbourhood of the point at infinity can be found by using the bases given by the series $\Psi^{\pm}(\varepsilon^k z_{\pm})$. As in the continuous case [2], the singularities of the vector function $\psi_n(Q) = \{\psi_n^i(Q)\}$ in the affine part of the spectral curve are described by matrix divisors.

In general position, when the poles ψ_n are simple, the corresponding matrix divisor $D = \{\gamma_s, \alpha_s\}$ is the set of non-coinciding points γ_s and the set of r-dimensional vectors $\alpha_s = \{\alpha_s^i\}$, defined up to proportionality $\alpha_s \to \lambda\alpha_s$. The points γ_s are the poles of ψ_n^i, and the parameters α_s determine the relationships between the residues:

$$\alpha_s^i \operatorname{res}_{\gamma_s} \psi_n^j(Q) = \alpha_s^j \operatorname{res}_{\gamma_s} \psi_n^i(Q). \tag{2.55}$$

In [1] and [2] the data (γ, α) were called *Tyurin parameters*, because, according to [27], in the general case they determine a stable framed bundle $\mathcal{E}$ over Γ of rank r and of degree $c_1(\det \mathcal{E}) = rg$.

We state the theorem in final form.

Theorem 2.3. *Let a pair of indecomposable commuting operators with separated infinities satisfy the condition* (2.38) *and have the rank* $r = r_\pm$. *Then the following assertions hold.*

(1) *The spectral curve* Γ *given by the characteristic equation* (2.53) *is irreducible and is compactified at infinity by two points* $P^\pm$ *with neighbourhoods in which one can take* $z_\pm = E^{-1/n_\pm}$ *as local coordinates.*

(2) *Let* $\psi(Q)$ *be the vector function whose coordinates are the joint eigenfunctions*,

$$L\psi_n^i(Q) = E\psi_n^i(Q), \qquad A\psi_n^i(Q) = w\psi_n^i(Q), \qquad Q = (w, E) \in \Gamma,$$

normalized by the conditions (2.54). *Then* $\psi(Q)$ *is a meromorphic vector function on* Γ *whose matrix divisor of poles away from the marked points* $P_i^\pm$ *does not depend on* n. *In general position*, *when the spectral curve is smooth*, *the degree of the divisor of poles* $D = \{\gamma_s, \alpha_s\}$ *is equal to* gr, *where* g *is the genus of the curve* Γ.

(3a) *In a neighbourhood of the point* P^+ *the function* $\psi_{kr+j}^i(Q)$ *is of the form*

$$\begin{aligned} \psi_{kr+j}^i &= O(z_+^{-k}), \quad i < j, \\ \psi_{kr+i}^i &= z_+^{-k}(1 + O(z_+)), \\ \psi_{kr+j}^i &= O(z_+^{-k+1}), \quad i > j. \end{aligned} \tag{2.56}$$

(3b) *In a neighbourhood of the point* P^- *the function* $\psi_{kr+j}^i(Q)$ *is of the form*

$$\begin{aligned} \psi_{kr+j}^i &= O(z_-^{k+1}), \quad i < j, \\ \psi_{kr+j}^i &= O(z_-^{k}), \quad i \geqslant j. \end{aligned} \tag{2.57}$$

The converse assertion is also true. For any set in general position formed by points γ_s and vectors $\alpha_s = (\alpha_s^i)$, $s = 1, \dots, gr$, $i = 0, \dots, r-1$, there is a unique set of functions $\psi_n^i(Q)$ such that:

(i) away from the marked points $P^\pm$ the function $\psi_n^i(Q)$ has at most simple poles at the points γ_s (if all these points are distinct), and the residues of these functions satisfy (2.55);

(ii) in a neighbourhood of the marked points $P^\pm$ the function $\psi_n^i(Q)$ has the form determined by (2.56) and (2.57).

Theorem 2.4. *Let $\psi^i(Q) = \{\psi^i_n(Q)\}$ be the functions constructed above from the data set $\{\Gamma, P^{\pm}, D = \{\gamma_s, \alpha_s\}\}$. Then for any function $f \in \mathcal{A}(\Gamma, P^{\pm})$ there is a unique difference operator L_f (whose coefficients do not depend on Q) such that*

$$L_f \psi^i(Q) = f(Q)\psi^i(Q).$$

If the function $f(Q)$ has poles of orders n_+ and n_- at the points $P_i^{\pm}$, respectively, then the operator L_f is of the form (2.2), where $r = r_+ = r_-$.

Example. $r = 2$, $g = 1$. Without loss of generality we can assume that the pair of marked points on an elliptic curve with periods $(2\omega, 2\omega')$ is identified with a pair of points of the form $\pm z_0$. Let us choose vectors α_1 and α_2 in the form $\alpha_s = (a_s, 1)$.

It follows from (2.56) and (2.57) that the function ψ^1_{2n} can be represented as

$$\psi^1_{2n}(z) = A_n \frac{\sigma(z - z_0)\sigma(z - \gamma_1 - \gamma_2 - (2n-1)z_0)}{\sigma(z-\gamma_1)\sigma(z-\gamma_2)} \left[\frac{\sigma(z+z_0)}{\sigma(z-z_0)}\right]^n, \tag{2.58}$$

where $\sigma(z) = \sigma(z; 2\omega, 2\omega')$ is the Weierstrass σ function. The similar expression for ψ^0_{2n} is of the form

$$\psi^0_{2n}(z) = \left(B_n \frac{\sigma(z - \gamma_1 - 2nz_0)}{\sigma(z-\gamma_1)} + C_n \frac{\sigma(z-\gamma_2 - 2nz_0)}{\sigma(z-\gamma_2)}\right)\left[\frac{\sigma(z+z_0)}{\sigma(z-z_0)}\right]^n. \tag{2.59}$$

The conditions (2.56) on the residues enable one to express the parameters B_n and C_n in terms of A_n:

$$B_n = a_1 A_n \frac{\sigma(\gamma_2 + (2n-1)z_0)\sigma(\gamma_1 - z_0)}{\sigma(\gamma_1 - \gamma_2)\sigma(2nz_0)}\,, \tag{2.60}$$

$$C_n = a_2 A_n \frac{\sigma(\gamma_1 + (2n-1)z_0)\sigma(\gamma_2 - z_0)}{\sigma(\gamma_2 - \gamma_1)\sigma(2nz_0)}\,. \tag{2.61}$$

In a neighbourhood of z_0 the function ψ^0_{2n} is of the form $(z - z_0)^{-n}$. This enables one to find an explicit expression for the coefficient A_n,

$$A_n = \frac{\sigma(2nz_0)\sigma(\gamma_1 - \gamma_2)}{(a_1 - a_2)\sigma^n(2z_0)\sigma((2n-1)z_0 + \gamma_2)\sigma((2n-1)z_0 + \gamma_1)}\,. \tag{2.62}$$

We can find the explicit form of the functions ψ^i_{2n+1} similarly:

$$\psi^0_{2n+1}(z) = A'_n \frac{\sigma(z+z_0)\sigma(z-\gamma_1-\gamma_2-(2n+1)z_0)}{\sigma(z-\gamma_1)\sigma(z-\gamma_2)}\left[\frac{\sigma(z+z_0)}{\sigma(z-z_0)}\right]^n, \tag{2.63}$$

$$\psi^1_{2n+1}(z) = \left(B'_n \frac{\sigma(z-\gamma_1-2nz_0)}{\sigma(z-\gamma_1)} + C'_n \frac{\sigma(z-\gamma_2-2nz_0)}{\sigma(z-\gamma_2)}\right)\left[\frac{\sigma(z+z_0)}{\sigma(z-z_0)}\right]^n, \tag{2.64}$$

where

$$B'_n = a_1^{-1} A'_n \frac{\sigma(\gamma_2 + (2n+1)z_0)\sigma(\gamma_1 + z_0)}{\sigma(\gamma_1 - \gamma_2)\sigma(2nz_0)}\,, \tag{2.65}$$

$$C'_n = a_2^{-1} A'_n \frac{\sigma(\gamma_1 + (2n+1)z_0)\sigma(\gamma_2 + z_0)}{\sigma(\gamma_2 - \gamma_1)\sigma(2nz_0)}\,, \tag{2.66}$$

$$A'_n = \frac{\sigma(2nz_0)\sigma(\gamma_1 - \gamma_2)}{\sigma^n(2z_0)} (I'_n - I''_n)^{-1}, \tag{2.67}$$

$$I'_n = \frac{\sigma(\gamma_2 + (2n+1)z_0)\sigma(\gamma_1 + (2n-1)z_0)\sigma(\gamma_1 + z_0)}{a_1 \sigma(z_0 - \gamma_1)}\,, \tag{2.68}$$

$$I''_n = \frac{\sigma(\gamma_1 + (2n+1)z_0)\sigma(\gamma_2 + (2n-1)z_0)\sigma(\gamma_2 + z_0)}{a_2 \sigma(z_0 - \gamma_2)}\,. \tag{2.69}$$

Similar expressions in terms of Riemann theta functions can be written also in the general case.

Corollary 2.2. *The coefficients of the operators L_f defined by virtue of the previous theorem are quasi-periodic functions of the variable n.*

Remark. It should be noted that, when we pass to the continuum limit

$$z_0 \to 0, \quad n \to \infty, \quad nz_0 \to x,$$

the functions $\psi_{2n}^{(i)}$ and $\psi_{2n+1}^{(i)}$ converge to different functions of the continuous variable x. This apparently explains why natural cases in which functional parameters are absent are not known in the problem of commuting differential operators.

2.5. Rank > 1. Combined infinities. Let us now consider the case of commuting operators of the form (2.2) of maximal possible rank $l = r_+ + r_-$. In this case there is a gluing together of the formal eigenvalues of the operator A at two points $\mathcal{L}_\pm$ at infinity. A necessary condition for at least one eigenvalue of $A(z)$ to coincide with one of the eigenvalues of $A^-(z_-)$ is given by the equalities $m_+ = m_- = m$ and $n_+ = n_- = n$, that is, the case of completely or partially combined infinities can occur only in the classification problem for commuting operators of the form

$$L = \sum_{i=-Nr_-}^{Nr_+} u_i(n) T^i, \qquad A = \sum_{i=-Mr_-}^{Mr_+} v_i(n) T^i, \qquad (n, m) = 1. \tag{2.70}$$

By the *type* of a commuting pair of operators we mean the tuples of pairs $(\mu_i^+ \,|\, \mu_i^-)$, where the index i ranges over the set I of all distinct eigenvalues of $A(z)$ and $A^-(z_-)$ and the numbers $\mu_i^\pm$ are the multiplicities of the corresponding eigenvalue for each of the operators separately. We note that the type introduced above for operators with separated infinities can be regarded as the special case of the general definition in which the set I is the union of the sets $I_\pm$ which parametrize the indices $i_\pm$ in (2.49), and all the pairs are of the form $(\mu_i^+ \,|\, 0)$ or $(0 \,|\, \mu_j^-)$.

In the opinion of the authors there are no restrictions on the types formed by pairs of two non-zero numbers $\mu_i^\pm > 0$. However, if there is a pair of the form $(\mu|\, 0)$, then, as above, there is seemingly only one restriction, namely, there must also be a pair of the form $(0|\,\mu)$. The complete solution of the classification problem for commuting difference operators requires the construction of commuting operators of different types. We leave this problem open and consider below the problem of constructing commuting operators only for the type consisting of a single pair $(r_+|\,r_-)$. We have already noted above that this corresponds to the case of operators of maximal possible scalar rank $l = r_+ + r_-$.

Direct spectral problem. It follows from the construction of formal eigenfunctions in a neighbourhood of the infinities that the maximal rank l can occur only if the conditions $v_i^+ = v_j^- = v$ hold, that is, both operators are of the form (2.70) and their highest and lowest coefficients under the gauge condition $u_{Nr_+} = 1$ are of the form

$$u_{-Nr_-} = h^{-1}(n - Nr_-)h(n), \qquad v_{Mr_+} = v, \qquad v_{-Mr_-} = vh^{-1}(n - Mr_-)h(n). \tag{2.71}$$

As above, we denote by Γ the curve given by the equation $R(w, E) = 0$, where $R(w, E)$ is the root of degree l of the characteristic polynomial,

$$\det(w - A(E)) = R^l(w, E).$$

In a neighbourhood of the point at infinity we have the factorization

$$R(w, E) = \left\lfloor \prod_{k=0}^{N-1} \bigl(w - v(z_k)\bigr) \right\rceil, \tag{2.72}$$

where the product is taken over all roots of E^{-1} of degree N. Thus, in the case of maximal rank the spectral curve is compactified at infinity by a *single* smooth point; hence, it is irreducible.

Corresponding to every point of the curve Γ is an l-dimensional space of joint eigenfunctions of the operators L and A. Let us fix a basis $\psi^i(Q)$ in this space by the conditions

$$\psi_n^i(Q) = \delta_{in}, \qquad -r_- \leqslant i, n < r_+. \tag{2.73}$$

We note that the choice of an interval of values of i and n used when fixing the normalization is fundamental for the closed description of the analytic properties of the joint eigenfunctions in a neighbourhood of the point at infinity.

Theorem 2.5. *In the case of general position the joint eigenfunctions ψ_n^i of a pair of commuting operators of rank l normalized by the conditions* (2.73) *have the following properties.*

1^0. *In the affine part of the spectral curve Γ the functions ψ_n^i have gl poles γ_s independent of n at which the following relations hold*:

$$\alpha_s^j \operatorname{res}_{\gamma_s} \psi_n^i(Q) = \alpha_s^i \operatorname{res}_{\gamma_s} \psi_n^j(Q). \tag{2.74}$$

2^0. *In a neighbourhood of the point P_0 'at infinity' the row vector $\psi_n = \{\psi_n^i\}$ is of the form*

$$\psi_n = \left(\sum_{s=0}^{\infty} \xi_s(n) z^s\right) \Psi_0(n, z), \qquad z^{-n} = E. \tag{2.75}$$

Here $\xi_s(n) = \{\xi_s^i(n)\}$ *are row vectors,*

$$\xi_0^i = \delta_0^i; \tag{2.76}$$

$\Psi_0(n,z)$ *is the Wronski matrix with* $\Psi_0^{j\,i}(n,z) = \phi_{n+j}^i(z)$, $-r_- \leqslant i,j < r_+$, *constructed from the basis* ϕ^i *of solutions of the equation*

$$\phi_{n+r_+} + \sum_{i=-r_-}^{r_+-1} f_i^{\,0}(n)\phi_{n+i} = z^{-1}\phi_n, \tag{2.77}$$

whose coefficients $f_i^{\,0}(n)$ *are polynomial functions of the coefficients of the operator* L. *The basis solutions* ϕ^i *are normalized by the conditions*

$$\Psi_0(0,z) = 1. \tag{2.78}$$

Proof. We denote by $\Psi(n,Q)$, $Q \in \Gamma$, the Wronski matrix $\Psi^{j\,i}(n,Q) = \psi_{n+j}^i(Q)$, $-r_- \leqslant i,j < r_+$. In a neighbourhood of the point P_0 at infinity, where the local coordinate is $z = E^{-1/n}$, this matrix can be regarded as a function of the variable z, that is, $\Psi(n,z)$. Using this function, we shall now find the polynomials ϕ_n^i of the variable z^{-1} by specifying their asymptotic behaviour as $z \to 0$.

Lemma 2.4. *In the case of general position there are functions* $\phi_n^i(z)$ *holomorphic on the extended* z*-plane away from the point* $z = 0$ *and such that in a neighbourhood of* $z = 0$ *the row vector* $\phi_n(z) = (\phi_n^i(z))$ *is of the form*

$$\phi_n(z) = r_n(z)\Psi(n,z), \tag{2.79}$$

where $r_n(z)$ *is a row vector holomorphic in a neighbourhood of* $z = 0$ *whose value at* $z = 0$ *is given by*

$$r_n^i(0) = \delta_0^i, \tag{2.80}$$

and these functions $\phi_n^i(z)$ *are uniquely determined by the above conditions.*

The problem of constructing $\phi_n(z)$ is a standard Riemann problem. Choosing a small neighbourhood of the point $z = 0$, we define holomorphic vector functions ϕ_n and r_n outside and inside this neighbourhood, respectively, such that the relation (2.79) holds on the boundary. If the argument of the determinant of the regluing matrix has zero increment when going around the contour, then in general position the Riemann problem has a unique solution if one fixes the value of the vector function at some point. Hence, to prove the lemma, it suffices to show that the determinant of $\Psi(n,z)$ is holomorphic in a neighbourhood of $z = 0$ and in general position is non-zero at $z = 0$. This fact is a consequence of the result of the next lemma, which is important in what follows.

Lemma 2.5. *In a neighbourhood of the point at infinity the matrix function*

$$\mathcal{X}(n,Q) = \Psi(n+1,Q)\Psi^{-1}(n,Q) \tag{2.81}$$

is of the form

$$\mathcal{X}(n,z)=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ \chi_{-r_-}(n,z) & \chi_{-r_-+1}(n,z) & \chi_{r_-+2}(n,z) & \dots & \chi_{r_+-1}(n,z) \end{pmatrix}, \tag{2.82}$$

$$\chi_i(n,z)=z^{-1}\delta_{i,0}-f_i(n,z), \tag{2.83}$$

where the functions $f_i(n,z)$ are regular series in the variable z.

Proof. The matrix $\mathcal{X}(n,Q)$ does not depend on the normalization of the basis functions ψ_n^i, and therefore to find the asymptotic behaviour of $\mathcal{X}(0,Q)$ in a neighbourhood of the point at infinity one can use the formal solutions constructed in subsection 2.2. It follows from the equalities (2.27) and (2.34) that the Wronski matrix $\Psi_\infty(0,z)$ constructed from these formal solutions has the block form

$$\Psi_\infty(0,z)=\begin{pmatrix} z^{-1}A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}, \tag{2.84}$$

where $A(z)$ and $D(z)$ are $r_-\times r_-$ and $r_+\times r_+$ matrices, respectively. All the matrices A, B, C, and D are series in the variable z. The constant term of the series $D(z)$ is a lower triangular matrix with 1 along the diagonal. The constant term of the series $A(z)$ is an upper triangular matrix. This implies that the inverse matrix is of the form

$$\Psi_\infty^{-1}(0,z)=\begin{pmatrix} zA_1(z) & zB_1(z) \\ zC_1(z) & D_1(z) \end{pmatrix}. \tag{2.85}$$

Moreover, the constant terms of the regular series A_1, B_1, C_1, and D_1 are

$$\begin{aligned} A_1(0)&=A^{-1}(0), & B_1(0)&=-A^{-1}(0)B(0)D^{-1}(0), \\ D_1(0)&=D^{-1}(0), & C_1(0)&=-D^{-1}(0)C(0)A^{-1}(0). \end{aligned} \tag{2.86}$$

For the formal solutions the terms $\psi^i_{r_+}$ are of the form $z^{-1}\delta^i_0+f^i(z)$, where the functions f^i are regular. Hence, the last row of the matrix $\mathcal{X}(0)$ (this row is equal to $\psi_r\Psi_\infty^{-1}(0)$) is of the form

$$\mathcal{X}^{r_+-1,i}(0)=z^{-1}\delta^i_0+f^i(z). \tag{2.87}$$

It follows from the translation invariance of the construction of formal solutions that the index n can be replaced by $n-n_0$. This shift does not change the matrix $\mathcal{X}(n)$. Hence, the last row of the matrix $\mathcal{X}(n_0)$ has the same structure as $\mathcal{X}(0)$ for any n_0. This proves the lemma.

The normalization conditions (2.73) are equivalent to the condition $\Psi(0,z)\equiv 1$. It follows from (2.82) that

$$\det\Psi(n,z)=\prod_{m=0}^{n-1}\det\mathcal{X}(m,z)=(-1)^n\prod_{m=0}^{n-1}f_{-r_-}(m,z),$$

and hence the determinant is holomorphic in a neighbourhood of the point $z = 0$ and in general position is non-zero at $z = 0$. This assertion completes the proof of Lemma 2.4.

We note that, by definition, the functions ϕ_n^i constructed above are entire functions of the variable z^{-1}. Since the regluing function $\Psi(n, z)$ is meromorphic in a neighbourhood of $z = 0$, it follows that the functions ϕ_n^i have a finite-order pole at $z = 0$, and hence are polynomials in the variable z^{-1}.

Let $\Psi_0(n, z)$ be the Wronski matrix of the functions $\phi_n^i(z)$, that is, $\Psi^{j,i}(n, z) = \phi_{n+j}^i(z)$.

Lemma 2.6. *In a neighbourhood of the point $z = 0$ the matrix function Ψ_0 is of the form*

$$\Psi_0(n, z) = R(n, z)\Psi(n, z), \tag{2.88}$$

where $R(n, z)$ is a matrix function holomorphic in a neighbourhood of $z = 0$ such that $R(n, 0)$ is of the block form

$$R(n, 0) = \begin{pmatrix} R_- & 0 \\ 0 & R_+ \end{pmatrix}, \tag{2.89}$$

where R_+ (R_-) is a lower (upper) triangular $r_\pm \times r_\pm$ matrix with 1 along the diagonal.

Proof. It follows from the definition of ϕ_n that the jth row R_j of the matrix R for $j > 0$ is

$$R_j(n, z) = r_{n+j}(n, z) \prod_{i=0}^{j-1} \mathcal{X}(n + i, z). \tag{2.90}$$

By (2.80) and (2.83), R_j is regular. Moreover, the coordinates $R_j^i(n, 0)$ of the vector $R_j(n, 0)$ can be non-zero only if $0 \leqslant i \leqslant j$. We note that $R_j^j(n, 0) = 1$. The inverse matrix $\mathcal{X}^{-1}$ is of the form

$$\mathcal{X}^{-1} = \chi_{-r_-}^{-1} \begin{pmatrix} \chi_{-r_-+1} & \chi_{-r_-+2} & \chi_{-r+3} & \cdots & \chi_{r_+-1} & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{2.91}$$

Repeating the above arguments replacing $\mathcal{X}$ by $\mathcal{X}^{-1}$, we obtain the desired assertion also for negative values of the index j. This completes the proof of the lemma.

By construction, $\Psi(0, z) = 1$. Since the elements of the Wronski matrix Ψ_0 are polynomials in the variable z^{-1}, it follows from the equality (2.88) that $\Psi_0(0, z)$ is a constant matrix equal to $R(0, 0)$. Using this fact, one can readily prove by induction on j that $\Psi_0(0, z)$ satisfies the normalization conditions (2.78), that is, $\Psi_0(0, z) = 1$.

Let us now prove that the functions ϕ_n^i satisfy an equation of the form (2.77). To this end, we consider the matrix

$$\Psi_0(n + 1, z)\Psi_0^{-1}(n, z) = \mathcal{X}_0(n, z) = R(n + 1, z)\mathcal{X}(n, z)R^{-1}(n, z). \tag{2.92}$$

It follows from (2.83) and the form of $R(n,0)$ that the only element in the last row of $\mathcal{X}_0$ which has a pole at $z=0$ is $\mathcal{X}_0^{r_+-1,0}$. The coefficient of the singular term z^{-1} of its expansion is equal to 1. Since $\mathcal{X}_0$ is regular for any $z\neq 0$, it follows that $\mathcal{X}_0$ is a polynomial function in the variable z^{-1}. Hence, the last row of $\mathcal{X}_0$ has the desired form.

To complete the proof of the second assertion of the theorem, it suffices to invert the equality (2.88). We have

$$\Psi = R^{-1}\Psi_0. \tag{2.93}$$

For the row with index $j=0$ this equality gives the desired equality (2.75) in which the first factor is the Taylor expansion of the corresponding row of the matrix $R^{-1}(n,z)$.

After finding the asymptotic behaviour of ψ_n in a neighbourhood of the point at infinity, the proof of the first assertion of the theorem, that is, the computation of the degree of the matrix divisor of poles, completely follows the lines of reasoning for the similar assertion in the continuous case [2].

In the general case the theorem proved above assigns a non-singular algebraic curve Γ with a marked point P_0, a set of Tyurin parameters of degree gl, and a set of $l-1$ arbitrary functions $f_i^0(n)$ of the discrete variable n to a pair of commuting operators of the form (2.70) with maximal possible rank $l=r_++r_-$:

$$L, A \mapsto \{\Gamma, (\gamma,\alpha), f_i^0(n)\}. \tag{2.94}$$

Inverse spectral problem. Let us consider an arbitrary smooth algebraic curve Γ with a fixed local parameter z in a neighbourhood of a marked point P_0. We take an arbitrary set of functions $f_i^0(n)$, $r_-\leqslant i<r_+$, and denote by $\Psi_0(k)$ the Wronski matrix $\Psi_0^{j,i}=\phi_{n+j}^i$, $r_-\leqslant j<r_+$, constructed from the solutions of the difference equation

$$\sum_{i=r_-}^{r_+} f_i^0(n)\phi_{n+i} = z^{-1}\phi_n, \qquad f_{r_+}^0=1, \tag{2.95}$$

of degree $l=r_++r_-$, normalized by the conditions

$$\phi_n^i=\delta_0^i, \qquad r_-\leqslant i<r_+. \tag{2.96}$$

Theorem 2.6 [21]. *For any set of Tyurin parameters in general position of degree lg and rank l, that is, for a set of lg points γ_s and a set of projective l-dimensional vectors $\alpha_s=(\alpha_s^i)$, $r_-\leqslant i\leqslant r_+$, there is a unique vector function $\psi_n(Q)$ whose coordinates away from the point P_0 have at most simple poles at the points γ_s. The residues of the functions at these points satisfy the relations* (2.74). *The row vector ψ_n is of the following form in a neighbourhood of P_0*:

$$\psi_n = \left(\sum_{s=0}^{\infty}\xi_s(n)z^s\right)\Psi_0(n,z), \qquad \xi_0^i=\delta_0^i. \tag{2.97}$$

For any meromorphic function $f\in\mathcal{A}(\Gamma,P_0)$ on Γ having a unique pole of order N at P_0 there is a unique operator L_f of the form

$$L_f=\sum_{i=-Nr_-}^{Nr_+} u_i(n)T^i, \qquad u_{Nr_+}=1, \tag{2.98}$$

such that

$$L_f\psi(Q) = f(Q)\psi(Q)\,. \tag{2.99}$$

The proof of the theorem is standard. The equality (2.97) is equivalent to the condition that ψ is a solution of the Riemann problem on Γ in which the function Ψ_0 is the regluing function in a neighbourhood of the marked point. In general position the existence and uniqueness of a solution of this problem follows from results in [28] and [29] or simply from the Riemann–Roch theorem for vector bundles (for details, see [2]). The same results enable us to prove the other assertion of the theorem.

Theorem 2.7. *Every commutative ring $\mathcal{A}$ of operators of the form* (2.98) *with maximal possible rank l is isomorphic to the ring $\mathcal{A}(\Gamma, P_0)$ of meromorphic functions on some algebraic curve Γ with a single pole at a marked point P_0. In the case of general position the isomorphism $\mathcal{A}(\Gamma, P_0) \cong \mathcal{A}$ is given by the equality* (2.99) *in which the Baker–Akhiezer vector function is given by some set of Tyurin parameters (γ, α).*

2.6. Discrete dynamics of the Tyurin parameters. Below we derive discrete equations for the Tyurin parameters for commuting operators with combined infinities and maximal possible rank l. In general position an arbitrary algebraic curve Γ with marked point P_0, a set of Tyurin parameters (γ, α), and arbitrary coefficients $f_i^0(n)$ of the difference equation (2.95) determine a vector function $\psi_n(Q)$ by Theorem 2.6. As above, let $\Psi(n, Q)$ be the corresponding Wronski matrix. The matrix function $\mathcal{X}(n, Q)$ given by (2.81) has asymptotic behaviour given by (2.82) and (2.83). We denote by

$$f_i(n) = f_i(n, 0) \tag{2.100}$$

the values at $z=0$ of the regular series $f_i(n, z)$, see (2.83). These functions of the discrete variable n can be expressed explicitly in terms of the original variables $f_i^0(n)$ and the first coefficients $\xi_1(n)$ of the expansion (2.97) for ψ_n. The corresponding relations are far from effective formulae because, as above, the expressions for $\xi_1(n)$ in terms of the original parameters $(\gamma_s, \alpha_s, f_i^0(n))$ require the solution of the corresponding Riemann problem. At the same time, as we shall see below, there is no need to obtain explicit formulae for $f_i(n)$, because these functions can be taken as independent parameters determining the coefficients of the commuting operators.

For $n \neq 0$ we denote by $\gamma_s(n)$ the zeros of $\det \Psi(n, Q)$. In general position these zeros are simple and their number is equal to gl. We denote by $\alpha_s(n)$ the corresponding left null-vector,

$$\alpha_s(n)\Psi(n, \gamma_s(n)) = 0. \tag{2.101}$$

For $n = 0$ we set

$$\gamma_s(0) = \gamma_s, \qquad \alpha_s(0) = \alpha_s. \tag{2.102}$$

The following assertion results immediately from the definition (2.81).

Lemma 2.7. *The matrix function $\mathcal{X}(n, Q)$ has simple poles at the points $\gamma_s(n)$. The following relations hold for the residues of its matrix elements*:

$$\alpha_s^j(n)\operatorname{res}_{\gamma_s(n)} \mathcal{X}^{m,i}(n, Q) = \alpha_s^i(n)\operatorname{res}_{\gamma_s(n)} \mathcal{X}^{m,j}(n, Q). \tag{2.103}$$

The points $\gamma_s(n+1)$ *are zeros of the determinant of the matrix* $\mathcal{X}(n,Q)$, *that is,*

$$\det \mathcal{X}(n,\gamma_s(n+1)) = 0. \tag{2.104}$$

The vector $\alpha_s(n+1)$ *is a left null-vector of the matrix* $\mathcal{X}(n,\gamma_s(n+1))$:

$$\alpha_s(n+1)\mathcal{X}(n,\gamma_s(n+1)) = 0. \tag{2.105}$$

A simple calculation of the dimensions using the Riemann–Roch theorem leads to the following assertion.

Lemma 2.8. *For any smooth algebraic curve* Γ *with a fixed local coordinate* $k^{-1}(Q)$ *in a neighbourhood of the marked point* P_0 *and for any data set* $(\gamma_s(n),\alpha_s(n),f_i(n))$ *in general position there is a unique meromorphic matrix function* $\mathcal{X}(n,Q)$, $Q\in\Gamma$, *for which there are at most simple poles at the points* P_0 *and* γ_s *and*:

(i) *the expansion of* $\mathcal{X}(n,Q)$ *in a neighbourhood of* P_0 *is of the form* (2.82), (2.83) *in which the regular series* $f_i(n,z)$ *satisfy the relation* (2.100);

(ii) *the residues of* $\mathcal{X}(n,Q)$ *at the points* γ_s *satisfy the relations* (2.103).

The equalities (2.104) and (2.105) can be regarded as equations determining the parameters $(\gamma_s(n+1),\alpha_s(n+1))$ for a given matrix function $\mathcal{X}(n,Q)$. Since the last function is uniquely determined by $(\gamma_s(n),\alpha_s(n),f_i(n))$, we obtain the following conclusion.

Corollary 2.3. *The parameters* $f_i(n)$ *and the Tyurin parameters* (γ,α) *giving the initial conditions* (2.102) *of the corresponding dynamical system form a complete data set parametrizing the commuting operators corresponding to a fixed spectral curve.*

Example. $g=1$, $l=2$. Let us consider a pair of commuting operators of the form

$$L=\sum_{i=-2}^{2} u_i(n)T^i, \qquad A=\sum_{i=-3}^{3} v_i(n)T^i, \tag{2.106}$$

and of maximal possible rank $l=2$. In this case the spectral curve Γ is elliptic. Let 2ω and $2\omega'$ be the periods of this curve. Fixing a fundamental domain, one can identify the marked point P_0 with the point $z=0$ without loss of generality.

The operators L and A are uniquely determined by the Tyurin parameters and also by the parameters $f_i(n)$, $i=-1,0$, which we denote as follows:

$$f_{-1}=c_{n+1}, \qquad f_0=v_{n+1}. \tag{2.107}$$

Our objective is to obtain explicit formulae for the coefficients of the commuting operators (2.106) by using equations describing the discrete *dynamics* of the Tyurin parameters

$$\gamma_n^1=\gamma_1(n), \qquad \gamma_n^2=\gamma_2(n). \tag{2.108}$$

For the vectors α_s^i, $i=-1,0$, which are defined up to proportionality, we choose the normalization under which the last coordinate is equal to one, $\alpha_s^0=1$, that is, in the example under consideration the vectors $\alpha_s(n)$ are two-dimensional row vectors with coordinates

$$\alpha_1(n) = (a_n^1, 1), \qquad \alpha_2(n) = (a_n^2, 1). \tag{2.109}$$

According to Lemma 2.8, the quantities $\gamma_n^{1,2}$, $a_n^{1,2}$, c_{n+1}, v_{n+1} uniquely determine the matrix $\mathcal{X}_n^{ji} = \mathcal{X}_n^{ji}(n,z)$, $i,j = -1,0$. Let us find an explicit form of this matrix in terms of the standard Weierstrass functions. By definition, the matrix $\mathcal{X}_n$ is of the form

$$\mathcal{X}_n = \begin{pmatrix} 0 & 1 \\ \chi_n^1(z) & \chi_n^2(z) \end{pmatrix}. \tag{2.110}$$

The elliptic function $\chi_n^1(z)$ has poles at the points $\gamma_n^{1,2}$ and is equal to $-c_{n+1}$ at the point $z=0$. Hence, this function can be represented *a priori* in the form

$$\chi_n^1 = -c_{n+1} + A_1\big(\zeta(z-\gamma_n^1) + \zeta(\gamma_n^1)\big) + B_1\big(\zeta(z-\gamma_n^2) + \zeta(\gamma_n^2)\big), \tag{2.111}$$

where $\zeta(z)$ is the standard Weierstrass zeta function.

At the marked point $z=0$ the function χ_n^2 is of the form $\chi^2 = z^{-1} - v_{n+1} + O(z)$, that is,

$$\chi_n^2 = -v_{n+1} + \zeta(z) + A_2\big(\zeta(z-\gamma_n^1) + \zeta(\gamma_n^1)\big) + B_2\big(\zeta(z-\gamma_n^2) + \zeta(\gamma_n^2)\big). \tag{2.112}$$

Since χ_n^i is elliptic, it follows that

$$A_1 + B_1 = 0, \qquad A_2 + B_2 = -1. \tag{2.113}$$

Moreover, we have

$$A_1 = a_n^1 A_2, \qquad B_1 = a_n^2 B_2. \tag{2.114}$$

It follows from (2.113) and (2.114) that

$$\chi_n^1 = -c_{n+1} + \frac{a_n^1 a_n^2}{a_n^1 - a_n^2}\big(\zeta(z-\gamma_n^1) - \zeta(z-\gamma_n^2) + \zeta(\gamma_n^1) - \zeta(\gamma_n^2)\big), \tag{2.115}$$

$$\begin{aligned} \chi_n^2 = \zeta(z) - v_{n+1} &+ \frac{a_n^2}{a_n^1 - a_n^2}\big(\zeta(z-\gamma_n^1) + \zeta(\gamma_n^1)\big) \\ &+ \frac{a_n^1}{a_n^2 - a_n^1}\big(\zeta(z-\gamma_n^2) + \zeta(\gamma_n^2)\big). \end{aligned} \tag{2.116}$$

According to Lemma 2.7, the points γ_{n+1}^s are determined from the equation

$$\det \mathcal{X}_n(\gamma_{n+1}^s) = \chi^1(\gamma_{n+1}^s) = 0.$$

Hence,

$$c_{n+1} = \frac{a_n^1 a_n^2}{a_n^1 - a_n^2}\big(\zeta(\gamma_{n+1}^s - \gamma_n^1) - \zeta(\gamma_{n+1}^s - \gamma_n^2) + \zeta(\gamma_n^1) - \zeta(\gamma_n^2)\big). \tag{2.117}$$

We note that the sum $\gamma_n^1+\gamma_n^2=2c$ does not depend on n, because the points γ_n^s are the poles and the points γ_{n+1}^s are the zeros of the elliptic function. Everywhere below, we can set

$$\gamma_n^1=\gamma_n+c, \qquad \gamma_n^2=-\gamma_n+c, \qquad c=\text{const.} \tag{2.118}$$

Using this, we can rewrite (2.115) and (2.116) as

$$\chi_n^1=\frac{a_n^1a_n^2}{a_n^1-a_n^2}\big[\zeta(z-\gamma_n-c)-\zeta(z+\gamma_n-c)-\zeta(\gamma_{n+1}-\gamma_n)-\zeta(\gamma_{n+1}+\gamma_n)\big], \tag{2.119}$$

$$\begin{aligned}\chi_n^2=-v_{n+1}+\zeta(z)&+\frac{a_n^2}{a_n^1-a_n^2}\big(\zeta(z-\gamma_n-c)+\zeta(\gamma_n+c)\big)\\&+\frac{a_n^1}{a_n^2-a_n^1}\big(\zeta(z+\gamma_n-c)-\zeta(\gamma_n-c)\big).\end{aligned} \tag{2.120}$$

If for the new independent data we choose the functions v_n and γ_n of the discrete variable, then the equation (2.117),

$$c_{n+1}=\frac{a_n^1a_n^2}{a_n^1-a_n^2}\big(\zeta(\gamma_{n+1}-\gamma_n)-\zeta(\gamma_{n+1}+\gamma_n)+\zeta(\gamma_n+c)+\zeta(\gamma_n-c)\big), \tag{2.121}$$

must simply be regarded as the definition of the variables c_{n+1}.

It follows from (2.105) that

$$a_s(n+1)=-\chi_n^2(\gamma_s(n+1)). \tag{2.122}$$

Using the formula (2.120), we obtain the following recursion expressions for the parameters $a_{n+1}^{1,2}$:

$$\begin{aligned}a_{n+1}^1=v_{n+1}-\zeta(\gamma_{n+1}+c)&-\frac{a_n^2}{a_n^1-a_n^2}\big(\zeta(\gamma_{n+1}-\gamma_n)+\zeta(\gamma_n+c)\big)\\&-\frac{a_n^1}{a_n^2-a_n^1}\big(\zeta(\gamma_{n+1}+\gamma_n)-\zeta(\gamma_n-c)\big),\end{aligned} \tag{2.123}$$

$$\begin{aligned}a_{n+1}^2=v_{n+1}+\zeta(\gamma_{n+1}-c)&+\frac{a_n^2}{a_n^1-a_n^2}\big(\zeta(\gamma_{n+1}+\gamma_n)-\zeta(\gamma_n+c)\big)\\&-\frac{a_n^1}{a_n^1-a_n^2}\big(\zeta(\gamma_{n+1}-\gamma_n)+\zeta(\gamma_n-c)\big).\end{aligned} \tag{2.124}$$

Thus, we have proved that an arbitrary set of functions γ_n and v_n and a constant determine the matrix function $\mathfrak{X}_n$, and hence also the coefficients of the commuting operators of rank 2 corresponding to an elliptic spectral curve. Every such operator corresponds to a function on the spectral curve having a pole at the marked point $z=0$. The simplest operator of this kind, L_4, is of order 4 and corresponds to the Weierstrass function $\wp(z)=z^{-2}+O(z^2)$.

To find the coefficients of this operator, one must take the vector function $\wp(z)\psi_n\Psi_n^{-1}$ and decompose it with respect to $\psi_{n+i}\Psi_n^{-1}$, $-1\leqslant i\leqslant 2$. To this end, it suffices to take only the singular terms of the expansions of all vectors in a neighbourhood of $z=0$. We denote by $\widetilde{\psi}_m$ the polynomials in the variable $k=z^{-1}$ such that $\psi_m\Psi_n^{-1}=\widetilde{\psi}_m+O(k^{-1})$. Then

$$\widetilde{\psi}_{n+2}=(-c_{n+1},k-v_{n+1}),\qquad \widetilde{\psi}_{n+1}=(0,1),\qquad \widetilde{\psi}_n=(1,0). \tag{2.125}$$

To find $\widetilde{\psi}_{n-1}$, we use the relations $\Psi_{n-1}=\mathcal{X}_{n-1}^{-1}\Psi_n$ and

$$\mathcal{X}_n^{-1}=\frac{1}{\chi_n^1}\begin{pmatrix}-\chi_n^2 & 1\\ \chi_n^1 & 0\end{pmatrix}. \tag{2.126}$$

Let us denote by ξ_n^{ij} the coefficients of the expansions

$$\chi_n^1=-c_{n+1}(1+\xi_n^{11}z+\xi_n^{12}z^2+\cdots), \tag{2.127}$$

$$\chi_n^2=k-v_{n+1}+\xi_n^{21}z+\cdots. \tag{2.128}$$

Then

$$\widetilde{\psi}_{n-1}=c_n^{-1}(k-v_n-\xi_{n-1}^{11},-1). \tag{2.129}$$

One can find the value of $\widetilde{\psi}_{n-2}$ similarly.

After straightforward but rather cumbersome manipulations, we see that the operator L_4 is equal to

$$L_4=L_2^2-(\xi_{n-1}^{11}+\xi_{n-2}^{11})T+c_n(\xi_{n-1}^{11}+\xi_{n-2}^{11})T^{-1}+u_n, \tag{2.130}$$

where L_2 is the Schrödinger difference operator

$$L_2=T+v_n+c_nT^{-1}, \tag{2.131}$$

and the function u_n is defined by the formula

$$u_n=v_n(\xi_{n-1}^{11}-\xi_{n-2}^{11})+\xi_{n-1}^{12}+\xi_{n-2}^{12}-(\xi_{n-2}^{11})^2-(\xi_{n-1}^{21}+\xi_{n-2}^{21}). \tag{2.132}$$

Symmetric case. Let the constant c in (2.118) vanish, $c=0$. Then χ_n^1 is an even function of the variable z. Thus, $\xi_n^{11}=0$. It follows from (2.119) that

$$\xi_n^{12}=-\frac{a_n^1a_n^2}{(a_n^1-a_n^2)}\,\frac{\wp'(\gamma_n)}{c_{n+1}}\,. \tag{2.133}$$

Using the formula (2.121), we obtain

$$\xi_n^{12}=\frac{\wp'(\gamma_n)}{\zeta(\gamma_{n+1}+\gamma_n)-\zeta(\gamma_{n+1}-\gamma_n)-2\zeta(\gamma_n)}=\wp(\gamma_n)-\wp(\gamma_{n+1}). \tag{2.134}$$

(To prove the last equality, one can use the addition formulae for the Weierstrass zeta function; however, one can also verify it directly by comparing the poles and residues of the functions on both sides of the equality.) It follows from (2.120) that

$$\xi_n^{21}=\wp(\gamma_n). \tag{2.135}$$

Substituting the last two formulae into (2.132), we see that the operator L_4 in the symmetric case is equal to

$$L_4 = L_2^2 - \wp(\gamma_n) - \wp(\gamma_{n-1}). \tag{2.136}$$

In the symmetric case the formulae for the coefficients of the Schrödinger operator L_2 defined in (2.131) are also substantially simplified. Let us denote by $F(u,v)$ the elliptic function

$$F(u,v) = \zeta(u+v) - \zeta(u-v) - 2\zeta(v) = \frac{\wp'(v)}{\wp(v) - \wp(u)}\,. \tag{2.137}$$

Then the formulae (2.121)–(2.124) for the symmetric case $c = 0$ can be represented in the form

$$c_{n+1} = -\frac{a_n^1 a_n^2}{a_n^1 - a_n^2} F(\gamma_{n+1}, \gamma_n), \tag{2.138}$$

$$a_{n+1}^1 = v_{n+1} + \frac{1}{2}\left(F(\gamma_n, \gamma_{n+1}) + \frac{a_n^1 + a_n^2}{a_n^1 - a_n^2} F(\gamma_{n+1}, \gamma_n)\right), \tag{2.139}$$

$$a_{n+1}^2 = v_{n+1} - \frac{1}{2}\left(F(\gamma_n, \gamma_{n+1}) - \frac{a_n^1 + a_n^2}{a_n^1 - a_n^2} F(\gamma_{n+1}, \gamma_n)\right). \tag{2.140}$$

The last two equalities are equivalent to

$$a_{n+1}^1 - a_{n+1}^2 = F(\gamma_n, \gamma_{n+1}), \tag{2.141}$$

$$a_{n+1}^1 + a_{n+1}^2 = 2v_{n+1} + \frac{a_n^1 + a_n^2}{a_n^1 - a_n^2}\, F(\gamma_{n+1}, \gamma_n). \tag{2.142}$$

We denote by s_n the expression

$$s_n = -\frac{a_n^1 + a_n^2}{a_n^1 - a_n^2}\,. \tag{2.143}$$

Then

$$a_n^1 + a_n^2 = -s_n F(\gamma_{n-1}, \gamma_n), \qquad \frac{a_n^1 a_n^2}{a_n^1 - a_n^2} = -\frac{1}{4}(s_n^2 - 1) F(\gamma_{n-1}, \gamma_n), \tag{2.144}$$

and the equalities (2.138) and (2.142) can be represented as

$$4c_{n+1} = (s_n^2 - 1) F(\gamma_{n+1}, \gamma_n) F(\gamma_{n-1}, \gamma_n), \tag{2.145}$$

$$2v_{n+1} = s_n F(\gamma_{n+1}, \gamma_n) - s_{n+1} F(\gamma_n, \gamma_{n+1}). \tag{2.146}$$

The last equality shows that in the symmetric case one can take γ_n, s_n as the independent variables. The formulae (2.113), (2.136), (2.145), and (2.146) then give closed explicit expressions for the coefficients of the operator L_4 which were given above in the introduction. The coefficients of the second commuting operator A_6 can be found in a similar way.

§ 3. Higher-Rank Solutions of the 2D Toda lattice

The key element of algebro-geometric constructions of solutions of non-linear equations is the construction of *multiparameter* Baker–Akhiezer functions. These functions, both scalar and vector, are determined by their analytic properties on the corresponding algebraic curve. Below we define the multiparameter Baker–Akhiezer vector functions which are deformations of eigenfunctions of commuting operators of arbitrary rank. For commuting operators with separated infinities and for those with combined infinities these constructions are different.

3.1. Separated infinities. As already noted more than once, commuting operators with separated infinities are uniquely determined by their algebro-geometric spectral data, that is, by the spectral curve with marked points and the Tyurin parameters. There are no functional parameters in the construction of these operators. Functional parameters arise in the construction of the corresponding solutions of the $2D$ Toda lattice. It should be stressed that these functional parameters, in contrast to the construction of commuting operators with combined infinities, are functions of a continuous variable rather than a discrete one. These parameters determine the *grafting* functions $\Psi_\pm(t^\pm, z)$, each of which depends on the corresponding half of the times of the hierarchy of $2D$ Toda equations and is an entire function of the variable z^{-1}.

Let us fix two arbitrary entire functions $\Psi_\pm(z)$ of z^{-1} such that the increment of the argument $\log\det\Psi_\pm$ vanishes upon going around the origin:

$$\oint_{|z|=\varepsilon} d(\log\det\Psi_\pm) = 0. \tag{3.1}$$

Lemma 3.1. *For any smooth algebraic curve Γ with fixed local coordinates $z_\pm$ in neighbourhoods of two marked points $P^\pm$ and for any set of Tyurin parameters of degree rg and rank r in general position, there is a unique vector function $\psi_n(Q)$ such that:*

(i) *its coordinates ψ_n^i, $i = 0, \dots, r-1$, away from the marked points $P^\pm$ have at most simple poles at the points γ_s at which the relations* (2.55) *hold*;

(ii) *in a neighbourhood of the marked points $P^\pm$ the function ψ_n is of the form*

$$\psi_{kr+j} = z_\pm^{\mp k} R_\pm(kr+j, z_\pm)\Psi_\pm(z_\pm), \tag{3.2}$$

where $R_\pm(n,z)$ are row vectors holomorphic in a neighbourhood of the origin and such that the values of their coordinates at the point $z=0$ satisfy the normalization conditions

$$R_-^i(kr+j,0) = 0, \quad i \leqslant j, \qquad R_+^i(kr+j,0) = \begin{cases} 0, & i > j, \\ 1, & i = j. \end{cases} \tag{3.3}$$

The proof of the lemma reduces to a simple computation of the dimension of the space of solutions of the Riemann problem which is equivalent to the conditions (3.2).

If the functions $\Psi_\pm = \Psi_\pm(t^\pm, z)$ depend on some independent variables $t^\pm = (t_j^\pm)$, then the Baker–Akhiezer vector functions ψ_n introduced above depend on the full set of variables, $\psi_n = \psi_n(t^+, t^-, Q)$. We now give the dependence of the grafting functions $\Psi_\pm$ on the variables $t^\pm$ in such a way that the corresponding Baker–Akhiezer functions lead to solutions of the hierarchy of $2D$ Toda equations.

If we restrict ourselves to the construction of solutions proper of the $2D$ Toda equations (1.1), then this dependence on $t_1^+ = \xi$ and $t_1^- = \eta$ can be given by ordinary differential equations whose coefficients are exactly the arbitrary functional parameters.

For any set of arbitrary functions $a_i(t_1^+)$, $y_i(t_1^-)$ we determine the grafting functions $\Psi_\pm$ by means of the equations

$$\partial_{t_1^\pm}\Psi_\pm = M_\pm^{0,1}\Psi_\pm, \qquad \Psi_\pm(0,z) = 1, \tag{3.4}$$

in which the matrices $M_\pm^{0,1}(t_1^\pm, z)$ are of the form

$$M_+^{0,1} = \begin{pmatrix} a_0 & 1 & 0 & \dots & 0 \\ 0 & a_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ z^{-1} & 0 & 0 & \dots & a_{r-1} \end{pmatrix}, \quad M_-^{0,1} = \begin{pmatrix} 0 & 0 & \dots & 0 & b_0 z^{-1} \\ b_1 & 0 & \dots & 0 & 0 \\ 0 & b_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{r-1} & 0 \end{pmatrix}, \tag{3.5}$$

where

$$b_i = e^{y_i - y_{i-1}}, \qquad y_{-1} = y_{r-1}. \tag{3.6}$$

Theorem 3.1. *The Baker–Akhiezer vector function ψ_n corresponding to an arbitrary data set $\{\Gamma, P^\pm, z_\pm, (\gamma,\alpha), a_i, y_i\}$ in general position satisfies the equations*

$$\partial_{t_1^+}\psi_n = \psi_{n+1} + v_n\psi_n, \qquad \partial_{t_1^-}\psi_n = c_n\psi_{n-1}, \tag{3.7}$$

whose coefficients are

$$v_n = \partial_{t_1^+}\varphi_n, \qquad c_n = e^{\varphi_n - \varphi_{n-1}},$$

where

$$\varphi_{kr+i} = y_i(t_1^-) + \log R_-^i(kr+i, 0, t_1^+, t_1^-). \tag{3.8}$$

For any function $f \in \mathcal{A}(\Gamma, P^\pm)$ having poles of orders n_+ and n_- at the points $P_i^\pm$, respectively, there is a unique difference operator L_f of the form (2.2) *with coefficients depending on t_1^+ and t_1^- and such that $L_f\psi^i = f\psi^i$.*

The consistency condition for the equations (3.7) is equivalent to the $2D$ Toda equations (1.1).

Corollary 3.1. *The functions φ_n given by the formula* (3.8) *in which the second summand is defined by the value at the origin of the regular factor in the R_- factorization* (3.2) *are solutions of the $2D$ Toda equations.*

The proof of the theorem is standard and reduces to showing that the functions determined by the formulae of the right- and left-hand sides of the equalities (3.7) have the same analytic properties on Γ. This proof depends on the special form (3.5) of the matrices $M^{0,1}_{\pm}$ only slightly. The assertions of the theorem and the corollary remain completely valid if one replaces $M^{0,1}_{\pm}$ by matrices of the form

$$\widetilde{M}^{0,1}_{\pm} = M^{0,1}_{\pm} + m_{\pm}(t^{\pm}_1), \tag{3.9}$$

where the matrix elements $m^{ij}_{\pm}$ do not depend on z and satisfy the conditions

$$m^{ij}_{+} = 0, \quad i < j, \qquad m^{ij}_{-} = 0, \quad i \geqslant j. \tag{3.10}$$

An extension of the class of grafting functions does not lead to any extension of the class of solutions thus constructed for the equations of the $2D$ Toda lattice. Indeed, the factorization (3.2) and the conditions (3.3) are invariant with respect to the transformations

$$\widetilde{\Psi}_{\pm} = g_{\pm}\Psi_{\pm}, \qquad \widetilde{R}_{\pm} = R_{\pm}g^{-1}_{\pm}, \tag{3.11}$$

where $g_{+}(t^{+}_1)$ is a lower triangular matrix with 1 along the main diagonal and $g_{-}(t^{-}_1)$ is an upper triangular matrix. Hence, these transformations do not modify the corresponding Baker–Akhiezer function. The transformations (3.11) lead to gauge transformations of the matrices,

$$\widetilde{M}^{0,1}_{\pm} \mapsto g^{-1}_{\pm}\partial_{t^{\pm}_1}g_{\pm} - g^{-1}_{\pm}\widetilde{M}^{0,1}_{\pm}g_{\pm}, \tag{3.12}$$

which can always be used to obtain the equalities $m_{\pm} = 0$.

To construct solutions of the full hierarchy of equations of the $2D$ Toda lattice, it suffices to indicate the dependence of the grafting functions $\Psi_{\pm}$ on all the times $t^{\pm}_p$ of the hierarchy by using the differential equations

$$\partial_{t^{\pm}_p}\Psi_{\pm} = M^{0,\,p}_{\pm}\,\Psi_{\pm}, \qquad \Psi_{\pm}(0,z) = 1, \tag{3.13}$$

where the matrices $M^{0,i}_{\pm}(t^{\pm},z)$ depend polynomially on the variable z^{-1}. The consistency conditions for the equations (3.13) for each half of the times, t^{+}_p or t^{-}_p, are gauge equivalent to one of the r-reductions of the KP hierarchy. Since our main objective is the construction of solutions of the equations (1.1), the explicit description of the structure of the matrices $M^{0,p}_{\pm}$, $p > 1$, and the subsequent analysis of the auxiliary soliton system thus arising are left outside the framework of the present paper.

3.2. One-point case. As in the case of separated infinities, the dependence of the one-point multiparameter Baker–Akhiezer function on the variables $t^{\pm}_p$ is completely determined by the dependence of the grafting function $\Psi_0(n,t,z)$ on them. The dependence on each of these variables is determined by the linear equation

$$\partial_{t^{\pm}_p}\Psi_0(n,t,z) = M^{0,p}_{\pm}(n,t,z)\Psi_0(n,t,z), \tag{3.14}$$

in which the matrix $M_{\pm}^{0,i}(n,t,z)$ depends polynomially on the variable z^{-1}. The essential difference between the one-point situation and the case of separated infinities is that, to construct the solutions proper of the equations of the $2D$ Toda lattice, the matrices $M_{\pm}^{0,1}$ must now satisfy the conditions for consistency of the equations (3.14), where $p=1$, and the difference equation

$$\Psi_0(n+1,t,z)=\mathcal{X}_0(n,t,z)\Psi_0(n,z), \tag{3.15}$$

which follows from the definition of Ψ_0 as the Wronski matrix of the solutions of the equations (2.95). This condition means that the matrix $\mathcal{X}_0=(\mathcal{X}_0^{ij})$, $r_-\leqslant i,j<r_+-1$, is of the form

$$\mathcal{X}_0=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & 1\\ \chi^0_{-r_-} & \chi^0_{-r_-+1} & \chi^0_{-r_-+2} & \cdots & \chi^0_{r_+-1}\end{pmatrix}, \tag{3.16}$$

where

$$\chi_i^0=z^{-1}\delta_{i,0}-f_i^{\,0}(n,t). \tag{3.17}$$

Let us consider the matrix functions $M_{\pm}^{0,1}$ of the form

$$M_+^{0,1}=\mathcal{X}_0+A(n,t),\qquad M_-^{0,1}=B(n,t)\mathcal{X}_0^{-1}, \tag{3.18}$$

where A and B are the diagonal matrices

$$\begin{aligned} A&=\operatorname{diag}\{a(n-r_-,t),\dots,a(n+r_+-1,t)\},\\ B&=\operatorname{diag}\{b(n-r_-,t),\dots,b(n+r_+-1,t)\}.\end{aligned} \tag{3.19}$$

In this case the conditions for consistency of the equations (3.14) with $p=1$ and the equation (3.15) are equivalent to the conditions for consistency of the linear system

$$\partial_{t_1^+}\phi_n=\phi_{n+1}+a(n,t)\phi_n,\qquad \partial_{t_1^-}\phi_n=b(n,t)\phi_{n-1}$$

and the equation (2.95). Hence, if the symbols $y_n(t)$ denote functions such that $b(n,t)=e^{y_n-y_{n-1}}$, then the conditions for consistency of the equations (3.14) and (3.15) are equivalent to reduction of the equations of the $2D$ Toda lattice for y_n to the stationary points of some linear combination of flows of the hierarchy that correspond to the times $t^-_{r_-},\dots,t^+_{r_+}$.

Let us fix some solution y_n of this reduction and denote by $\Psi_0(n,t_1^+,t_1^-,z)$ the corresponding solution of the auxiliary linear system (3.14), (3.15). Then the following assertion holds.

Theorem 3.2 [21]. *For each smooth algebraic curve Γ of genus g with a fixed local parameter z in a neighbourhood of a marked point P_0 and for any set of Tyurin parameters (γ,α) in general position of degree lg and rank l there exists a unique vector function $\psi_n(t_1^+,t_1^-,Q)$ whose coordinates away from the point P_0 have at most simple poles at the points γ_s. The residues of these functions at the points γ_s*

satisfy the conditions (2.74). *In a neighbourhood of* P_0 *the row vector* ψ_n *is of the form*

$$\psi_n = \left(\sum_{s=0}^{\infty} \xi_s(n, t_1^+, t_1^-) z^s\right) \Psi_0(n, t_1^+, t_1^-, z), \qquad \xi_0^i = \delta_0^i. \tag{3.20}$$

This function satisfies the equations

$$\partial_{t_1^+} \psi_n = \psi_{n+1} + (\partial_{t_1^+} \varphi_n)\psi_n, \qquad \partial_{t_1^-} \psi_n = (e^{\varphi_n - \varphi_{n-1}})\psi_{n-1}, \tag{3.21}$$

where the functions φ_n *are given by the formula*

$$\varphi_n = y_n(t_1^+, t_1^-) + \log\bigl(1 + \xi_1^{(-1)}(n, t_1^+, t_1^-)\bigr), \tag{3.22}$$

in which $\xi_1^{(-1)}$ *is the coordinate of the vector* ξ_1 *with index* $i = -1$ *in the expansion* (3.20).

Example. In the case of rank $l = 2$ and $r_\pm = 1$ the grafting function Ψ_0 can be determined from any solution of the one-dimensional Toda lattice

$$\ddot{y}_n = e^{y_n - y_{n-1}} - e^{y_{n+1} - y_n} \tag{3.23}$$

and is of the form

$$\Psi_0 = \Phi(n, t, z) e^{x z^{-1}}, \qquad x = t_1^+ + t_1^-, \quad t = t_1^+ - t_1^-, \tag{3.24}$$

where Φ is the Wronski matrix of the solutions of the auxiliary linear system for (3.23).

3.3. Deformations of Tyurin parameters. The problem of recovering a Baker–Akhiezer vector function from its data reduces to the solution of a linear Riemann problem in which the grafting function Ψ_0 determines the regluing function in a neighbourhood of the distinguished point P_0. It was already noted above that this problem cannot be solved explicitly in the general case. At the same time, in some cases one can obtain more explicit expressions for the corresponding solutions of the two-dimensionalized lattice by using the deformation equations for the Tyurin parameters.

Let us denote by $\Psi(n, t, Q)$ the Wronski matrix whose rows are the Baker–Akhiezer vector functions $\psi_{n+j}(t, Q)$. As above, we define a deformation of the Tyurin parameters as follows. In general position the determinant $\det \Psi(n, t, Q)$ has gl simple zeros $\gamma_s(n, t)$. We denote by $\alpha_s(n, t)$ the corresponding left zero-vector:

$$\alpha_s(n, t)\Psi(n, t, \gamma_s(n, t)) = 0. \tag{3.25}$$

The difference equations describing the dynamics of the Tyurin parameters with respect to the discrete variable n were obtained above in subsection 2.5. The equations for continuous deformations of the Tyurin parameters follow from previous results of the authors [21].

Let us consider the logarithmic derivative of Ψ with respect to any of the times of the hierarchy:

$$\partial_{t_p^\pm} \Psi = M_\pm^p \Psi. \tag{3.26}$$

This logarithmic derivative $M^p_\pm$ is a meromorphic function on Γ, and away from the marked point this function has simple poles at the points $\gamma_s = \gamma_s(n,t)$. Its Laurent expansion in a neighbourhood of γ_s has the form

$$M = \frac{m_s\alpha_s}{z - z(\gamma_s)} + \mu_s + O(z - z(\gamma_s)), \tag{3.27}$$

where m_s is some column vector. (For brevity, here and henceforth we omit the indices p and $\pm$ in our formulae.) The first two coefficients of this expansion give the right-hand sides of the deformation equations with respect to the variable $t = t^p_\pm$,

$$\partial_t\, z(\gamma_s) = -\operatorname{Tr}(m_s\alpha_s) = -(\alpha_s m_s), \qquad \partial_t \alpha_s = -\alpha_s\mu_s + \kappa_s\alpha_s. \tag{3.28}$$

Here κ_s stands for some constant. Its presence on the right-hand side of the equation reflects the fact that the vectors α_s are determined up to proportionality. The equations (3.28) unambiguously determine some dynamics on the space of Tyurin parameters, which is the symmetric power $S^{gl}(\Gamma \times CP^{l-1})$.

The consistency conditions

$$\partial_t \mathfrak{X}_n = M_{n+1}\mathfrak{X}_n - \mathfrak{X}_n M_n \tag{3.29}$$

of the linear problems

$$\Psi_{n+1} = \mathfrak{X}_n\Psi_n, \qquad \partial_t\Psi_n = M_n\Psi_n \tag{3.30}$$

give a well-defined system of non-linear equations for the parameters in the *singular coefficients* of the expansion of the matrices $\mathfrak{X}_n$ and M_n in a neighbourhood of the distinguished point. Here and below we use the notation $\Psi_n = \Psi(n,t,Q)$, $\mathfrak{X}_n = \mathfrak{X}(n,t,Q)$, and $M_n = M(n,t,Q)$.

Discrete analogue of the Krichever–Novikov equation. As an illustrating example we consider the non-linear equations arising in the case of rank $l = 2$ and genus $g = 1$. We recall that in this case the coefficients of the linear system defining the grafting function Φ in (3.24) have the form

$$\mathfrak{X}^0_n = \begin{pmatrix} 0 & 1 \\ -c^0_{n+1} & k - v^0_{n+1} \end{pmatrix}, \qquad M^0_n = \begin{pmatrix} -k + 2v^0_n & 2 \\ -2c^0_{n+1} & k \end{pmatrix}, \qquad k = z^{-1}. \tag{3.31}$$

The Lax equations for this system lead to the equations of the one-dimensional Toda lattice.

The leading parts of the 'dressed' matrices $\mathfrak{X}_n$ (see (2.110)) and M_n have the same form but with other functions c_n and v_n. In particular, in a neighbourhood of $z = 0$ the matrix M_n has the form

$$M_n = \begin{pmatrix} 2v_n - k & 2 \\ -2c_{n+1} & k \end{pmatrix} + m_n k^{-1} + O(k^{-2}), \qquad k = z^{-1}. \tag{3.32}$$

The equations (3.29) lead to the system

$$\dot c_{n+1} = 2c_{n+1}(v_{n+1} - v_n), \qquad \dot v_{n+1} = 2(c_{n+2} - c_{n+1}) + m^{22}_n - m^{22}_{n+1}. \tag{3.33}$$

The additional terms m_n^{ij} in this system can be expressed explicitly in terms of c_n, v_n and the Tyurin parameters γ_n^s, a_n^s. Our objective is to obtain a closed system of equations by using the equations for the Tyurin parameters.

For simplicity we consider the *symmetric case* in which the constant c in the formulae of the example in subsection 2.6 must be taken to be zero in (2.118): $c = 0$. It follows from the definition of M_n that

$$M_n^{21} = -c_{n+1} + \mathfrak{X}_n^{21}, \qquad M_n^{22} = v_{n+1} + \mathfrak{X}_n^{22}. \tag{3.34}$$

Thus,

$$m_n^{22} = \xi_n^{21} = \wp(\gamma_n). \tag{3.35}$$

Substitution of this formula into (3.33) leads to the equation

$$\dot v_{n+1} = 2(c_{n+2} - c_{n+1}) + \wp(\gamma_n) - \wp(\gamma_{n+1}). \tag{3.36}$$

It follows from (3.28) that

$$\dot\gamma_n = -\operatorname{res}_{\gamma_n} M_n = -\frac{a_n^1 + a_n^2}{a_n^1 - a_n^2}\,, \tag{3.37}$$

and this equality enables us to identify $\dot\gamma_n$ with the variables s_n defined in (2.143). After this identification, the formulae (2.145) and (2.146) become

$$4c_{n+1} = (\dot\gamma_n^2 - 1)F(\gamma_{n+1}, \gamma_n)F(\gamma_{n-1}, \gamma_n), \tag{3.38}$$

$$2v_{n+1} = \dot\gamma_n F(\gamma_{n+1}, \gamma_n) - \dot\gamma_{n+1} F(\gamma_n, \gamma_{n+1}). \tag{3.39}$$

We give the following two identities needed below:

$$\partial_u \log F(u, v) = -F(v, u), \tag{3.40}$$

$$\partial_v \log F(u, v) = -F(u, v) + 2\zeta(2v) - 4\zeta(v), \tag{3.41}$$

where the elliptic function $F(u, v)$ is defined by the formula (2.137). Both the identities can be verified directly by comparing the singularities on the right- and left-hand sides. Substituting (3.38) and (3.39) into the first equality in (3.33), we can see by using (3.40) and (3.41) that

$$\ddot\gamma_n = (\dot\gamma_n^2 - 1)\big(V(\gamma_n, \gamma_{n+1}) + V(\gamma_n, \gamma_{n+1})\big), \tag{3.42}$$

where

$$V(u, v) = \zeta(u + v) + \zeta(u - v) - \zeta(2u). \tag{3.43}$$

Using the same relations, we see immediately that substitution of (3.38) and (3.39) into the equality (3.36) leads to the same system (3.42).

The system (3.42) is a Hamiltonian system with Hamiltonian

$$H = \sum_n \big[\log\big(\sinh^{-2}(p_n/2)\big) + \log\big(\wp(x_n - x_{n-1}) - \wp(x_n + x_{n-1})\big)\big]. \tag{3.44}$$

This system was obtained by one of the authors in [30] as a solution of the inverse problem of recovering an integrable system from a given system of spectral curves. A similar problem is natural in the Witten–Seiberg theory in which such families parametrize moduli spaces of physically non-equivalent vacuum states in supersymmetric gauge models.

The system (3.42), which in [30] was called an *elliptic analogue of the Toda lattice*, coincides after a change of variables with one of the equations obtained in [31] in the framework of the classification problem for integrable chains. In [30] the system (3.42) was identified with a pole system describing solutions of the two-dimensionalized Toda lattice that are elliptic with respect to the variable x. The appearance of the same system in the theory of rank-two solutions of the two-dimensionalized Toda lattice was quite unexpected for the authors of the present paper.

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