Abelian solutions of the soliton equations and geometry of abelian varieties.

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Abstract

We introduce the notion of abelian solutions of the $2D$ Toda lattice equations and the bilinear discrete Hirota equation and show that all of them are algebro-geometric.

1 Introduction

The first goal of this paper to extend a theory of the abelian solutions of the Kadomtsev-Petviashvili (KP) equation developed recently in [23] to the case of the $2D$ Toda lattice

$$\partial_\xi \partial_\eta \varphi_n = e^{\varphi_{n-1}-\varphi_n} - e^{\varphi_n - \varphi_{n+1}} \tag{1.1}$$

We call a solution $\varphi_n(\xi, \eta)$ of the equation *abelian* if it is of the form

$$\varphi_n(\xi, \eta) = \ln \frac{\tau((n+1)U + z, \xi, \eta)}{\tau(nU + z, \xi, \eta)}, \tag{1.2}$$

where $n \in \mathbb{Z}$, ξ, $\eta \in \mathbb{C}$ and $z \in \mathbb{C}^d$ are the independent variables, $0 \neq U \in \mathbb{C}^d$, and for all ξ, η the function $\tau(\cdot, \xi, \eta)$ is a holomorphic section of a line bundle $\mathcal{L} = \mathcal{L}(\xi, \eta)$ on an abelian variety $X = \mathbb{C}^d/\Lambda$, i.e., it satisfies the monodromy relations

$$\tau(z + \lambda, \xi, \eta) = e^{a_\lambda \cdot z + b_\lambda} \tau(z, \xi, \eta), \quad \lambda \in \Lambda, \tag{1.3}$$

for some $a_\lambda \in \mathbb{C}^d$, $b_\lambda = b_\lambda(\xi, \eta) \in \mathbb{C}$.

A concept of abelian solutions of soliton equations provides an unifying framework for the theory of elliptic solutions of soliton equations and the theory of their rank 1 algebro-geometric solutions. The former corresponds to the case when the τ-function is a section of

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line bundle on an elliptic curve ($d = 1$), and the latter corresponds to the case when X is the Jacobian of an auxiliary algebraic curve and τ is the corresponding Riemann θ-function.

Theory of elliptic solutions of the KP equation goes back to the work [1], where it was found that the dynamics of poles of the elliptic solutions of the Korteweg-de Vries equation can be described in terms of the elliptic Calogero-Moser (CM) system with certain constraints. In [14] it was shown that when the constraints are removed this correspondence becomes a full isomorphism between the solutions of the elliptic CM system and the elliptic solutions of the KP equation.

Elliptic solutions of the $2D$ Toda lattice were considered in [24] where it was shown that if $\tau(z, \xi, \eta)$ in (1.2) is an elliptic polynomial, i.e., if the τ-function of the $2D$ Toda lattice equation is of the form

$$\tau(z, \xi, \eta) = c(\xi, \eta) \prod_{i=1}^{N} \sigma(z - x_i(\xi, \eta)) , \tag{1.4}$$

then its zeros as functions of the variables ξ and η satisfy the equations of motion of the Ruijsenaars-Schneider (RS) system [27]:

$$\ddot{x}_i = \sum_{s \neq i} \dot{x}_i \dot{x}_s (V(x_i - x_s) - V(x_s - x_i)) , \quad V(x) = \zeta(x) - \zeta(x + \eta) ,$$

which is a relativistic version of the elliptic CM system. Here and below $\sigma(z) = \sigma(z, 2\omega, 2\omega')$ and $\zeta(z) = \zeta(z, 2\omega, 2\omega')$ are the Weierstrass σ- and ζ-functions, respectively.

The correspondence between finite-dimensional integrable systems and pole systems of various soliton equations has been extensively studied in [4, 17, 18, 22] (see [5, 10, 19] and references therein for connections with the Hitchin type systems).

A general scheme of constructing Lax representations with a spectral parameter, for systems using a specific inverse problem for linear equations with elliptic coefficients, is presented in [17]. Roughly speaking, this inverse problem is the problem of characterization of linear difference or differential equations with *elliptic* coefficients having solutions that are meromorphic sections of some line bundle on the corresponding elliptic curve (double-Bloch solutions).

Analogous problems for linear equations with coefficients that are meromorphic functions expressed in terms of the Riemann theta function of an indecomposable principally polarized abelian variety (ppav) X were a starting point in the recent proof in [20, 21] of Welters' remarkable trisecant conjecture: *an indecomposable principally polarized abelian variety X is the Jacobian of a curve if and only if there exists a trisecant of its Kummer variety $K(X)$.*

Welters' conjecture, first formulated in [30], was motivated by Gunning's celebrated theorem [9] and by another famous conjecture: the Jacobians of curves are exactly the indecomposable principally polarized abelian varieties whose theta-functions provide explicit solutions of the KP equation. The latter was proposed earlier by Novikov and was unsettled at the time of the Welters' work. It was proved later in [25].

Let B be an indecomposable symmetric matrix with positive definite imaginary part. It defines an indecomposable principally polarized abelian variety $X = \mathbb{C}^g/\Lambda$, where the lattice

Λ is generated by the basis vectors $e_m \in \mathbb{C}^g$ and the column-vectors B_m of B. The Riemann theta-function $\theta(z) = \theta(z|B)$ corresponding to B is given by the formula

$$\theta(z) = \sum_{m\in\mathbb{Z}^g} e^{2\pi i(z,m)+\pi i(Bm,m)}, \quad (z,m) = m_1 z_1 + \cdots + m_g z_g\,. \tag{1.5}$$

The Kummer variety $K(X)$ is an image of the Kummer map

$$K\colon X \ni Z \longmapsto (\Theta[\varepsilon_1,0](Z) : \cdots : \Theta[\varepsilon_{2^g},0](Z)) \in \mathbb{CP}^{2^g-1} \tag{1.6}$$

where $\Theta[\varepsilon,0](z) = \theta[\varepsilon,0](2z|2B)$ are level two theta-functions with half-integer characteristics ε.

A trisecant of the Kummer variety is a projective line which meets $K(X)$ at least at three points. Fay's well-known trisecant formula [8] implies that if B is a matrix of b-periods of normalized holomorphic differentials on a smooth genus g algebraic curve Γ, then a set of three arbitrary distinct points on Γ defines a *one-parameter family* of trisecants parameterized by a fourth point of the curve. In [9] Gunning proved under certain non-degeneracy assumptions that the existence of such a family of trisecants characterizes Jacobian varieties among indecomposable principally polarized abelian varieties.

Gunning's geometric characterization of the Jacobian locus was extended by Welters who proved that the Jacobian locus can be characterized by the existence of a formal one-parameter family of flexes of the Kummer varieties [29, 30]. A flex of the Kummer variety is a projective line which is tangent to $K(X)$ at some point up to order 2. It is a limiting case of trisecants when the three intersection points come together.

In [2] Arbarello and De Concini showed that the Welters' characterization is equivalent to an infinite system of partial differential equations representing the KP hierarchy, and proved that only a finite number of these equations is sufficient. Novikov's conjecture that just the first equation of the hierarchy is sufficient for the characterization of the Jacobians is much stronger. It is equivalent to the statement that the Jacobians are characterized by the existence of length 3 formal jet of flexes.

Welter's conjecture that requires the existence of only one trisecant is the strongest. In fact, there are three particular cases of the Welters' conjecture, which are independent and have to be considered separately. They correspond to three possible configurations of the intersection points (a,b,c) of $K(X)$ and the trisecant:

(i) all three points coincide,

(ii) two of them coincide;

(iii) all three intersection points are distinct.

In all of these cases the classical addition theorem for the Riemann theta-functions directly imply that secancy conditions are equivalent to the existence of certain solutions for the auxiliary linear problems for the KP, the $2D$ Toda, and the bilinear discrete Hirota equations, respectively.

For example, one of the Lax equations for the $2D$ Toda equation is the differential-difference equation

$$\partial_t \psi_n(t) = \psi_{n+1}(t) - u_n(t)\psi_n(t) \tag{1.7}$$

with the potential u of the form

$$u_n(t) = \partial_t \ln \tau(n,t) - \partial_t \ln \tau(n+1,t) \tag{1.8}$$

Let us assume that

$$\tau(n,t) = \theta(nU + tV + z) \tag{1.9}$$

and equation (1.7) has a solution of the form

$$\psi_n(t) = \frac{\theta(A + nU + tV + Z)}{\theta(nU + tV + z)} \, e^{np+tE}, \tag{1.10}$$

where p, E are constants and z is arbitrary. Then a direct substitution of (1.9) and (1.10) into (1.7) gives the equation

$$E\theta(A+z)\theta(U+z) - e^p\theta(A+U+z)\theta(z) = \partial_V\theta(U+z)\theta(A+z) - \partial_V\theta(A+z)\theta(U+Z) \tag{1.11}$$

which is equivalent to the condition that the projective line passing through the points $\{K((A \pm U)/2)\}$ is tangent to the Kummer variety at the point $K((A-U)/2)$ (the case (ii) above).

The characterization of the Jacobian locus via (1.11) is the statement: *an indecomposable, principally polarized abelian variety* (X, θ) *is the Jacobian of a smooth curve of genus g if and only if there exist non-zero g-dimensional vectors* $U \neq A \,(\bmod \Lambda)$, V*, such that equation (1.11) holds* ([21]).

The "only if" part of the statement follows from the construction of solutions of the $2D$ Toda lattice equations in [15], from which it also follows that the vector A in (1.11) is a point of $\Gamma \subset J(\Gamma)$, the vector U is of the form $U = P_- - P_+$, where $P_\pm \in \Gamma$ are points on Γ, and the vector V is a tangent vector to Γ at one of the points.

In geometric terms the spectral curves of the elliptic RS system, that give elliptic solutions of (1.1) are singled out by the condition that there exist a pair of points such that the corresponding vector U *spans an elliptic curve* in $J(\Gamma)$.

For any curve Γ and any pair of points $P_\pm \in \Gamma$ the Zariski closure of the group $\{Un \,|\, n \in \mathbb{Z},\ U = P_- - P_+\}$ in $J(\Gamma)$ is an abelian subvariety $X \subset J(\Gamma)$. When X is a proper subvariety, i.e., $\dim X = d < g = \dim J(G)$, the restrictions of $\theta(tV + z)$ and $\theta(A + tV + z)$ on the corresponding linear subspace $\mathbb{C}^d \subset \mathbb{C}^g$, i.e., the component through the origin of $\pi^{-1}(X)$, where $\pi\colon \mathbb{C}^g \to J(\Gamma)$ is the covering map, can be seen as sections $\tau(z,t), \tau_A(z,t)$ of some line bundles on X, i.e. they satisfy the monodromy properties with respect to the lattice $\Lambda \subset \mathbb{C}^d$ defining X

$$\tau(z+\lambda,t) = e^{a_\lambda \cdot z + b_\lambda}\tau(z,t)\,, \quad \tau_A(z+\lambda,t) = e^{a_\lambda \cdot z + c_\lambda}\tau_A(z,t)\,, \quad \lambda \in \Lambda,\ z \in \mathbb{C}^d \tag{1.12}$$

for some $a_\lambda \in \mathbb{C}^d$, $b_\lambda = b_\lambda(t), c_\lambda = c_\lambda(t) \in \mathbb{C}$.

Equation (1.11) restricted to $z \in \mathbb{C}^d$ takes the form

$$E\,\tau_A(z,t)\,\tau(U+z,t) - e^p\,\tau_A(U+z,t)\,\tau(z,t) = \dot\tau(z+U,t)\,\tau_A(z,t) - \tau(z+U,t)\,\dot\tau_A(z,t) \tag{1.13}$$

Here and below "dot" stands for the derivative with respect to the variable t.

At first sight equation (1.13) considered as an equation for two unknown sections $\tau(z,t)$ and $\tau_A(z,t)$ of some line bundles $\mathcal{L}(t)$ and $\mathcal{L}_A(t)$ on an arbitrary abelian variety X is not as restrictive as finite-dimensional equation (1.11). Nevertheless, our first main result is that at least under certain genericity assumptions all the *abelian* solutions of equation (1.13) arise in way described above, i.e., they are rank one algebro-geometric, and we have $X \subset J(\Gamma)$ for some algebraic curve Γ, which in general might be singular.

Theorem 1.1 *Suppose that the equation (1.13) with some p, $E \in \mathbb{C}$ and $0 \neq U \in \mathbb{C}^n$, is satisfied with $\tau(z,t), \tau_A(z,t)$, such that for all t the functions $\tau_A(z,t)$ and $\tau(z,t)$ are holomorphic functions satisfying the monodromy properties (1.12). Assume, moreover, that*

(i) Λ is maximal with this property, i.e., any $\lambda \in \mathbb{C}^n$ satisfying (1.12) for some $a_\lambda \in \mathbb{C}^n$ and $b_\lambda(t)$, $c_\lambda(t) \in \mathbb{C}$ must belong to Λ, and that,

(ii) for each t the divisor $\mathcal{T}^t := \{z \in X \mid \tau(z,t) = 0\}$ is reduced and irreducible;

(iii) the group $\{Un \mid n \in \mathbb{Z}\}$ is Zariski dense in X.

Then there exist a unique irreducible algebraic curve Γ, smooth points $P_\pm \in \Gamma$, an injective homomorphism $j_0 \colon X \to J(\Gamma)$ and a torsion-free rank 1 sheaf $\mathcal{F} \in \overline{\mathrm{Pic}^{g-1}}(\Gamma)$ of degree $g-1$, where $g = g(\Gamma)$ is the arithmetic genus of Γ, such that setting $j(z) = j_0(z) \otimes \mathcal{F}$ we have

$$\tau(Un + z, t) = \rho(t)\, \widehat{\tau}_n(t, 0 \mid \Gamma, P, j(z)), \tag{1.14}$$

where, $\widehat{\tau}_n(t_1^+, t_1^- \mid \Gamma, P, \mathcal{F})$ is the 2D Toda tau-function defined by the data $(\Gamma, P_i, \mathcal{F})$.

Note that when Γ is smooth:

$$\widehat{\tau}_n(t_1^+, t_1^- \mid \Gamma, P, j(z)) = \theta\Big(nU + t_1^+ V_+ + t_1^- V_- + j(z) \;\Big|\; B(\Gamma)\Big)\, e^{Q(n, t_1^+, t_1^-)}, \tag{1.15}$$

where $V_\pm \in \mathbb{C}^n$, Q is a quadratic form, $B(\Gamma)$ is the matrix of B-periods of Γ, and θ is the Riemann theta function. Linearization in the Jacobian $J(\Gamma)$ of nonlinear t-dynamics for $\tau(z,t)$ provides some evidence that there might be underlying integrable systems on the spaces of higher level theta-functions on ppav. The RS system is an example of such a system for $d = 1$.

Almost till the very end the proof of Theorem 1.1 goes along the lines of [21]. We would like to stress that the proof of the trisecant conjecture in [21] uses nighter of the assumptions above. We include the assumption (iii) in the statement of the theorem only to avoid unnecessary at this stage analytical difficulties.

The second goal of this paper, discussed in the last section, is to study abelian solutions of the BDHE. The latter is a difference equation of the form

$$\tau_n(l+1,m)\tau_n(l,m+1) - \tau_n(l,m)\tau_n(l+1,m+1) + \tau_{n+1}(l+1,m)\tau_{n-1}(l,m+1) = 0 \tag{1.16}$$

One of its auxiliary Lax equations is the two dimensional linear difference equation

$$\psi(m, n+1) = \psi(m+1, n) + u(m,n)\psi(m,n) \tag{1.17}$$

with the potential u of the form

$$u(m,n) = \frac{\tau(n+1,m+1)\,\tau(n,m)}{\tau(n+1,m)\,\tau(n,m+1)} \tag{1.18}$$

Under the light-cone change of variables

$$x = m - n, \quad \nu = m + n \tag{1.19}$$

and under the assumption that $\tau(n,m)$ is of the form $\tau(Wx+z,\nu)$ with $z, W \in \mathbb{C}^d$, equation (1.7) get transformed to the difference-functional equation

$$\psi(z-W,\nu) = \psi(z+W,\nu) + u\psi(z,\nu-1)\,. \tag{1.20}$$

with

$$u(z,\nu) = \frac{\tau(z,\nu+1)\,\tau(z,\nu-1)}{\tau(z-W,\nu)\,\tau(z+W,\nu)} \tag{1.21}$$

Equation (1.20) for ψ of the form

$$\psi(x,\nu) = \frac{\tau_A(z,\nu)}{\tau(z,\nu)}\,e^{p\cdot z+\nu E} \tag{1.22}$$

is equivalent to the discrete analog of (1.13)

$$e^{-p\cdot W}\tau(z+W,\nu)\tau_A(z-W,\nu) = e^{p\cdot W}\tau(z-W,\nu)\tau_A(z+W,\nu) + e^{-E}\tau(z,\nu+1)\tau_A(z,\nu-1)\,, \tag{1.23}$$

where, as before, $\tau(z,\nu)$ and $\tau_A(z,\nu)$ are sections of some line bundles on X, i.e. they are holomorphic functions satisfying the monodromy properties

$$\tau(z+\lambda,\nu) = e^{a_\lambda\cdot z+b_\lambda(\nu)}\tau(z,\nu)\,, \quad \tau_A(z+\lambda,\nu) = e^{a_\lambda\cdot z+c_\lambda(\nu)}\tau_A(z,\nu)\,, \quad \lambda \in \Lambda, \tag{1.24}$$

with respect to the lattice Λ of an abelian variety $X = \mathbb{C}^n/\Lambda$. If X is ppav and $\tau(z,\nu) = \theta(z+V\nu)$, $\tau_A(z,\nu) = \theta(A+z+V\nu)$ then (1.23) is equivalent to the trisecant equation

$$e^{-p\cdot W}\theta(z+W)\theta(z+A-W) = e^{p\cdot W}\theta(z+A+W)\theta(z-W) + e^{-E}\theta(z+V)\theta(z+A-V)\,. \tag{1.25}$$

We conjecture that under the assumption that $\tau(z,\nu), \tau_A(z,\nu)$ are *meromorphic quasiperiodic functions of the variable* ν all the *abelian* solutions of equation (1.23) are rank one algebro-geometric, and we have $X \subset J(\Gamma)$ for some algebraic curve Γ, (which in general might be singular). The main result of the last section is a proof of this conjecture in the case when $\tau(z,\nu)$ is periodic in the variable ν with some sufficiently large prime period N. More precisely,

Theorem 1.2 *Suppose that the equation (1.23) with some p, $E \in \mathbb{C}$ and $0 \neq W \in \mathbb{C}^n$, is satisfied with $\tau(z,\nu), \tau_A(z,\nu)$, such that for all ν the functions $\tau_A(z,\nu)$ and $\tau(z,\nu)$ are holomorphic functions satisfying the monodromy properties (1.24) with respect to the lattice Λ of an abelian variety $X = \mathbb{C}^n/\Lambda$. Assume, moreover, that* *(i) Λ is maximal with this property, i.e., any $\lambda \in \mathbb{C}^n$ satisfying (1.24) for some $a_\lambda \in \mathbb{C}^n$ and $b_\lambda(\nu)$, $c_\lambda(\nu) \in \mathbb{C}$ must belong to Λ, and that,*

(ii) for each ν the divisor $\mathcal{T}^\nu := \{z \in X \mid \tau(z,\nu) = 0\}$ is reduced and is irreducible;

(iii) the Zariski closure of the group $\{2Wm | \, m \in \mathbb{Z}\}$ in X coincides with X;

(iv) the functions $\tau(z,\nu), \tau_A(z,\nu)$ are meromorphic functions of the variable $\nu \in \mathbb{C}$ and $\tau(z,\nu)$ is a quasiperiodic function of ν, satisfying the monodromy relation

$$\tau(z, \nu + N) = e^{a\cdot z + c\nu}\tau(z,\nu) \tag{1.26}$$

with an integer prime period $N > \dim H^0(\mathcal{T}^\nu)$ and with some $a \in \mathbb{C}^n, \ c \in \mathbb{C}$.

Then there exist a unique irreducible algebraic curve Γ, smooth points $P_0, P_1, P_2 \in \Gamma$, an injective homomorphism $j_0 \colon X \to J(\Gamma)$ and a torsion-free rank 1 sheaf $\mathcal{F} \in \overline{\mathrm{Pic}^{g-1}}(\Gamma)$ of degree $g-1$, where $g = g(\Gamma)$ is the arithmetic genus of Γ, such that setting $j(z) = j_0(z) \otimes \mathcal{F}$ we have

$$\tau(Wx + z, \nu) = \rho(\nu)\, \widehat{\tau}(x, \nu, 0, \ldots \mid \Gamma, P, j(z)) \,, \tag{1.27}$$

where, $\widehat{\tau}(t_1, t_2, t_3, \ldots \mid \Gamma, P, \mathcal{F})$ is the BDHE tau-function defined by the data $(\Gamma, P_i, \mathcal{F})$.

2 Construction of the wave function

Equation (1.13) is equivalent to equation (1.7) with

$$u_n = -\partial_t \ln \frac{\tau((n+1)U + z, t)}{\tau(nU + z, t)}, \quad \psi_n = \frac{\tau_A(nU + z, t)}{\tau(nU + z, t)} e^{P\cdot z + Et}, \tag{2.1}$$

where $P \in \mathbb{C}^d$ is a vector such that $P \cdot U = p$. In the core of the proof of Theorem is the construction of quasiperiodic wave function as in (2.9,2.10) below, which contains much more information than the function ψ in (2.1) having no spectral parameter. We would like to emphasize once again that the construction of wave function follows closely the argument from the beginning of Section 2 in [21] but is drastically simplified by the assumption (ii) in the formulation of the theorem.

The construction is presented in two steps. First we show that the existence of a holomorphic solutions of equation (1.23) implies certain relations on the tau divisor $\mathcal{T}^t$.

Lemma 2.1 *If equation (1.23) has holomorphic solutions whose divisors have no common components (or if the τ-divisor is irreducible), then the equation*

$$\partial_t^2 \tau(z,t)\, \tau(z+U,t)\, \tau(z-U,t) = \partial_t \tau(z,t)\, \partial_t\left(\tau(z+U,t)\, \tau(z-U,t)\right) \tag{2.2}$$

is valid on the divisor $\mathcal{T}^t = \{\, z \in \mathbb{C}^d \mid \tau(z,t) = 0\}$.

In [21] equation (2.2) was derived with the help of pure local consideration. Let us show that they can be easy obtained globally.

Proof. The evaluations of (1.13) at the divisors $\mathcal{T}^t$ and $\mathcal{T}^t - U$ give

$$(\dot{\tau}_A(z) + E\tau_A(z))\tau(z+U) = \dot{\tau}(z+U)\tau_A(z), \quad z \in \mathcal{T}^t, \tag{2.3}$$

$$\tau_A(z)\tau(z-U) + \dot{\tau}(z)\tau_A(z-U)e^{-p} = 0, \quad z \in \mathcal{T}^t\,. \tag{2.4}$$

Here and below for brevity we omit the notations for explicit dependence of functions on the variable t, i.e. $\tau(z) = \tau(z,t), \tau_A(z) = \tau_A(z,t)$.

The evaluation of the derivative of (1.13) at $\mathcal{T}^t - U$ gives an another equation

$$(E\tau_A(z) + \dot{\tau}_A(z))\tau(z-U) + \dot{\tau}(z-U)\tau_A(z) + \ddot{\tau}(z)\tau_A(z-U)e^{p} = 0, \quad z \in \mathcal{T}^t \tag{2.5}$$

Eliminating $\tau_A(z-U)$ and $\dot{\tau}_A(z)$ from (2.3-2.5) we obtain the equation

$$\left[\ddot{\tau}(z)\,\tau(z+U)\,\tau(z-U) - \dot{\tau}(z,t)\,\partial_t\left(\tau(z+U)\,\tau(z-U)\right)\right]\tau_A(z) = 0, \quad z \in \mathcal{T}^t. \tag{2.6}$$

which implies (2.2) due to the assumption that the divisors of τ and τ_A have no common components (or under the assumption that $\mathcal{T}^t$ is irreducible).

In [21] it was shown that equation (2.2) is sufficient for the existence of *local* meromorphic wave solutions of (1.7) which are holomorphic outside of zeros of τ. Let us show that in a global setting they are sufficient for the existence of quasi-periodic wave solutions of the differential-functional equation:

$$\partial_t \psi(z,t) = \psi(z+U,t) - u(z,t)\psi(z,t) \tag{2.7}$$

with

$$u = \partial_t \ln \tau(z,t) - \partial_t \ln \tau(z+U,t)\,, \tag{2.8}$$

which restricted to the points $z + Un$ takes the form (1.7).

The wave solution of (2.7) is a formal solution of the form

$$\psi = k^{l\cdot z} e^{kt}\phi(z,t,k)\,, \tag{2.9}$$

where l is a vector $l \in \mathbb{C}^d$ such that $l \cdot U = 1$ and ϕ is a formal series

$$\phi(z,t,k) = e^{bt}\left(1 + \sum_{s=1}^{\infty} \xi_s(z,t)\,k^{-s}\right) \tag{2.10}$$

Lemma 2.2 *Let equation (2.2) for $\tau(z,t)$ holds, and let $\lambda_1, \ldots, \lambda_d$ be a set of linear independent vectors of the lattice Λ Then equation (2.7) with u as in (2.8) has a unique, up to a z-independent factor, wave solution such that:*

(i) the coefficients $\xi_s(z,t)$ of the formal series (2.10) are meromorphic functions of the variable $z \in \mathbb{C}^d$ with a simple pole at the divisor $\mathcal{T}^t$, i.e.

$$\xi_s(z,t) = \frac{\tau_s(z,t)}{\tau(z,t)}\,; \tag{2.11}$$

and $\tau_s(z,t)$ is a holomorphic function of z;

(ii) $\phi(z,t,k)$ is quasi-periodic with respect to the lattice Λ

$$\phi(z+\lambda,t,k)=\phi(z,t,k)\,B^{\lambda}(k),\quad \lambda\in\Lambda; \tag{2.12}$$

and is periodic with respect to the vectors $\lambda_1,\ldots,\lambda_d$, i.e.,

$$B^{\lambda_i}(k)=1,\quad i=1,\ldots,d. \tag{2.13}$$

Proof. The functions $\xi_s(z)$ are defined recursively by the equations

$$\Delta_U\,\xi_{s+1}=\dot{\xi}_s+(u+b)\,\xi_s. \tag{2.14}$$

Here and below Δ_U stands for the difference derivative $e^{\partial_U}-1$. The quasi-periodicity conditions (2.12) for ϕ are equivalent to the equations

$$\xi_s(z+\lambda,t)-\xi_s(z,t)=\sum_{i=1}^{s}B_i^{\lambda}\xi_{s-i}(z,t)\,,\quad \xi_0=1. \tag{2.15}$$

The general quasi-periodic solution of the first equation $\Delta_U\,\xi_1=u+b$ is given by the formula

$$\xi_1=-\partial_t\ln\tau+l_1(z,t)\,b+c_1(t), \tag{2.16}$$

where $l_1(z,t)$ is a linear form on $\mathbb{C}^d$ such that $l_1(U,t)=1$. It satisfies the monodromy relations (2.15) with

$$B_1^{\lambda}=l_1(\lambda)\,b-\partial_t\ln\tau(z+\lambda,t)+\partial_t\ln\tau(z,t)=l_1(\lambda,t)\,b-\dot{b}_{\lambda}(t)\,, \tag{2.17}$$

where $b_\lambda=b_\lambda(t)$ are defined in (1.12). The normalizing conditions $\mathcal{B}_1^{\lambda_i}=0,\ i=1,\ldots,d$ uniquely define the constant b and the linear form $l_1(z)$.

Let us assume that the coefficient ξ_{s-1} of the series (2.10) is known, and that there exists a solution ξ_s^0 of the next equation, which is holomorphic outside of the divisor $\mathcal{T}^t$, and which satisfies the quasi-periodicity conditions (2.15) with $B_s^{\lambda_j}=0$ and possibly t-dependent coefficient $B_s^{\lambda}(t)$, for $\lambda\neq\lambda_j$, i.e.

$$\xi_s(z+\lambda,t)-\xi_s(z,t)=B_s^{\lambda}(t)+\sum_{i=1}^{s-1}B_i^{\lambda}\xi_{s-i}(z,t),\quad B_s^{\lambda_j}=0. \tag{2.18}$$

We assume also that ξ_s^0 is unique up to the transformation $\xi_s=\xi_s^0+c_s(t)$, where $c_s(t)$ is a time-dependent constant.

Let us define a function $\tau_{s+1}^0(z)$ on $\mathcal{T}^t$ with the help of the formula

$$\tau_{s+1}^0=-\partial_t\tau_s(z,t)-b\tau_s(z,t)+\frac{\partial_t\tau(z+U,t)}{\tau(z+U,t)}\tau_s(z,t),\quad z\in\mathcal{T}^t. \tag{2.19}$$

Let us show that the formula (2.19) can be written also in the alternative form:

$$\tau_{s+1}^0=-\partial_t\tau(z,t)\frac{\tau_s(z-U,t)}{\tau(z-U,t)},\quad z\in\mathcal{T}^t. \tag{2.20}$$

By the induction assumption, $\xi_s = (\tau_s/\tau)$ is a solution of (2.14) for $s-1$, i.e. the function τ_s satisfies the equation

$$[\dot\tau_{s-1}(z-U) + \tau_s(z-U) + b\tau_{s-1}(z-U)]\,\tau(z) = \tau_s(z)\tau(z-U) + \dot\tau(z)\,\tau_{s-1}(z-U), \quad (2.21)$$

where once again we omit notations for explicit dependence of all the functions on the variable t.

From (2.21) it follows that

$$\tau_s(z)\tau(z-U) + \dot\tau(z)\,\tau_{s-1}(z-U) = 0. \ \ z \in \mathcal{T}^t\,. \quad (2.22)$$

The evaluation of the derivative of (2.21) at $\mathcal{T}^t$ implies

$$(\tau_s(z-U) + b\tau_{s-1}(z-U))\,\dot\tau(z) = \dot\tau_s(z)\,\tau(z-U) + \tau_s(z)\,\dot\tau(z-U) + \ddot\tau(z)\tau_{s-1}(z-U), \quad z \in \mathcal{T}^t. \quad (2.23)$$

Then, using (2.2) and (2.22) we obtain the equation

$$\frac{\dot\tau(z)\tau_s(z-U)}{\tau(z-U)} = b\tau_s(z) + \dot\tau_s(z) - \frac{\dot\tau(z+U)\tau_s(z)\,.}{\tau(z+U)} \quad (2.24)$$

Hence, the expressions (2.19) and (2.20) do coincide.

The expression (2.19) is certainly holomorphic when $\tau(z+U)$ is non-zero, i.e. is holomorphic outside of $\mathcal{T}^t \cap (\mathcal{T}^t - U)$. Similarly from (2.20) we see that $\tau^0_{s+1}(z,t)$ is holomorphic away from $\mathcal{T}^t \cap (\mathcal{T}^t + U)$.

We claim that $\tau^0_{s+1}(z,t)$ is holomorphic everywhere on $\mathcal{T}^t$. Indeed, by the assumption the abelian subgroup generated by U is Zariski dense. Therefore, for any point $z_0 \in \mathcal{T}^t$ there exists an integer $k > 0$ such that $z_k = z_0 - kU$ is in $\mathcal{T}^t$, and $\tau(z_{k+1}, t) \neq 0$. Then, from equation (2.20) it follows that τ^0_{s+1} is regular at the point $z = z_k$. Using equation (2.19) for $z = z_k$, we get that $\partial_t\tau(z_{k-1}, t)\tau_s(z_k, t) = 0$. The last equality and the equation (2.20) for $z = z_{k-1}$ imply that τ^0_{s+1} is regular at the point z_{k-1}. Regularity of τ^0_{s+1} at z_{k-1} and equation (2.19) for $z = z_{k-1}$ imply $\partial_t\tau(z_{k-2}, t)\tau_s(z_{k-1}, t) = 0$. Then equation (2.20) for $z = z_{k-2}$ implies that τ^0_{s+1} is regular at the point z_{k-2}. By continuing these steps we get finally that τ^0_{s+1} is regular at $z = z_0$. Therefore, τ^0_{s+1} is regular on $\mathcal{T}^t$.

Recall, that an analytic function on an analytic divisor in $\mathbb{C}^d$ has a holomorphic extension onto $\mathbb{C}^d$ ([28]). Therefore, there exists a holomorphic function $\tilde\tau(z,t)$ such that $\tilde\tau_{s+1}|_{\mathcal{T}^t} = \tau^0_{s+1}$. Consider the function $\chi_{s+1} = \tilde\tau_{s+1}/\tau$. It is holomorphic outside of the divisor $\mathcal{T}^t$. From (2.15) and (2.20) it follows that the function $f^\lambda_{s+1}(z)$ defined by the equation

$$\chi_{s+1}(z+\lambda) - \chi_{s+1}(z) = f^\lambda_{s+1}(z) + \sum_{i=1}^{s} B_i^\lambda \xi_{s+1-i}(z)\,, \quad (2.25)$$

has no pole at $\mathcal{T}^t$, i.e. it is a holomorphic function of $z \in \mathbb{C}^d$. It satisfies the twisted homomorphism relations

$$f^{\lambda+\mu}_{s+1}(z) = f^\lambda_{s+1}(z+\mu) + f^\mu_{s+1}(z), \quad (2.26)$$

i.e., it defines an element of the first cohomology group of Λ_U with coefficients in the sheaf of holomorphic functions, $f \in H^1_{gr}(\Lambda_U, H^0(\mathcal{C}, \mathcal{O}))$. The same arguments, as that used in the proof of the part (b) of the Lemma 12 in [25], show that there exists a holomorphic function $h_{s+1}(z)$ such that

$$f_{s+1}^{\lambda}(z) = h_{s+1}(z+\lambda) - h_{s+1}(z) + \widetilde{B}_{s+1}^{\lambda}, \tag{2.27}$$

where $\widetilde{B}_{s+1}^{\lambda} = \widetilde{B}_{s+1}^{\lambda}(t)$ is a time-dependent constant. Hence, the function $\zeta_{s+1} = \chi_{s+1} + h_{s+1}$ has the following monodromy properties

$$\zeta_{s+1}(z+\lambda) - \zeta_{s+1}(z) = \widetilde{B}_{s+1}^{\lambda} + \sum_{i=1}^{s} B_i^{\lambda} \xi_{s+1-i}(z), \tag{2.28}$$

Let us consider the function

$$R_{s+1} = \zeta_{s+1}(z+U) - \zeta_{s+1}(z) - \dot{\xi}_s(z) - (u(z)+b)\,\xi_s(z) \tag{2.29}$$

From equation (2.19,2.20) it follows that it has not poles at $\mathcal{T}^t$ and $\mathcal{T}^t - U$, respectively. Hence, $R_{s+1}(z)$ is a holomorphic function.

From (2.28) it follows that it satisfies the following monodromy properties

$$R_{s+1}(z+\lambda) = R_{s+1}(z) - \dot{B}_s^{\lambda}\,. \tag{2.30}$$

Recall, that by the induction assumption $B_s^{\lambda_j} = 0$, where $\lambda_j, j = 1, \ldots, d$, are linear independent. Therefore, R_{s+1} is a constant (z-independent) and B_s^{λ} for all λ are in fact t-independent.

The function

$$\widetilde{\xi}_{s+1}(z,t) = \zeta_{s+1}(z,t) + l_{s+1}(z,t) + c_{s+1}(t)\,, \tag{2.31}$$

where l_{s+1} is a linear form such that

$$l_{s+1}(U,t) = -R_{s+1}(t)\,,$$

is a solution of (2.14).

Under the transformation $\xi_s \longmapsto \xi_s(z,t) + c_s(t)$ which does not change the monodromy properties of ξ_s, the solution $\widetilde{\xi}_{s+1}$ gets transformed to

$$\xi_{s+1} = \widetilde{\xi}_{s+1} + \dot{c}_s(t) l_1(z,t) + c_s(t)\xi_1(z,t), \tag{2.32}$$

where $l_1(z,t)$ is the linear form defined above in the initial step of the induction. The new solution ξ_{s+1} satisfies the monodromy relations (2.15) with constant B_i^{λ} for $i \le s$ and with t-dependent coefficient

$$B_{s+1}^{\lambda}(t) = \widetilde{B}_{s+1}^{\lambda}(t) + l_{s+1}(\lambda,t) + \dot{c}_s(t) l_1(\lambda,t) + c_s(t) B_1^{\lambda}\,. \tag{2.33}$$

The normalization condition (2.13) for $B_{s+1}^{\lambda_i} = 1, i = 0, \ldots, d$ defines uniquely l_{s+1} and $\partial_t c_s$, i.e. the time-dependence of $c_s(t)$. The induction step is completed.

Note that the remaining ambiguity in the definition of ξ_s on each step is the choice of a time-independent constant c_s. That corresponds to the multiplication of ψ by a constant formal series and thus the lemma is proven.

3 Commuting difference operators.

Our next goal is to construct rings $\mathcal{A}^z$ of commuting difference operators parameterized by points $z \in X$. In fact the construction of such operators completes the proof of Theorem 1.1 because as shown in ([26, 13]) there is a natural correspondence

$$\mathcal{A} \longleftrightarrow \{\Gamma, P_\pm, \mathcal{F}\} \tag{3.1}$$

between commutative rings $\mathcal{A}$ of ordinary linear difference operators containing a pair of monic operators of co-prime orders, and sets of algebro-geometric data $\{\Gamma, P_\pm, [k^{-1}]_1, \mathcal{F}\}$, where Γ is an algebraic curve with a fixed first jet $[k^{-1}]_1$ of a local coordinate k^{-1} in the neighborhood of a smooth point $P_+ \in \Gamma$ and $\mathcal{F}$ is a torsion-free rank 1 sheaf on Γ such that

$$h^0(\Gamma, \mathcal{F}(nP_+ - nP_-)) = h^1(\Gamma, \mathcal{F}(nP_+ - nP_-)) = 0. \tag{3.2}$$

The correspondence becomes one-to-one if the rings $\mathcal{A}$ are considered modulo conjugation $\mathcal{A}' = g(x)\mathcal{A}g^{-1}(x)$.

The construction of the correspondence (3.1) depends on a choice of initial point $x_0 = 0$. The spectral curve and the sheaf $\mathcal{F}$ are defined by the evaluations of the coefficients of generators of $\mathcal{A}$ at a finite number of points of the form $x_0 + n$. In fact, the spectral curve is independent on the choice of x_0, but the sheaf does depend on it, i.e. $\mathcal{F} = \mathcal{F}_{x_0}$.

Using the shift of the initial point it is easy to show that the correspondence (3.1) extends to the commutative rings of operators whose coefficients are *meromorphic* functions of x. The rings of operators having poles at $x = 0$ correspond to sheaves for which the condition (3.2) for $n = 0$ is violated.

The algebraic curve Γ is called the spectral curve of $\mathcal{A}$. The ring $\mathcal{A}$ is isomorphic to the ring $A(\Gamma, P_+, P_-)$ of meromorphic functions on Γ with the only pole at the points P_+ and which vanish at P_-. The isomorphism is defined by the equation

$$L_a\psi_0 = a\psi_0, \quad L_a \in \mathcal{A}, \; a \in A(\Gamma, P_+, P_-). \tag{3.3}$$

Here ψ_0 is a common eigenfunction of the commuting operators. At $x = 0$ it is a section of the sheaf $\mathcal{F} \otimes \mathcal{O}(P_+)$.

In order to construct rings of commutative operators we first introduce a unique pseudo-difference operator

$$\mathcal{L}(z,t) = T + \sum_{s=0}^{\infty} w_s(z,t)\, T^{-s}, \quad T = e^{\partial_U}, \tag{3.4}$$

such that the equation

$$\left(T + \sum_{s=0}^{N} w_s(z,t)\, T^{-s} \right) \psi(z,t) = k\psi(z,t)\,, \tag{3.5}$$

with ψ is given by (2.9), holds. The coefficients $w_s(z,t)$ of $\mathcal{L}$ are difference polynomials in terms of the coefficients of ϕ. Due to quasiperiodicity of ψ they are meromorphic functions on the abelian variety X.

Consider now the strictly positive difference parts of the operators $\mathcal{L}^m$. Let $\mathcal{L}^m_+$ be the difference operator such that $\mathcal{L}^m_- = \mathcal{L}^m - \mathcal{L}^m_+ = F_m + F^1_m T^{-1} + O(T^{-2})$. By definition the leading coefficient F_m of $\mathcal{L}^m_-$ is the residue of $\mathcal{L}^m$:

$$F_m = \mathrm{res}_T\ \mathcal{L}^m,\ F^1_m = \mathrm{res}_T\ \mathcal{L}^m\, T. \tag{3.6}$$

From the construction of $\mathcal{L}$ it follows that $[\partial_t - T + u, \mathcal{L}^n] = 0$. Hence,

$$[\partial_t - T + u, \mathcal{L}^m_+] = -[\partial_t - T + u, \mathcal{L}^m_-] = (\Delta_U F_m)\, T. \tag{3.7}$$

Indeed, the left hand side of (3.7) shows that the right hand side is a difference operator with non-vanishing coefficients only at the positive powers of T. The intermediate equality shows that this operator is at most of order 1. Therefore, it has the form $f_m T$. The coefficient f_m is easy expressed in terms of the leading coefficient $\mathcal{L}^m_-$. Note, that the vanishing of the coefficient at T^0 and T^{-1} implies the equation

$$\Delta_U\, F^1_m = \partial_t F_m, \tag{3.8}$$

$$\Delta_U\, F^2_m = \partial_t F^1_m + uF_1 - F_1(T^{-1}u), \tag{3.9}$$

which we will use later.

The functions $F_m(z)$ are difference polynomials in the coefficients w_s of $\mathcal{L}$. Hence, $F_m(z)$ are meromorphic functions on X.

Lemma 3.1 *There exist holomorphic functions $q_m(z,t)$ such that the equation*

$$F_m = \frac{q_m(z+U,t)}{\tau(z+U,t)} - \frac{q_m(z,t)}{\tau(z,t)}\,. \tag{3.10}$$

holds.

Proof. If ψ is as in Lemma 3.1, then there exists a unique pseudo-difference operator Φ such that

$$\psi = \Phi k^{P\cdot z} e^{kt},\quad \Phi = 1 + \sum_{s=1}^{\infty} \varphi_s(s,t) T^{-s}. \tag{3.11}$$

The coefficients of Φ are universal difference polynomials in ξ_s. Therefore, $\varphi_s(z,t)$ is a meromorphic function of z. Note, that $\mathcal{L} = \Phi\, T \Phi^{-1}$.

Consider the dual wave function defined by the left action of the operator Φ^{-1}: $\psi^+ = \left(k^{-P\cdot z} e^{-kt}\right)\Phi^{-1}$. Recall that the left action of a pseudo-difference operator is the formal adjoint action under which the left action of T on a function f is $(fT) = T^{-1}f$. If ψ is a formal wave solution of (2.7), then ψ^+ is a solution of the adjoint equation

$$(-\partial_t - T^{-1} + u)\,\psi^+ = 0. \tag{3.12}$$

The same arguments, as before, prove that if equation (2.2) holds then ξ^+_s have simple poles on the divisor $\mathcal{T}^t - U$. Therefore, if ψ as in Lemma 2.2, then the dual wave solution is of the form $\psi^+ = k^{-P\cdot z} e^{-kt} \phi^+(Ux + Z, t, k)$, where the coefficients $\xi^+_s(z+Z,t)$ of the formal series

$$\phi^+(z,t,k) = e^{-bt}\left(1 + \sum_{s=1}^{\infty} \xi^+_s(z,t)\, k^{-s}\right) \tag{3.13}$$

have simple poles along the divisor $\mathcal{T}^t - U$.

The ambiguity in the definition of ψ does not affect the product

$$\psi^+\psi = \left(k^{-x}e^{-kt}\Phi^{-1}\right)\left(\Phi k^x e^{kt}\right). \tag{3.14}$$

Therefore, the coefficients J_s of the product

$$\psi^+\psi = \phi^+(z,t,k)\,\phi(z,t,k) = 1 + \sum_{s=1}^{\infty} J_s(z,t)\,k^{-s} \tag{3.15}$$

are meromorphic functions on X. The factors in the left hand side of (3.15) have the simple poles on $\mathcal{T}^t$ and $\mathcal{T}^t - U$. Hence, $J_s(z)$ is a meromorphic function on X with the simple poles at $\mathcal{T}^t$ and $\mathcal{T}^t - U$. Moreover, the left and right action of pseudo-difference operators are formally adjoint, i.e., for any two operators the equality $(k^{-x}\mathcal{D}_1)(\mathcal{D}_2 k^x) = k^{-x}(\mathcal{D}_1\mathcal{D}_2 k^x) + (T-1)(k^{-x}(\mathcal{D}_3 k^x))$ holds. Here $\mathcal{D}_3$ is a pseudo-difference operator whose coefficients are difference polynomials in the coefficients of $\mathcal{D}_1$ and $\mathcal{D}_2$. Therefore, from (3.14-3.19) it follows that

$$\psi^+\psi = 1 + \sum_{s=1}^{\infty} J_s k^{-s} = 1 + \Delta\left(\sum_{s=2}^{\infty} Q_s k^{-s}\right). \tag{3.16}$$

The coefficients of the series Q are difference polynomials in the coefficients φ_s of the wave operator. Therefore, they are meromorphic functions of z with poles on $\mathcal{T}^t$, i.e. $Q_s = q_s/\tau$.

From the definition of $\mathcal{L}$ it follows that

$$\mathrm{res}_k\left(\psi^+(\mathcal{L}^n\psi)\right)k^{-1}dk = \mathrm{res}_k\left(\psi^+ k^n\psi\right)k^{-1}dk = J_n. \tag{3.17}$$

On the other hand, using the identity

$$\mathrm{res}_k\left(k^{-x}\mathcal{D}_1\right)(\mathcal{D}_2 k^x)\,k^{-1}dk = \mathrm{res}_T\,(\mathcal{D}_2\mathcal{D}_1)\,, \tag{3.18}$$

we get

$$\mathrm{res}_k(\psi^+\mathcal{L}^n\psi)k^{-1}dk = \mathrm{res}_k\left(k^{-x}\Phi^{-1}\right)(\mathcal{L}^n\Phi k^x)\,k^{-1}dk = \mathrm{res}_T\,\mathcal{L}^n = F_n. \tag{3.19}$$

Therefore, $F_n = J_n$ and the lemma is proved.

Important remark. In [21] the statement that F_m has poles only along $\mathcal{T}^t$ and $\mathcal{T}^t - U$ was crucial for the proof of the existence of commuting difference operators associated with u. Namely, it implies that for all but a finite number of positive integers $i \notin A$ there exist constants $c_{n,\alpha}$ such that

$$F_i(z,t) - \sum_{\alpha\in A} c_{i,\alpha}F_\alpha(z,t) = 0\,, \tag{3.20}$$

hence (3.7) would imply that the corresponding linear combinations $L_i := \mathcal{L}_+^i - \sum c_{i,\alpha}\mathcal{L}_+^\alpha$ commutes with $P := \partial_t - T - u$. Not so: since these constants $c_{i,\alpha}$ might depend on t, we might not have $[P, L_n] = 0$, and we cannot immediately make the next step and claim the existence of commuting operators (!).

So our next goal is to show that these constants in fact are t-independent. For that let us consider the functions $F_i^1(z,t)$. From (3.8) and (3.10) it follows that

$$F_i^1 = \partial_t \left(\frac{q_i(z,t)}{\tau(z,t)} \right) \tag{3.21}$$

Let $\{F_\alpha^1 \mid \alpha \in A\}$, for finite set A, be a basis of the space $\mathcal{F}(t)$ spanned by $\{F_m^1\}$. Then for all $n \notin A$ there exist constants $c_{n,\alpha}(t)$ such that

$$F_n^1(z,t) = \sum_{\alpha \in A} c_{n,\alpha}(t) F_\alpha^1(z,t)\,. \tag{3.22}$$

Due to (3.21) it is equivalent to the equations

$$q_n(z,t) = \sum_\alpha c_{n,\alpha}(t) q_\alpha(z,t)\,, \quad z \in \mathcal{T}^t, \tag{3.23}$$

$$\dot q_n^1(z,t) = \sum_\alpha c_{n,\alpha}(t) \dot q_\alpha^1(z,t) \quad z \in \mathcal{T}^t\,, \tag{3.24}$$

from which we get

$$\sum_\alpha (\dot c_{n,\alpha}) q_\alpha(z,t) = 0 \qquad z \in \mathcal{T}^t. \tag{3.25}$$

From (3.9) we obtain

$$\Delta_U \left(F_n^2 - \sum_{\alpha \in A} c_{n,\alpha}(t) F_\alpha^2(z,t) \right) = \dot c_{n,\alpha} F_\alpha^1\,. \tag{3.26}$$

The left hand side is Δ_U derivative of a meromorphic function. The right hand side has pole only at $\mathcal{T}^t$. Therefore, both sides of the equation must vanish. Then the assumption that the set F_α^1 is minimal imply $\dot c_{n,\alpha} = 0$.

Lemma 3.2 *Let ψ be a wave function corresponding to u, and let L_i, $i \notin A$ be the difference operator given by the formula*

$$L_i = \mathcal{L}_+^i - \sum_{\alpha \in A} c_{i,\alpha} \mathcal{L}_+^\alpha, \ i \notin A, \tag{3.27}$$

where the constants $c_{i,\alpha}$ are defined by equations (3.22).

Then the equation

$$L_i\,\psi = a_i(k)\,\psi, \quad a_i(k) = k^i + \sum_{s=1}^\infty a_{s,i} k^{n-s} \tag{3.28}$$

where $a_{s,i}$ are constants, hold.

Proof. First note that from (3.7) it follows that

$$[\partial_t - T - u, L_i] = 0. \tag{3.29}$$

Hence, if ψ is the wave solution of (1.7) then $L_i\psi$ is also a wave solution of the same equation. By uniqueness of the wave function up to a constant in z-factor we get (3.28) and thus the lemma is proven.

The operator L_i can be regarded as a z-parametric family of ordinary difference operators L_i^z.

Corollary 3.1 *The operators L_i^z commute with each other,*

$$[L_i^z, L_j^z] = 0\,. \tag{3.30}$$

From (3.28) it follows that $[L_i^z, L_j^z]\psi = 0$. The commutator is an ordinary difference operator. Hence, the last equation implies (3.30).

4 The fully discrete case

The main goal of this section is to characterize under some nondegeneracy assumptions all the abelian solutions of equation (1.23. As above we begin with the construction of the corresponding quasiperiodic wave function. We would like to emphasize once again that the construction of wave function follows closely the argument from the beginning of Section 5 in [21] but is simplified by the assumption (*iii*) in the formulation of Theorem 1.2.

4.1 Construction of the wave function

First let us show that the existence of a holomorphic solutions of equation (1.23) implies certain relations on $\mathcal{T}^\nu$.

Lemma 4.1 *([21]) If equation (1.23) has holomorphic solutions, then the equation*

$$\frac{\tau(z+W,\nu+1)\,\tau(z-2W,\nu)\,\tau(z+W,\nu-1)}{\tau(z-W,\nu+1)\,\tau(z+2W,\nu)\,\tau(z-W,\nu-1)} = -1 \tag{4.1}$$

is valid on the divisor $\mathcal{T}^\nu = \{\, z \in \mathbb{C}^m \mid \tau(z,\nu) = 0\}$.

Proof. The evaluations of (1.23) at the divisors $\mathcal{T}^\nu \pm W$ give two different expressions for the restriction of $\tau_A(z,\nu)$ on $\mathcal{T}^\nu$:

$$\tau_A(z,\nu) = e^{p\cdot W - E}\,\frac{\tau(z+W,\nu+1)\,\tau_A(z+W,\nu-1)}{\tau(z+2W,\nu)}\,, \quad z \in \mathcal{T}^\nu, \tag{4.2}$$

$$\tau_A(z,\nu) = -e^{-p\cdot W - E}\,\frac{\tau(z-W,\nu+1)\,\tau_A(z-W,\nu-1)}{\tau(z-2W,\nu)}\,, \quad z \in \mathcal{T}^\nu\,. \tag{4.3}$$

The evaluation of equation (1.23) for $\nu-1$ at $\mathcal{T}^\nu$ implies

$$e^{-p\cdot W}\tau(z+W,\nu-1)\,\tau_A(z-W,\nu-1)=e^{p\cdot W}\tau(z-W,\nu-1)\tau_A(z+W,\nu-1),\quad z\in\mathcal{T}^\nu. \tag{4.4}$$

Taking the ratio of (4.2,4.3) and using (4.4) we get (4.1). The lemma is proved.

Equation (4.1) is all what we need for the rest.

Lemma 4.2 *Let $\tau(z,\nu)$ be a sequence of non-trivial quasiperiodic holomorphic functions on $\mathbb{C}^m$. Suppose that the group $\{2W\nu|\,\nu\in\mathbb{Z}\}$ is Zariski dense in X and equation (4.1) holds. Then there exist wave solutions $\psi(z,\nu,k)=k^\nu\phi(z,\nu,k)$ of the equation (1.20) with u as in (1.21) such that:*

(i) the coefficients $\xi_s(z,\nu)$ of the formal series

$$\phi(z,\nu,k)=\xi_0(\nu)+\sum_{s=1}^{\infty}\xi_s(z,\nu)\,k^{-s} \tag{4.5}$$

are meromorphic functions of the variable $z\in\mathbb{C}^m$ with simple poles at the divisor $\mathcal{T}^\nu$, i.e.

$$\xi_s(z,\nu)=\frac{\tau_s(z,\nu)}{\tau(z,\nu)}, \tag{4.6}$$

where $\tau_s(z,\nu)$ is now a holomorphic function;

(ii) $\xi_s(z,\nu)$ satisfy the following monodromy properties

$$\xi_s(z+\lambda,\nu)-\xi_s(z,\nu)=\sum_{i=1}^{s}B^\lambda_{i,\,\nu-s+i}\,\xi_{s-i}(z,\nu)\,,\quad \lambda\in\Lambda, \tag{4.7}$$

where $B^\lambda_{i,\,\nu}$ are z-independent.

Proof. The functions $\xi_s(z,\nu)$ are defined recursively by the equations

$$\xi_{s+1}(z-W,\nu)-\xi_{s+1}(z+W,\nu)=u(z,\nu)\,\xi_s(z,\nu-1). \tag{4.8}$$

The first equation for $s=-1$ is satisfied by an arbitrary z-independent function $\xi_0=\xi_0(\nu)$. In what follows it will be assumed that $\xi_0(\nu)\neq 0$.

We will now prove lemma by induction in s. Let us assume inductively that for $r\le s$ the functions ξ_r are known and satisfy (4.7). Note, that the evaluation of (4.8) for $s-1$ and $\nu-1$ at the divisor $\mathcal{T}^\nu$ gives the equation

$$\tau_s(z-W)\tau(z+W)=\tau_s(z+W)\tau(z-W)\,,\quad z\in\mathcal{T}^\nu. \tag{4.9}$$

From (4.1) and (4.9) it follows that the two formulae by which we define the residue of ξ_{s+1} on $\mathcal{T}^\nu$

$$\tau^0_{s+1}(z,\nu)\ =\ \frac{\tau(z+W,\nu+1)\,\tau_s(z+W,\nu-1)}{\tau(z+2W,\nu)}\,,\quad z\in\mathcal{T}^\nu, \tag{4.10}$$

$$-\tau^0_{s+1}(z,\nu)\ =\ \frac{\tau(z-W,\nu+1)\,\tau_s(z-W,\nu-1)}{\tau(z-2W,\nu)}\,,\quad z\in\mathcal{T}^\nu\,. \tag{4.11}$$

do coincide.

The expression (4.10) is certainly holomorphic when $\tau(z+2W)$ is non-zero, i.e. is holomorphic outside of $\mathcal{T}^\nu \cap (\mathcal{T}^\nu - 2W)$. Similarly from (4.11) we see that $\tau^0_{s+1}(z,\nu)$ is holomorphic away from $\mathcal{T}^\nu \cap (\mathcal{T}^\nu + 2W)$.

We claim that $\tau^0_{s+1}(z,\nu)$ is holomorphic everywhere on $\mathcal{T}^\nu$. Indeed, by assumption the closure of the abelian subgroup generated by $2W$ is everywhere dense. Thus for any $z \in \mathcal{T}^\nu$ there must exist some $N \in \mathbb{N}$ such that $z - 2(N+1)W \notin \mathcal{T}^\nu$; let N moreover be the minimal such N. From (4.11) it then follows that $\tau^0_{s+1}(z,\nu)$ can be extended holomorphically to the point $z - 2NW$. Thus expression (4.10) must also be holomorphic at $z - 2NW$; since its denominator there vanishes, it means that the numerator must also vanish. But this expression is equal to the numerator of (4.11) at $z - 2(N-1)W$; thus τ^0_{s+1} defined from (4.11) is also holomorphic at $z - 2(N-1)W$ (the numerator vanishes, and the vanishing order of the denominator is one, since we are talking exactly about points on its vanishing divisor). Note that we did not quite need the fact $z - 2(N+1)W \notin \mathcal{T}^\nu$ itself, but the consequences of the minimality of N, i.e., $z - 2kW \in \mathcal{T}^\nu$, $0 \le k \le N$, and the holomorphicity of $\tau^0_{s+1}(z,\nu)$ at $z - 2NW$." Therefore, in the same way, by replacing N by $N-1$, we can then deduce holomorphicity $\tau^0_{s+1}(z,\nu)$ at $z - 2(N-2)W$ and, repeating the process N times, at z.

Recall that an analytic function on an analytic divisor in $\mathbb{C}^d$ has a holomorphic extension to all of $\mathbb{C}^d$ ([28]). Therefore, there exists a holomorphic function $\widetilde{\tau}_{s+1}(z,\nu)$ extending the $\tau^0_{s+1}(z,\nu)$. Consider then the function $\chi_{s+1}(z,\nu) = \widetilde{\tau}_{s+1}(z,\nu)/\tau(z,\nu)$, holomorphic outside of $\mathcal{T}^\nu$.

From (4.7) and (4.10) it follows that the function

$$f^\lambda_{s+1}(z,\nu) = \chi_{s+1}(z+\lambda,\nu) - \chi_{s+1}(z,\nu) - \sum_{i=1}^{s} B^\lambda_{i,\,\nu-1-s+i}\,\xi_{s+1-i}(z,\nu) \tag{4.12}$$

has no pole at the divisor $\mathcal{T}^\nu$. Hence, it is a holomorphic function. It satisfies the twisted homomorphism relations

$$f^{\lambda+\mu}_{s+1}(z,\nu) = f^\lambda_{s+1}(z+\mu,\nu) + f^\mu_{s+1}(z,\nu), \tag{4.13}$$

i.e., it defines an element of the first cohomology group of Λ_0 with coefficients in the sheaf of holomorphic functions, $f \in H^1_{gr}(\Lambda_0, H^0(\mathbb{C}^m, \mathcal{O}))$. Once again using the same arguments, as that used in the proof of the part (b) of the Lemma 12 in [25], we get that there exists a holomorphic function $h_{s+1}(z,\nu)$ such that

$$f^\lambda_{s+1}(z,\nu) = h_{s+1}(z+\lambda,\nu) - h_{s+1}(z,\nu) + \widetilde{B}^\lambda_{s+1,\nu}\xi_0(\nu), \tag{4.14}$$

where $\widetilde{B}^{\lambda,}_{s+1,\nu}$ is z-independent. Hence, the function $\zeta_{s+1} = \chi_{s+1} + h_{s+1}$ has the following monodromy properties

$$\zeta_{s+1}(z+\lambda,\nu) - \zeta_{s+1}(z,\nu) = \widetilde{B}^\lambda_{s+1,\nu}\,\xi_0(\nu) + \sum_{i=1}^{s} B^\lambda_{i,\,\nu-1-s+i}\,\xi_{s+1-i}(z,\nu)\,. \tag{4.15}$$

Let us consider the function R_{s+1} defined by the equation

$$R_{s+1} = \zeta_{s+1}(z-W,\nu) - \zeta_{s+1}(z+W,\nu) - u(z,\nu)\,\xi_s(z,\nu-1)\,. \tag{4.16}$$

Equation (4.10) and (4.11) imply that the r.h.s of (4.16) has no pole at $\mathcal{T}^\nu \pm W$. Hence, $R_{s+1}(z,\nu)$ is a holomorphic function of z. From (4.7,4.15) it follows that it is periodic with respect to the lattice Λ, i.e., it is a function on X. Therefore, R_{s+1} is a constant.

Hence, the function

$$\xi_{s+1}(z,\nu) = \zeta_{s+1}(z,\nu) + l_{s+1}(z,\nu)\xi_0(\nu) + c_{s+1}(\nu)\xi_0(\nu)\,, \tag{4.17}$$

where $c_{s+1}(\nu)$ is a constant, and l_{s+1} is a linear form such that

$$l_{s+1}(2W,\nu)\xi_0(\nu) = -R_{s+1}(\nu)\,,$$

is a solution of (4.8). It satisfies the monodromy relations (4.7) with

$$B^{\lambda}_{s+1,\nu} = \widetilde{B}^{\lambda}_{s+1,\nu} + l_{s+1}(\lambda,\nu)\,. \tag{4.18}$$

The induction step is completed and thus the lemma is proven.

On each step the ambiguity in the construction of ξ_{s+1} is a choice of linear form $l_{s+1}(z,\nu)$ and constants $c_{s+1}(\nu)$. As in Section 2, we would like to fix this ambiguity by normalizing monodromy coefficients $B^{\lambda}_{i,\nu}$ for a set of linear independent vectors $\lambda_1,\ldots,\lambda_d \in \Lambda$. As it was revealed in [21] in the fully discrete case there is an obstruction for that. This obstruction is a possibility of the existence of periodic solutions of (4.8), $\xi_{s+1}(z+\lambda,\nu) = \xi_{s+1}(z,\nu),\ \lambda \in \Lambda$, for $0 \le s \le r-1$.

Note, that there are no periodic solutions of (4.8) for all s. Indeed, the functions $\xi_s(z,\nu)$ as solutions of non-homogeneous equations are linear independent. Suppose not. Take a smallest nontrivial linear relation among $\xi_s(z,\nu)$, and apply (5.24) to obtain a smaller linear relation. The space of meromorphic functions on X with simple pole at $\mathcal{T}^\nu$ is finite-dimensional. Hence, there exists minimal r such that equation (4.8) for $s=r$ has no periodic solutions.

Let $\lambda_1,\ldots,\lambda_d$ be a set of linear independent vectors in Λ. Without loss of generality throughout the rest of the paper it will be assumed that there is no linear form $l(z),\ z\in\mathbb{C}^m$, with $l(\lambda_j)=1$ and $l(2W)=0$.

Lemma 4.3 *Suppose equations (4.8) has periodic solutions for $s<r$ and has a quasi-periodic solution ξ_r whose monodromy relations for λ_j have the form*

$$\xi_r(z+\lambda_j,\nu) - \xi_r(z,\nu) = b\,\xi_0(\nu), \quad j=1,\ldots,d, \tag{4.19}$$

where $b\neq 0$ is a constant. Then for all s equations (4.8) has solutions of the form (4.6) satisfying (4.7) with $B^{\lambda_j}_{i,\nu} = b\,\delta_{i,r}$, i.e.,

$$\xi_s(z+\lambda_j,\nu) - \xi_s(z,\nu) = b\,\xi_{s-r}(z,\nu). \tag{4.20}$$

Proof. We will now prove the lemma by induction in $s \ge r$. Let us assume inductively that ξ_{s-r} is known, and for $1\le i\le r$ there are solutions $\tilde{\xi}_{s-r+i}$ of (4.8) satisfying (4.7) with

$B^{\lambda_j}_{i,\nu} = b\,\delta_{i,r}$. Then, according to the previous lemma, there exists a solution $\tilde\xi_{s+1}$ of (4.8) having the form (4.6) and satisfying monodromy relations (4.7), which for λ_j have the form

$$\tilde\xi_{s+1}(z+\lambda_j,\nu) - \tilde\xi_{s+1}(z,\nu) = b\,\tilde\xi_{s-r+1}(z,\nu) + B^{\lambda_j}_{s+1,\,\nu}\xi_0(\nu)\,. \tag{4.21}$$

If ξ_{s-r} is fixed, then the general quasi-periodic solution ξ_{s-r+1} with the normalized monodromy relations is of the form

$$\xi_{s-r+1}(z,\nu) = \widetilde{\xi}_{s-r+1}(z,\nu) + c_{s-r+1}(\nu)\xi_0(\nu)\,. \tag{4.22}$$

It is easy to see that under the transformation (4.22) the functions $\widetilde{\xi}_{s-r+i}$ get transformed to

$$\xi_{s-r+i}(z,\nu) = \widetilde{\xi}_{s-r+i}(z,\nu) + c_{s-r+1}(\nu-i+1)\,\xi_{i-1}(z,\nu)\,. \tag{4.23}$$

This transformation does not change the monodromy properties of ξ_{s-r+i} for $i \le r$, but changes the monodromy property of ξ_{s+1}:

$$\begin{aligned}\xi_{s+1}(z+\lambda_j,\nu) - \xi_{s+1}(z,\nu) = b\,\xi_{s-r+1}(z,\nu) + B^{\lambda_j}_{s+1,\,\nu}\,\xi_0(\nu) + \\ b\,(c_{s-r+1}(\nu-r) - c_{s-r+1}(\nu))\,\xi_0(\nu).\end{aligned} \tag{4.24}$$

Recall, that $\widetilde{\xi}_{s+1}$ was defined up to a linear form $l_{s+1}(z,\nu)$ which vanishes on $2W$. Therefore the normalization of the monodromy relations for ξ_{s+1} uniquely defines this form and the differences $(c_{s-r+1}(\nu-r) - c_{s-r+1}(\nu))$. The induction step is completed and the lemma is thus proven.

Note, the following important fact: if ξ_{s-r} is fixed then ξ_{s-r+1}, such that there exists quasi-periodic solution ξ_{s+1} with normalized monodromy properties, is defined uniquely up to the transformation:

$$\xi_{s-r+1}(z,\nu) \longmapsto \xi_{s-r+1}(z,\nu) + c_{s-r+1}(\nu)\xi_0(\nu), \quad c_{s-r+1}(\nu+r) = c_{s-r+1}(\nu). \tag{4.25}$$

Our next goal is to show that the assumption of Lemma 4.3 holds for some r, and then to fix the remaining ambiguity (4.25) in the definition of the wave function. At this moment we are going to use for the first time the assumption that τ is a meromorphic periodic function of the variable ν.

Let r be the minimal integer such that there exist solutions $\xi_0^0 = 1, \xi_1^0, \ldots, \xi_{r-1}^0$ of (4.8) that are periodic functions of z with respect to Λ, and there is no periodic solution ξ_r of (4.8). As it was noted above, the functions τ_s are linear independent. Hence, $r \le h^0(Y, \theta|_Y)$.

If ξ_{r-1}^0 is periodic, then the monodromy relation for ξ_r has the form

$$\xi_r^0(z+\lambda,\nu) - \xi_r^0(z,\nu) = B_r^\lambda(z,\nu)\,, \quad \lambda\in\Lambda. \tag{4.26}$$

The function B_r^λ is independent of the ambiguities in the definition of $\xi_i,\ i<r$, and therefore, it is a well-defined holomorphic function of $z \in X$. Hence, it is z-independent, $B_r^\lambda(z,\nu) = B_r^\lambda(\nu)$. The function ξ_r^0 is defined up to addition of a linear form $l_r(z,\nu)$ such that $l(2W,\nu) = 0$. Therefore, there exist the solution ξ_r^0 such that $B_r^{\lambda_j}(\nu) = B_r(\nu)$. There is no ξ_r^0 which is periodic for all ν. Hence, $B_r(\nu) \ne 0$ at least for one value of ν. By assumption the function

τ is a meromorphic function of ν. Therefore, $B_r(\nu)$ is a meromorphic function of ν. Shifting $\nu \to \nu + \nu_0$ if needed, we may assume without loss of generality that $B_r(\nu) \neq 0$ for *all* $\nu \in \mathbb{Z}$. From (1.26) it follows that $u(z, \nu + N) = u(z, \nu)$. Hence, $B_r(\nu)$ is a periodic function of ν, i.e.

$$B_r(\nu + N) = B_r(\nu). \tag{4.27}$$

Under the transformation

$$\xi_0^0 = 1 \longmapsto \xi_0(\nu) \tag{4.28}$$

the solutions ξ_r^0 get transformed to

$$\xi_s(z, \nu) = \xi_s^0(z, \nu)\, \xi_0(\nu - s). \tag{4.29}$$

From (4.26) it follows that the transformed function ξ_r satisfies the relations

$$\xi_r(z + \lambda, \nu) - \xi_r(z, \nu) = B_r^\lambda(\nu)\xi_0(z, \nu - r)\,, \quad \lambda \in \Lambda. \tag{4.30}$$

The equation

$$b\, \xi_0(\nu) = B_r(\nu)\xi_0(\nu - r), \quad \xi_0(\nu + N) = \xi_0(\nu). \tag{4.31}$$

restricted to the space of periodic functions ξ_0 can be regarded as a finite-dimensional linear equation. The vanishing of the determinant of this equation defines the constant b. With b fixed equation (4.31) defines ξ_0 uniquely up to multiplication by a function $c_0(\nu)$ such that $c_0(\nu + N) = c_0(\nu + r) = c_0(\nu)$. By the assumption of Theorem 1.2 the period N is prime and $N > H^0(\mathcal{T}^\nu)$. As it was mentioned above $r \leq H^0(\mathcal{T}^\nu)$. Hence, two periods of c_0 are coprime, i.e.,$(r, N) = 1$. Therefore, ξ_0 is defined uniquely up to a constant factor.

Lemma 4.4 *Suppose that the assumptions of Theorem 1.2 hold. Then there exists a formal solution*

$$\phi = \xi_0(\nu) + \sum_{s=1}^{\infty} \xi_s(z, \nu)\, k^{-s} \tag{4.32}$$

of the equation

$$k\phi(z - W, \nu, k) = k\phi(z + W, \nu, k) + u(z, \nu)\, \phi(z, \nu - 1, k)\,, \tag{4.33}$$

with u as in (1.21) such that:

(i) the coefficients ξ_s of the formal series ϕ are of the form $\xi_s = \tau_s/\theta$, where $\tau_s(Z)$ are holomorphic functions;

(ii) $\phi(z, \nu, k)$ is quasi-periodic with respect to the lattice Λ and for the basis vectors λ_j in $\mathbb{C}^m$ its monodromy relations have the form

$$\phi(z + \lambda_j, \nu, k) = (1 + b\, k^{-r})\, \phi(z, \nu, k), \quad j = 1, \ldots, m, \tag{4.34}$$

where b are constants defined by (4.31);

(iii) $\phi(z, \nu, k)$ is a quasi-periodic function of the variable ν, i.e.

$$\phi(z, \nu + N, k) = \phi(z, \nu, k)\mu(k) \tag{4.35}$$

(iv) ϕ is unique up to the multiplication by a constant in z factor $\rho(k)$.

Proof. We prove the lemma by induction in s. Let us assume inductively that ξ_{s-r} is known. As shown above the normalization of the relations for ξ_{s+1} uniquely defines ξ_{s-r+1} up to the transformation (4.25), i.e. up to a r-periodic function $c_{s-r+1}(\nu+r) = c_{s-r+1}(\nu)$. The quasiperiodicity condition (iii) is equivalent to the condition that this function of c_{s-r+1} is N-periodic. As it was mentioned above the periods r and N are coprime. Hence, on each step ξ_{s-r+1} is defined up to the additive constant. This ambiguity corresponds to the multiplication of ϕ be a constant factor $\rho(k)$, and thus the lemma is proven.

4.2 Commuting difference operators

As it Section 3 we are now going to construct rings $\mathcal{A}^z$ of commuting difference operators. First we introduce pseudo-difference operator in one of the original variable m depending on the second variable n and a point $z \in \mathbb{C}^d$. (Recall, that the variables n, m are related to x, ν via (1.19).

The formal series $\phi(z, \nu, k)$ defines a unique pseudo-difference operator

$$\mathcal{L}(z,\nu) = w_0(\nu)T + \sum_{s=0}^{\infty} w_{s+1}(z,\nu)\, T^{-s}, \quad T = e^{\partial_m}, \tag{4.36}$$

such that the equation

$$\left(w_0(m+n)T + \sum_{s=0}^{N} w_s(z+(m-n)W,(m+n))\, T^{-s} \right) \psi = k\psi\,. \tag{4.37}$$

holds. Here $\psi = k^{n+m}\phi(z+(m-n)W,(m+n),k)$. The coefficients $w_s(z,\nu)$ of $\mathcal{L}$ are difference polynomials in terms of the coefficients of ϕ. Due to quasiperiodicity of ψ they are meromorphic functions on the abelian variety X.

From equations $(4.33, 4.37)$ it follows that

$$\left((\Delta_1 \mathcal{L}^i)\, T_1 - (\Delta \mathcal{L}^i)\, T - [u, \mathcal{L}^i] \right) \psi = 0\,, \tag{4.38}$$

where $\Delta_1 \mathcal{L}^i$ and $\Delta \mathcal{L}^i$ are pseudo-difference operator in T, whose coefficients are difference derivatives of the coefficients of $\mathcal{L}^i$ in the variables n and m respectively. Using the equation $(T_1 - T - u)\,\psi = 0$, we get

$$\left(\left(\Delta_1 \mathcal{L}^i\right) T - \left(\Delta \mathcal{L}^i\right) T + \left(\Delta_1 \mathcal{L}^i\right) u - [u, \mathcal{L}^i] \right) \psi = 0. \tag{4.39}$$

The operator in the left hand side of (4.39) is a pseudo-difference operator in the variable m. Therefore, it has to be equal to zero. Hence, we have the equation

$$\left(\Delta_0 \mathcal{L}^i\right) T + \left(\Delta_1 \mathcal{L}^i\right) u - [u, \mathcal{L}^i] = 0, \ \Delta_0 = T_1 - T \tag{4.40}$$

Let $\mathcal{L}^i_+$ be the strictly positive difference part of the operator $\mathcal{L}^i$, i.e.,

$$\mathcal{L}^i = \mathcal{L}^i_+ + \mathcal{L}^i_- = \mathcal{L}^i_+ + \sum_{s=0}^{\infty} F_{i,s} T^{-s} \tag{4.41}$$

Then,

$$\left(\Delta_0\mathcal{L}_+^i\right)T+\left(\Delta_1\mathcal{L}_+^i\right)u-[u,\mathcal{L}_+^i]=-\left(\Delta_0\mathcal{L}_-^i\right)T-\left(\Delta_1\mathcal{L}_-^i\right)u+[u,\mathcal{L}_-^i] \tag{4.42}$$

The left hand side of (4.42) is a difference operator with non-vanishing coefficients only at the positive powers of T. The right hand side is a pseudo-difference operator of order 1. Therefore, it has the form f_iT. The coefficient f_i is easy expressed in terms of the leading coefficient $\mathcal{L}_-^i$. Finally we get the equation

$$\left(\Delta_0\mathcal{L}_+^i\right)T+\left(\Delta_1\mathcal{L}_+^i\right)u-[u,\mathcal{L}_+^i]=-(\Delta_0F_i)\,T, \tag{4.43}$$

where $F_i=F_i=\text{res }\ \mathcal{L}^i$.

By definition of $\mathcal{L}$ we have that the functions F_i in (4.41) are of the form

$$F_i=\text{res}_T\,\mathcal{L}^i=F_i(z+(m-n)W,(m+n)) \tag{4.44}$$

where for each ν the functions $F_i(z,\nu)$ are abelian functions, i.e., periodic functions of the variable $z\in\mathbb{C}^d$.

Lemma 4.5 *The abelian functions F_i have the form*

$$F_i(z,\nu)=\frac{q_i(z+W,\nu+1)}{\tau(z+W,\nu+1)}-\frac{q_i(z,\nu)}{\tau(z,\nu)}\,, \tag{4.45}$$

where $q_i(z,\nu)$ are holomorphic functions of the variable $z\in\mathbb{C}^d$.

Proof. The wave solution ψ define the unique operator Φ such that

$$\psi=\Phi k^{n+m},\quad \Phi=1+\sum_{s=1}^{\infty}\varphi_s((m-n)W+z,(m+n)\,T^{-s}\,, \tag{4.46}$$

where $\varphi_s(z,\nu)$ are meromorphic functions of $z\in\mathbb{C}^d$. The dual wave function

$$\psi^+=k^{-n-m}\left(1+\sum_{s=1}^{\infty}\xi_s^+((n-m)W+z,(n+m))\,k^{-s}\right) \tag{4.47}$$

is defined by the formula

$$\psi^+=k^{-n-m}\,T_1\,\Phi^{-1}\,T_1^{-1}. \tag{4.48}$$

It satisfies the equation

$$(T_1^{-1}-T^{-1}-u)\,\psi^+=0, \tag{4.49}$$

which implies that the functions $\xi_s^+(z,\nu)$ have the form $\xi_s^+(z,\nu)=\tau_s^+(z,\nu)/\theta(z+W,\nu+1)$, where $\tau_s^+(z,\nu)$ are holomorphic functions of $z\in\mathbb{C}^d$. Therefore, the functions $J_s(z,\nu)$ such that

$$(\psi^+T_1)\,\psi=k+\sum_{s=1}^{\infty}J_s((n-m)W+z,(n+m))\,k^{-s+1} \tag{4.50}$$

are meromorphic function on X with the simple poles at $\mathcal{T}^\nu$ and $\mathcal{T}^{\nu+1} - W$.

The same arguments as that used for the proof of (3.16) show that

$$(\psi^+ T_1)\psi = (k^{-x} T_1 \Phi^{-1})(\Phi k^x) = k + (\Delta Q) \tag{4.51}$$

where the coefficients of the series Q are of the form

$$Q = \sum_{s=0}^{\infty} Q_s(n-m)W + z, (n+m))k^{-s}\,, \tag{4.52}$$

and the functions $Q_s(z,\nu)$ are difference polynomials in the coefficients φ_s of the wave operator. Therefore, they are well-defined meromorphic functions of z. As shown above, the functions

$$J_s(z,\nu) = Q_s(z+W, \nu+1) - Q_s(z,\nu) \tag{4.53}$$

have simple poles at $\mathcal{T}^\nu$ and $\mathcal{T}^{\nu+1} - W$. Hence, $Q_s(z,\nu)$ have poles only at $\mathcal{T}^\nu$, i.e.

$$Q_s = \frac{q_s(z,\nu)}{\tau(z,\nu)}\,, \tag{4.54}$$

where $q_s(z,\nu)$ are holomoprhic functions of z.

From the definition of $\mathcal{L}$ it follows that

$$\mathrm{res}_k \left((\psi^+ T_1) (\mathcal{L}^i \psi) \right) k^{-2} dk = \mathrm{res}_k \left((\psi^+ \, T_1) \, \psi \right) k^{i-2} dk = J_i. \tag{4.55}$$

On the other hand, using (3.18) we get

$$\mathrm{res}_k((\psi^+ \, T_1) \, (\mathcal{L}^i \psi) \, k^{-2} dk = \mathrm{res}_k \left(k^{-n-m} \Phi^{-1} \right) \left(\mathcal{L}^i \Phi k^{n+m} \right) k^{-1} dk = \mathrm{res}_T \, \mathcal{L}^i = F_i. \tag{4.56}$$

Equation (4.45) is a direct corollary of (4.53-4.56). The lemma is proved.

The function ψ is quasiperiodic function of the variable ν. Then, from the definition of ψ^+ it follows that

$$\phi^+(z, \nu+N, k) = \phi^+(z,\nu,k)\mu^{-1}(k)\,, \tag{4.57}$$

where $\mu(k)$ is defined in (4.35). Therefore, the functions J_s are periodic functions of ν. Hence,

$$F_i(z, \nu+N) = F_i(z,\nu). \tag{4.58}$$

For each ν the space of functions spanned by the abelian functions $F_i(z,\nu)$ is finite-dimensional. Due to periodicity of F_i in ν the total space $\mathcal{F}$ spanned by sequences $F_i(z,\nu)$ is also finite-dimensional. Let $\{F_\alpha \mid \alpha \in A\}$, for finite set A, be a basis of the factor- space of $\mathcal{F}$ modulo z-independent sequences. Then for all $i \notin A$ there exist constants $c_{i,\alpha}, d_i(\nu)$ such that

$$F_i(z,\nu) - \sum_{\alpha\in A} c_{i,\alpha} F_\alpha(z,\nu) = d_i(\nu)\,. \tag{4.59}$$

The rest of the proof of Theorem 1.2 is identical to that in the proof of Theorem 1.1. Namely,

Lemma 4.6 *Let ψ be a wave function corresponding to u, and let L_i, $i \notin A$ be the difference operator given by the formula*

$$L_i = \mathcal{L}_+^i - \sum_{\alpha \in A} c_{i,\alpha} \mathcal{L}_+^\alpha, \ i \notin A, \tag{4.60}$$

where the constants $c_{i,\alpha}$ are defined by equations (4.59).

Then the equations

$$L_i \, \psi = a_i(k) \, \psi, \quad a_i(k) = k^i + \sum_{s=1}^{\infty} a_{s,i} k^{n-s}\,, \tag{4.61}$$

where $a_{s,i}$ are constants, hold.

Proof. From (4.43) it follows that

$$[T_1 - T - u, L_i] = 0. \tag{4.62}$$

Hence, if ψ is the wave solution of (1.20) then $L_i\psi$ is also a wave solution of the same equation. By uniqueness of the wave function up to a constant in z-factor we get (3.28) and thus the lemma is proven.

Corollary 4.1 *The operators L_i^z commute with each other,*

$$[L_i^z, L_j^z] = 0\,. \tag{4.63}$$

References

[1] H. Airault, H. McKean, and J. Moser. *Rational and elliptic solutions of the Korteweg-de Vries equation and related many-body problem*, Commun. Pure Appl. Math., **30** (1977), no. 1, 95–148.

[2] E. Arbarello, C. De Concini, *On a set of equations characterizing Riemann matrices.* Ann. of Math. (2) 120 (1984), no. 1, 119–140.

[3] E. Arbarello, C. De Concini, *Another proof of a conjecture of S.P. Novikov on periods of abelian integrals on Riemann surfaces*, Duke Math. Journal, 54 (1987), 163–178.

[4] O. Babelon, E. Billey, I. Krichever and M. Talon. *Spin generalisation of the Calogero-Moser system and the matrix KP equation*, in "Topics in Topology and Mathematical Physics", Amer. Math. Soc. Transl. Ser. 2 **170**, Amer. Math. Soc., Providence, 1995, 83–119.

[5] David Ben-Zvi, Thomas Nevins, Flows of Calogero-Moser Systems. arXiv:math/0603722

[6] J.L.Burchnall, T.W. Chaundy, *Commutative ordinary differential operators.I*, Proc. London Math Soc. **21** (1922), 420–440.

[7] J.L.Burchnall, T.W. Chaundy, *Commutative ordinary differential operators.II*, Proc. Royal Soc. London **118** (1928), 557–583.

[8] J.D. Fay, *Theta functions on Riemann surfaces.* Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin-New York, 1973.

[9] R. Gunning, *Some curves in abelian varieties,* Invent. Math. 66 (1982), no. 3, 377–389.

[10] Gorsky, A.; Nekrasov, N. Hamiltonian systems of Calogero-type, and two-dimensional Yang-Mills theory. Nuclear Phys. B 414 (1994), no. 1–2, 213–238.

[11] I. M. Krichever, *Integration of non-linear equations by methods of algebraic geometry*, Funct. Anal. Appl., 11 (1977), n. 1, 12–26.

[12] I. M. Krichever, *Methods of algebraic geometry in the theory of non-linear equations*, Russian Math. Surveys, 32 (1977), n. 6, 185–213.

[13] I.Krichever, *Algebraic curves and non-linear difference equation*, Uspekhi Mat. Nauk 33 (1978), n 4, 215-216.

[14] I. Krichever. *Elliptic solutions of Kadomtsev-Petviashvili equations and integrable systems of particles* (In Russian), Funct. Anal. Appl., **14** (1980), no. 1, 45–54. Transl. 282–290.

[15] I.Krichever, *The periodic nonabelian Toda lattice and two-dimensional generalization*, appendix to: B.Dubrovin,*Theta-functions and nonlinear equations*, Uspekhi Mat. Nauk **36** (1981), n 2, 72-77.

[16] I. Krichever, *Two-dimensional periodic difference operators and algebraic geometry*, Doklady Akad. Nauk USSR 285 (1985), n 1, 31-36.

[17] I. Krichever. *Elliptic solutions to difference non-linear equations and nested Bethe ansatz equations*, Calogero-Moser-Sutherland models (Montreal, QC, 1997), 249–271, CRM Ser. Math. Phys., Springer, New York, 2000.

[18] I. Krichever. *Elliptic analog of the Toda lattice*, Internat. Math. Res. Notices (2000), no. 8, 383–412.

[19] Krichever, I. Vector bundles and Lax equations on algebraic curves. Comm. Math. Phys. 229 (2002), no. 2, 229–269.

[20] I.Krichever, *Integrable linear equations and the Riemann-Schottky problem*, Algebraic geometry and number theory, 497–514, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006.

[21] I. Krichever, *Characterizing Jacobians via trisecants of the Kummer Variety*, math.AG /0605625.

[22] I. Krichever, O. Lipan, P. Wiegmann, and A. Zabrodin. *Quantum integrable models and discrete classical Hirota equations*, Comm. Math. Phys., **188** (1997), no. 2, 267–304.

[23] I.Krichever, T.Shiota, *Abelian solutions of the KP equation*, arXiv:0804.0274

[24] I. Krichever, A. Zabrodin. *Spin generalisation of the Ruijsenaars-Schneider model, the nonabelian two-dimensionalized Toda lattice, and representations of the Sklyanin algebra* (in Russian), Uspekhi Mat. Nauk, **50** (1995), no. 6, 3–56.

[25] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*, Invent. Math., 83(2):333–382, 1986.

[26] D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg-de Vries equation and related non-linear equations* – Proceedings Int.Symp. Algebraic Geometry, Kyoto, 1977, 115–153, Kinokuniya Book Store, Tokyo, 1978.

[27] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Physics 170 (1986) 370–405.

[28] J-P. Serre, *Faisceaux algébriques cohérents*, (French) Ann. of Math. (2) 61, (1955). 197–278.

[29] G.E. Welters, *On flexes of the Kummer variety (note on a theorem of R. C. Gunning).* Nederl. Akad. Wetensch. Indag. Math. 45 (1983), no. 4, 501–520.

[30] G.E. Welters, *A criterion for Jacobi varieties*, Ann. of Math., 120 (1984), n. 3, 497–504.