Chazy–Ramanujan Type Equations

Mark J. Ablowitz

Department of Applied Mathematics

University of Colorado, Boulder

Soliton Theory–1979

Soliton Theory–1979-II

Outline

Chazy–Ramanujan Type Equations

- Introduction
- Painlevé equations–connection to integrable systems
- Reductions of self-dual Yang-Mills (SDYM) to “DH-9” which relates to Darboux-Halphen systems
- Solution of DH-9 via Schwarzian Eq./ Schwarzian ‘triangle” fcn’s
- Reductions of DH-9 and solutions to Chazy equations

Outline–con't

- Classical Chazy eq.— solution in terms of quasi-modular forms $\in SL_2(\mathbb{Z}) : \Gamma$
- Connection to differential equations of Ramanujan
- Discuss relation of Chazy type eq. with other eq. of number theoretic interest: $\Gamma_0(2)$
- If time permits short discussion of water waves, asymptotic reductions and physical realization of KP solutions
- Conclusion

Introduction

Wide interest in integrable systems; many mathematically and physically interesting systems; some of the best known are listed below

$1+1$ dimension

- KdV: $u_t + 6uu_x + u_{xxx} = 0$
- mKdV: $u_t \pm 6u^2u_x + u_{xxx} = 0$
- NLS: $iu_t + u_{xx} \pm 2|u|^2u = 0$

$2+1$ dimension

- KP: $(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} = 0$
- DS: $iu_t + u_{xx} + \sigma_1 u_{yy} + \phi u = 0$

$$\phi_{xx} - \sigma_1\phi_{yy} = 2\sigma_2(|u|^2)_{xx} \qquad \sigma_j = \pm 1;\, j = 1,2$$

Solutions

- Rapid decay:
  Riemann-Hilbert BVP; DBAR $\Rightarrow$
  Linear integral equations
  Soliton solutions
- Periodic/quasi-periodic solutions
  => expressed via multidimensional theta functions
- Self-similar solutions:
  ODE-Painlevé type
- Automorphic functions:
  Darboux-Halphen-Chazy-Ramanujan class

KdV: Self-similar Sol'n

$$u_t + 6uu_x + u_{xxx} = 0$$

Self-similar (similarity) solution

$u(x,t) \sim \frac{1}{(3t)^{2/3}} f(z), \quad z = \frac{x}{(3t)^{1/3}}$

$$f''' + 6ff' - (zf' + 2f) = 0 \quad (E)$$

1973: MJA & A. Newell: $t \to \infty$ for $|\frac{x}{(3t)^{1/3}}| = O(1)$ => Eq. (E)

Note: $f = -(w' + w^2)$ => 2nd Painlevé equation:

$$w'' - (zw + 2w^3) = \alpha \quad PII$$

α =const

mKdV => PII

Prototype

$$u_t - 6u^2u_x + u_{xxx} = 0$$

Asymptotic analysis $t \to \infty$ => slowly varying (modulated) self-similar sol'n (cf. MJA & H. Segur, '77-'81)

$$u(x,t) \sim \frac{1}{(3t)^{1/3}} w(z; c_1, c_2), \quad z = \frac{x}{(3t)^{1/3}} \quad \text{where } c_i = c_i(\xi), \ \xi = x/t$$

$$w'' - (zw + 2w^3) = 0$$

From slowly varying similarity solution: when $\xi = x/t \to 0$ => connection formulae for PII

Connection Formulae– PII

$$w'' - (zw + 2w^3) = 0$$

$w(z) \sim r_0 Ai(z), \;\; z \to \infty$

$w(z) \sim \frac{d_0}{|z|^{1/4}} sin\theta, \;\; z \to -\infty$

where: $\theta = \frac{2}{3}|z|^{3/2} - \frac{3}{2}d_0^2 log|z| + \theta_0; \;\; |r_0| < 1$

Find connection formulae (here PII => 'NL Airy' fcn)

$d_0(r_0) = -\frac{1}{\pi} log(1 - |r_0|^2)$

$\theta_0(r_0) = \frac{\pi}{4} - \frac{3log2}{2} d_0^2(r_0) - arg\{\Gamma(1 - i\frac{d_0(r_0)^2}{2})\}$

Thus given the constant r_0 as $z \to \infty$ we have explicit formulae for the values of the constants as $z \to -\infty$, i.e.

$d_0 = d_0(r_0)$

$\theta_0 = \theta_0(r_0)$

(cf. MJA & H. Segur, '81)

Applicability of Similarity Sol'ns

self-similar solutions arise frequently in physics and math

$t \to \infty$ analysis => self-similar solutions

e.g. linear wave problems, integrable systems: KdV, mKdV, NLS, and their hierarchies; (note: asymptotic techniques of Deift, Zhou, co-workers...)

Broad context of slowly varying (modulated) similarity solutions associated with asymptotic solutions of NL PDEs is still open

Integrable systems–ODE's of P-Type

Self-similar reductions of integrable systems
MJA, Ramani, Segur: '77-'81

Reductions: KdV=> PI, mKdV => PII; Sine-Gordon => PIII; ... SDYM => all six Painlevé equations in gen'l position (Mason and Woodhouse '93), hierarchies of KdV=> hierarchies of Painlevé eq.,...

Painlevé (P) type equations have no movable branch points

NLPDE's solvable by inverse scattering transform (IST) deeply connected to P- type equations

Sol'ns of the underlying linear integral equations only yield movable poles

P-Type Equations

- P-Type: ODE has no movable branch points
  Fuch's, Kovalevskaya (cf. Golubev), Painlevé, Chazy, ...
- 1st order ODE:

  $$y' = F(z, y)$$

  Rational in y, locally analytic (l.a.) in z
  Find: only Ricatti equation of P-Type:

  $$\frac{dy}{dz} = a_0(z) + a_1(z)y + a_2(z)y^2$$

- 2nd order ODE:

  $$y'' = F(z, y, y')$$

  Rational in y, y', l.a. in z. Some 50 classes of equations; including linear eq., reductions to Ricatti, Eq. with elliptic fcn sol'ns, and and 6 Painlevé eq.

Painlevé equations

$$y'' = 6y^2 + z, \quad PI$$

$$y'' = zy + y^3 + \alpha, \;\; \alpha \text{ const.}, \quad PII$$

$$y'' = \frac{y'^2}{y} - \frac{y'}{z} + \frac{\alpha y^2 + \beta}{z} + \gamma y^3 + \frac{\delta}{y}, \;\; \alpha, \ldots \delta \text{ const.}, \quad PIII$$

... Third order equations: full classification of

$$y''' = F(y, y', y'', z)$$

still open. Chazy (1909-1911), Bureau (1987) found interesting systems with movable natural boundaries

Painlevé

Painlevé (1863-1933): Studied/taught at at École Normale; member French Academy of Sciences; President of the French Mathematical Society: 1903

Held major political offices: Minister of War and Prime Minister; an aircraft carrier was named in his honor

Reduction SDYM

SDYM:

$$F_{\alpha\beta} = 0, \quad F_{\bar{\alpha}\bar{\beta}} = 0$$

$$F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0$$

where

$$F_{\alpha\beta} = \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha - [\gamma_\alpha, \gamma_\beta]$$

and $[\gamma_\alpha, \gamma_\beta] = \gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha$

Cartesian cood.: $\alpha = t + iz, \bar{\alpha} = t - iz, \beta = x + iy, \bar{\beta} = x - iy$

Reductions of SDYM:

1. $\gamma_a(\alpha, \bar{\alpha}, \beta, \bar{\beta}) -> \gamma_a(\alpha), \gamma_a(\alpha, \beta), ...$
2. choice of algebra: $gl(N), su(N)$...
3. gauge freedom: $\gamma_a -> (f\gamma_a - \partial_a f) f^{-1}$

1D Reductions of SDYM

Use:

$\gamma_\alpha = \gamma_t + i\gamma_z = \gamma_0 + i\gamma_3$

$\gamma_\beta = \gamma_x + i\gamma_y = \gamma_1 + i\gamma_2$

Take one indep. variable: t and use guage: $\gamma_0 = 0$ =>

$$\gamma_j = \gamma_j(t), j = 1, 2, 3$$

$$F_{\alpha\beta} = \partial_\alpha \gamma_\beta - \partial_\beta \gamma_\alpha - [\gamma_\alpha, \gamma_\beta] = \partial_t(\gamma_1 + i\gamma_2) - [i\gamma_3, \gamma_1 + i\gamma_2] = 0$$

Formally, real, imaginary parts => Nahm system:

$$\partial_t \gamma_1 = [\gamma_2, \gamma_3], \quad 1, 2, 3 \;\; cyclic$$

Simplest case: $\gamma_l(t) = \omega_l(t) X_l; \;\; su(2): \;\; [X_j, X_k] = \sum_l \epsilon_{jkl} X_l$
where ϵ_{jkl} is antisym tensor ($\epsilon_{123} = 1$); find

$$\partial_t \omega_1 = \omega_2 \omega_3, \quad 1, 2, 3 \;\; cyclic$$

1D Reductions of SDYM–con’t

$$\partial_t \omega_1 = \omega_2 \omega_3, \quad 1,2,3 \ \ cyclic$$

Note:

$$\omega_1 = E\cosh\phi(t),\ \omega_2 = E\sinh\phi(t),\ \omega_3 = \frac{d\phi(t)}{dt}$$

E=const. find:

$$\frac{d^2\phi}{dt^2} = \frac{E^2}{2}\sinh\phi$$

Solution is in terms of elliptic functions

Darboux-Halphen Systems

$$\partial_t \gamma_1 = [\gamma_2, \gamma_3], \quad 1,2,3 \;\; cyclic$$

Set $\gamma_l(t) = \sum_{j,k} O_{lj} M_{jk}(t) X_k$ where:

$[X_j, X_k] = \sum_l \epsilon_{jkl} X_l, \;\; OO^T = I, \;\; O \in \;\; so(3)$
$X_l(O_{jk}) = \sum_p \epsilon_{lkp} O_{jp}, \quad sdiff(S^3)$

Find $M = \{M_{jk}(t)\}$ satisfies:

$$\frac{dM}{dt} = (detM)(M^{-1})^T + M^T M - (TrM)M \qquad \text{(DH-9)}$$

(Chakravarty, MJA, Takhtajan, '92) If $M = diag(\omega_1, \omega_2, \omega_3)$ find

$$\partial_t \omega_1 = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3), \;\; 1,2,3 \;\; cyclic \quad \text{(DH)}$$

(Chakravarty, MJA, Clarkson, '90)

DH and Chazy Eq.

From DH eq. let $y = -2(\omega_1 + \omega_2 + \omega_3)$ find classical Chazy eq. (Chazy 1909)

$$\frac{d^3 y}{dt^3} - 2y\frac{d^2 y}{dt^2} + 3(\frac{dy}{dt})^2 = 0 \quad \text{(C)}$$

Later discuss automorphic character of (C) and relation to modular forms

Other cases of reductions to eq. with automorphic solutions:

Gibbons and Pope ('79), Hitchin '85 relativity: Bianchi IX cosmological models; Dubrovin Top. field th'y '96

Buchstaber, Leikin, Pavlov '03; Pavlov '04; Ferapontov and Marshall '07: Egorov Chains

Ferapontov, Odesski '10: integrable Lagrangian flows; Burovskiy, Ferapontov, Tsarev, '09 integrable 2+1d flows

Chazy

J. Chazy (1882–1955): Studied at École Normale and taught at the Sorbonne

Major contributions to study of differential eq. and celestial mechanics

Member of French Academy of Sciences

1912 shared Grand Prix des Sciences (differential eq.) with P. Boutroux and R. Garnier and in 1922 awarded Prix Benjamin Valz (Celestial Mechanics)

President French Mathematical Society: 1934

Solution of DH-9

$$\frac{dM}{dt} = (detM)(M^{-1})^T + M^T M - (TrM)M \quad \text{(DH-9)}$$

(MJA,Chakravarty, Halburd, '99)

If $M = P(D + a)P^{-1}$ find P, D, a satisfy:

$\frac{dP}{dt} = -Pa,\quad D = \text{diag}(\omega_1, \omega_2, \omega_3),\quad a_{ij} = \sum_k \epsilon_{ijk}\tau_k$

$$\partial_t \omega_1 = \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \quad 1,2,3 \;\; cyclic$$

$$\tau^2 = \sum_k \tau_k^2, \qquad \partial_t \tau_1 = -\tau_1(\omega_2 + \omega_3), \quad 1,2,3 \;\; cyclic$$

Solution of DH-9– con't

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}\log\frac{\dot{s}}{s(s-1)}, \quad \omega_2 = -\frac{1}{2}\frac{d}{dt}\log\frac{\dot{s}}{s-1}, \quad \omega_3 = -\frac{1}{2}\frac{d}{dt}\log\frac{\dot{s}}{s}$$

$$\tau_1 = \frac{\kappa_1\dot{s}}{[s(s-1)]^{1/2}}, \quad \tau_2 = \frac{\kappa_2\dot{s}}{s(s-1)^{1/2}}, \quad \tau_3 = \frac{\kappa_3\dot{s}}{s^{1/2}(s-1)}$$

$\kappa_j =$ const, $j = 1, 2, 3$ where $s(t)$ satisfies:

$$\{s, t\} + \frac{\dot{s}^2}{2}V(s) = 0$$

where

$$\{s, t\} = \left(\frac{s''}{s'}\right)' - \frac{1}{2}\left(\frac{s''}{s'}\right)^2, \qquad V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}$$

$$\alpha = -2\kappa_1^2, \quad \beta = 2\kappa_2^2, \quad \gamma = -2\kappa_3^2$$

Schwarzian Eq.

Schwarzian 'triangle' functions $s(t) = s(\alpha, \beta, \gamma, t)$ satisfy

$$\{s,t\} + \frac{\dot{s}^2}{2} V(s) = 0$$

where $\{s,t\} = (\frac{s''}{s'})' - \frac{1}{2}(\frac{s''}{s'})^2, \quad V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}$

Schwarzian triangle function are automorphic functions. If $s(t)$ is a sol'n of Schwarzian eq., so is

$$\tilde{s}(t) = s(\gamma(t)), \quad \gamma(t) = \frac{at+b}{ct+d}, \quad ad - bc = 1, \quad \gamma \in SL_2(\mathbb{C})$$

Schwarzian eq. can be linearized.

Use inversion of variables $\{s,t\} = -\dot{s}^2\{t,s\}$ =>

Linearization of Schwarzian

$$\{t,s\} - \frac{V(s)}{2} = (\frac{t''}{t'})' - \frac{1}{2}(\frac{t''}{t'})^2 - \frac{V(s)}{2} = 0$$

Then solution in terms of: $t(s) = \frac{y_1(s)}{y_2(s)}$ where y_1, y_2 are 2 l.i. solutions of:

$$y'' + \frac{1}{4}V(s)y = 0$$

Note: $t'(s) = \frac{y_2 y_1' - y_1 y_2'}{y_2^2} = \frac{W}{y_2^2},\ W =$ const.; $\frac{t''}{t'} = -2\frac{y_2'}{y_2}$

$s(t)$ single valued if

$\alpha = \frac{1}{l}, \beta = \frac{1}{m}, \gamma = \frac{1}{n},\ \ l, m, n \,\epsilon\, \mathbf{Z}^+$ and $0 \leq \alpha + \beta + \gamma < 1$

Moreover find $s(t)$ has a movable natural boundary which is a circle. Radius and center depend on I.C.'s

Darboux-Halphen -Chazy

When $M = diag(\omega_1, \omega_2, \omega_3), \alpha = \beta = \gamma = 0$ DH-9 reduces to:

$$\partial_t \omega_1 = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3), \quad 1,2,3 \ \ cyclic \quad \text{(DH)}$$

With $y = -2(\omega_1 + \omega_2 + \omega_3)$ find classical Chazy eq.

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = 0 \quad \text{(C)}$$

When: $\alpha = \beta = \gamma = \frac{2}{n}$ (DH-9) yields:

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = \frac{4}{36-n^2}(6\frac{dy}{dt} - y^2)^2 \quad \text{(GC)}$$

GC: Generalized Chazy eq.: $n = \infty$ => (C)
C and GC eq. has movable natural boundary–circle
$y(t)$ single valued for $n > 6,$ integer

Chazy Eq. – Modular Forms

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3(\frac{dy}{dt})^2 = 0 \quad \text{(C)}$$

(C) admits symmetry:

$$y \longrightarrow \tilde{y} = \frac{1}{(ct+d)^2} y(\gamma(t)) - \frac{6c}{ct+d}, \quad \gamma(t) = \frac{at+b}{ct+d}$$

$ad - bc = 1; \gamma \in SL_2(\mathbb{C})$; special solution of (C)

$$y(t) = i\pi E_2(t) = i\pi(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n), \quad q = e^{2\pi i t}$$

$\sigma_1(n) = \sum_{d/n} d$ = sum of divisors of n; $E_2(t) \in SL_2(\mathbf{Z})$ satisfies above symmetry —it is a quasi-modular form weight 2

(MJA, Chakravarty,Takhtajan '91)

Modular Forms

If $f(z)$ (note $t \to z$) satisfies

$$f(z) = \frac{1}{(cz+d)^k} f(\gamma(z)), \quad \gamma(z) = \frac{az+b}{cz+d}$$

where $ad - bc = 1$; $\gamma \in SL_2(\mathbb{Z})$ and $f(z)$ has a q exp'n

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z}$$

i.e. it is analytic in upper half z plane, then $f(z)$ is said to be a modular form of weight k

$E_2(z)$ satisfies: $E_2(z) = \frac{1}{(ct+d)^2} E_2(\gamma(z)) - \frac{6c}{cz+d}$

$E_2(z)$ is said to be a quasi-modular form weight 2

Chazy Eq. – Modular Form.–con't

From properties of q series find sol'n of Chazy ($E_2(z)$) may be written as:

$$y(z) = \frac{1}{2}\frac{d}{dz}\log\Delta(z)$$

where

$$\Delta(z) = \frac{\Delta(\gamma(z))}{(cz+d)^{12}} = Cq\prod_1^{\infty}(1-q^n)^{24} = C\sum_1^{\infty}\tau(n)q^n$$

$\gamma \in SL_2(\mathbb{Z}),\ \ q = e^{2\pi iz},\ C = (2\pi)^{12},\ \tau(n) =$ Ramanujan coef.

$\Delta(z)$ is a modular form weight 12; from Chazy eq. it satisfies a homogeneous NL ODE, 4th order in derivatives and powers:

$$\Delta''''\Delta^3 - 5\Delta'''\Delta'\Delta^2 - \frac{3}{2}\Delta''^2\Delta^2 + 12\Delta\Delta'^2\Delta'' - \frac{13}{2}\Delta'^4 = 0$$

(Rankine: '56)

Eisenstein series

Consider the Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{1}^{\infty} \sigma_{k-1}(n) q^n$$

$k \geq 2$, even integer, B_k is the k-th Bernoulli number, $q = e^{2\pi i z}$ and

$\sigma_k(n) = \sum_{d/n} d^k =$ sum of divisors of n to kth power

$E_k(z)$ are modular forms weight k for $k \geq 4$; $E_2(z)$ is quasi-modular form weight 2

Ramanujan (1916) showed that E_2, E_4, E_6 satisfies a 3rd order coupled system of ODEs

Chazy and Ramanujan Eq.

Ramanujan found:

$P(q) = E_2(q), \quad Q(q) = E_4(q), \quad R(q) = E_6(q)$ satisfy

$$qP'(q) = \frac{P^2 - Q}{12} \qquad \text{(i)}$$

$$qQ'(q) = \frac{PQ - R}{3} \qquad \text{(ii)}$$

$$qR'(q) = \frac{PR - Q^2}{2} \qquad \text{(iii)}$$

From (i): $Q = P^2 - 12qP'(q)$; then (ii) => $R = R[P, P', P'']$

So eq. (iii) is a 3rd order eq. for $P(q)$

Using $q = e^{2\pi i z}$ and letting $P(z) = \frac{1}{i\pi} y(z)$ =>

$$y''' - 2yy'' + 3(y')^2 = 0 \qquad \text{Chazy!}$$

(MJA, Chakravarty, Halburd, ’03)

Number Theoretic Fcn's–$\Gamma_0(2)$

There are other interesting ODEs associated with number theoretic functions. With the subgroup $\Gamma_0(2)$:

$$\Gamma_0(2) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid c \equiv 0 \text{ (mod 2)} \right\}$$

are Eisenstein series; with $q = e^{2\pi i z}$ for even integer $k \geq 2$:

$$\mathcal{E}_k(q) = 1 + \frac{2k}{(1-2^k)B_k} \sum_1^\infty \frac{(-)^n n^{k-1} q^n}{1-q^n}$$

which are modular forms of weight $k \geq 4$; further $\mathcal{E}_2(z)$ is a quasi-modular form

E-Fcn's in $\Gamma_0(2)$-con't

Another function $\tilde{\mathcal{E}}_2(z)$

$$\tilde{\mathcal{E}}_2(z) = 1 + 24\sum_1^\infty \frac{nq^n}{1+q^n}, \quad q = e^{2\pi i z}$$

$\tilde{\mathcal{E}}_2(z)$ is modular form of wt. 2 $\in \Gamma_0(2)$

Ramamani ('70) showed that:

$$\mathcal{P} = \mathcal{E}_2, \;\; \mathcal{Q} = \mathcal{E}_4, \;\; \tilde{\mathcal{P}} = \tilde{\mathcal{E}}_2$$

associated with $\Gamma_0(2)$ satisfy a 3rd order coupled ODE system

ODEs and $\Gamma_0(2)$

$$q\mathcal{P}'(q) = \frac{\mathcal{P}^2 - \mathcal{Q}}{4} \qquad \text{(i)}$$

$$q\mathcal{Q}'(q) = \mathcal{P}\mathcal{Q} - \tilde{\mathcal{P}}\mathcal{Q} \qquad \text{(ii)}$$

$$q\tilde{\mathcal{P}}'(q) = \frac{\tilde{\mathcal{P}}\mathcal{P} - \mathcal{Q}}{2} \qquad \text{(iii)}$$

From (i) $\mathcal{Q} = \mathcal{P}^2 - 4q\mathcal{P}'$

From (ii) $\tilde{\mathcal{P}} = \mathcal{P} - q\mathcal{Q}'/\mathcal{Q}$ so $\tilde{\mathcal{P}} =$ fcn of $(\mathcal{P}, \mathcal{P}', \mathcal{P}'')$

=> from (iii) find a 3rd order NL ODE for $\mathcal{P}$

In terms of $y(z) = i\pi\mathcal{P}(z) = i\pi\mathcal{E}_2(z)$,

$$y''' - 2yy'' + (y')^2 + 2\frac{(y'' - yy')^2}{y^2 - 2y'} = 0$$

(MJA, Chakravarty and Hahn ’06); this eq. was also found by Bureau (1987) in his study of 3rd order ODE of ‘Chazy-type’

ODEs and Number Theor. Fcn's–con't

Also from properties of q series:

$$y = i\pi\mathcal{E}_2(z) = \frac{\mathcal{D}'}{2\mathcal{D}} = \frac{1}{2}(\log\mathcal{D}(z))'$$

where $\mathcal{D}$ is a modular form weight 4. $\mathcal{D}$ satisfies a homog. NL ODE 6th order in derivatives and powers:

$$\mathcal{D}''''(8\mathcal{D}''\mathcal{D}^4 - 10\mathcal{D}'^2\mathcal{D}^3) + 8\mathcal{D}'''^2\mathcal{D}^4 + \mathcal{D}'''(10\mathcal{D}'^3\mathcal{D}^2 + 16\mathcal{D}''\mathcal{D}'\mathcal{D}^3)$$

$$-20\mathcal{D}''^3\mathcal{D}^3 - 48\mathcal{D}''^2\mathcal{D}'^2\mathcal{D}^2 - 60\mathcal{D}''\mathcal{D}'^4\mathcal{D} + 25\mathcal{D}'^6 = 0$$

DH systems and $\Gamma_0(2)$

The gDH system below can be related to the $\mathcal{P} = \mathcal{E}_2, \ \ \mathcal{Q} = \mathcal{E}_4, \ \ \tilde{\mathcal{P}} = \tilde{\mathcal{E}}_2$ system in $\Gamma_0(2)$

$$w_1' = -w_2 w_3 + w_1(w_2 + w_3) + \tau^2$$
$$w_2' = -w_3 w_1 + w_2(w_3 + w_1) + \tau^2$$
$$w_3' = -w_1 w_2 + w_3(w_1 + w_2) + \tau^2$$

$$\tau^2 = \alpha^2(w_1-w_2)(w_2-w_3)+\beta^2(w_2-w_1)(w_1-w_3)+\gamma^2(w_3-w_1)(w_2-w_3)$$

The $w_j, j = 1, 2, 3$ can be written in terms of a Schwarz triangle function $s = s(\alpha, \beta, \gamma, z)$

DH systems and $\Gamma_0(2)$-con't

The triangle function satisfies

$$\{s,z\} + \frac{s'^2}{2}V(s) = 0, \quad \{s,z\} = (\frac{s''}{s})' - \frac{1}{2}\left(\frac{s''}{s'}\right)^2$$

$$V(s) = \frac{1-\alpha^2}{s^2} + \frac{1-\beta^2}{(s-1)^2} + \frac{\alpha^2+\beta^2-\gamma^2-1}{s(s-1)}$$

and the solution of gDH is given in terms of s below

$$w_1 = -\frac{1}{2}\Big[\log\Big(\frac{s'}{s}\Big)\Big]', w_2 = -\frac{1}{2}\Big[\log\Big(\frac{s'}{s-1}\Big)\Big]', w_3 = -\frac{1}{2}\Big[\log\Big(\frac{s'}{s(s-1)}\Big)\Big]'$$

DH systems and $\Gamma_0(2)$ -con't

The solution of the $\mathcal{P} = \mathcal{E}_2, \;\; \mathcal{Q} = \mathcal{E}_4, \;\; \tilde{\mathcal{P}} = \tilde{\mathcal{E}}_2$ system is given by

$$y(z) = i\pi\mathcal{P}(z) = -(w_2 + w_3)(z), \;\; i\pi\tilde{\mathcal{P}}(z) = (w_1 - w_3)(z)$$

$$\pi^2\mathcal{Q}(z) = (w_1 - w_3)(w_3 - w_2)(z)$$

with $\alpha = \frac{1}{2}, \beta = \gamma = 0.$ Further the *general* solution is obtained due to the automorphic nature of $s(z)$:

$$\tilde{s}(z) = s\left(\frac{az+b}{cz+d}\right) \;\; => \tilde{y}(z) = \frac{1}{(cz+d)^2}y\left(\frac{az+b}{cz+d}\right) - \frac{c}{cz+d}$$

where a,b,c,d in $\gamma \in SL_2(\mathbb{C})$; $\;\; w_j, \;\; j = 1,2,3$ have a similar transformation property

Chazy and DH systems

Another direction: can find many representations of sol'ns of the classical Chazy eq.

$$y''' - 2yy'' + 3(y')^2 = 0 \quad \text{Chazy}$$

in terms of solutions of a gDH system: $w_j,\, j = 1, 2, 3$ and its triangle fcn

$$y(z) = a_1 w_1 + a_2 w_2 + a_3 w_3, \quad a_1 + a_2 + a_3 = 6$$

where a_j are const. Employ the analytic properties of Chazy solutions.

Chazy and DH systems–con't

$$y(z) = a_1 w_1 + a_2 w_2 + a_3 w_3, \quad a_1 + a_2 + a_3 = 6$$

Below some of them are given (Chakravarty, MJA '10) in terms of $s = s(\alpha, \beta, \gamma, z)$

i) $s(0, \frac{1}{2}, \frac{1}{3}, z); \quad a_1 = 3, a_2 = 1, a_3 = 2$ Chazy's case

ii) $s(0, \frac{1}{3}, \frac{1}{3}, z); \quad a_1 = a_2 = a_3 = 2$ Takhtajan '93

iii) $s(0, \frac{1}{2}, 0, z); \quad a_1 = 3, a_2 = 2, a_3 = 1$

iv) $s(0, \frac{1}{3}, 0, z); \quad a_1 = 2, a_2 = 3, a_3 = 1$

v) $s(0, \frac{2}{3}, 0, z); \quad a_1 = 4, a_2 = 1, a_3 = 1$ may transf y to case (i)

vi) $s(0, 0, 0, z); \quad a_1 = a_2 = a_3 = 2$

Chazy and Hypergeometric

Another form of linearization of Chazy. Consider:

$$s(s-1)\chi'' + [(a+b+1)s - c]\chi' + ab\chi = 0,$$

with $a = (1-\alpha-\beta-\gamma)/2,\ b = (1-\alpha-\beta+\gamma)/2,\ c = 1-\alpha$

Let $z(s) = \frac{\chi_2}{\chi_1}$, $\chi_j, j = 1,2$ are two l.i. sol'ns

One sol'n is: $\chi_1 = {}_2F_1(a,b,c,s)$; χ_2 is obtained from χ_1

$z'(s) = 1/s'(z) = W/\chi_1^2$, $W = Cs^{\alpha-1}(s-1)^{\beta-1}$ is the Wronskian, $C \neq 0$

Use gDH with $y = a_1 w_1 + a_2 w_2 + a_3 w_3$, $a_1 + a_2 + a_3 = 6$ =>

Chazy and Hypergeometric–con't

gDH =>

$$y(s(z)) = \frac{3}{C} s^{-\alpha}(s-1)^{-\beta}\Big(2s(s-1)\chi_1\chi_1' - [(\tilde{b}_1 + \tilde{b}_2)s - \tilde{b}_2]\chi_1^2\Big)$$

where $\tilde{b}_j$ depend on α, β, a_j and $\chi_1 = {}_2F_1(a, b, c, s)$

With different triangle fnc's $s(\alpha, \beta, \gamma, z)$ all Chazy sol'ns can be expressed in terms of hypergeometric fcn's

Ramanujan - Hypergeometric

Ramanujan: ‘classical’:

$$P(z) = (1-5x)\chi^2 + 12x(1-x)\chi\chi' = \frac{1}{i\pi}y$$

with $\chi(x) := {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; x)$ and

$$z = \frac{i}{2}\,\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}1, 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; x)}, \quad q = e^{2\pi i z}$$

Find $x = x(z)$; via modular fc’ns (${}_2F_1(x) \to K(x)$). He also found ‘alternative parametrizations’ of P

$$z_r = \frac{i}{2\sin(\frac{\pi}{r})}\,\frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}), \quad r = 2, 3, 4, 6$$

with $P(z)$ depending on χ, χ'

May relate R and C sol’ns (transf of hypergeometric fcns)...

Chazy - Ramanujan

Those solutions of P(z); i.e. sol'ns of Chazy eq., which were written down by Ramanujan correspond to:

$s(0, \frac{1}{2}, 0, z): \quad r = 4$

$s(0, \frac{1}{3}, 0, z): \quad r = 3$

$s(0, \frac{2}{3}, 0, z): \quad r = 6$ may transform to $s(0, \frac{1}{2}, \frac{1}{3}, z)$ (Chazy's case)

$s(0, 0, 0, z)$: r=2

A case the Ramanujan formulae do not correspond to:

$s(0, \frac{1}{3}, \frac{1}{3}, z):$ Takhtajan '93

Conclusion

- Reductions of integrable systems yield: Painlevé and Chazy type equations
- In particular, reduction of SDYM => $3x3$ matrix system: DH-9–which can be solved in terms of Schwarzian triangle functions
- Special cases include Classical Chazy and Generalized Chazy eq.
- Classical Chazy also has solution $E_2(z)$ from which gen'l sol'n can be obtained

Conclusion–con't

- Ramanujan found a 3rd order system for $E_j(z),\ j = 2, 4, 6$ which reduces to Classical Chazy (MJA,Chakravarty, Halburd, '03)
- Chazy (1909-'11) and Ramanujan ('16) worked on the same eqs.; but from different perspectives
- Ramamani (1970) found number theoretic functions in $\Gamma_0(2)$ satisfy a 3rd order system of eq.
- From above system one can find a NL scalar eq. in Bureau's class of 'Chazy-type' eq.- and can find the gen'l sol'n
- The Ramamani system and Bureau's eq. can be related to gDH systems (MJA, Chakravarty, Hahn, '06)
- Can extend to other number theoretic fcn's in $\Gamma_0(N), N = 3, 4$ (Maier '10)

Conclusion–con’t

- Classical Chazy sol’n represented by many different triangle functions; they all can be linearized via hypergeomtric f’cns (Chakravraty, MJA, ’10)
- Many paramterizations were written down by Ramanujan; they can be related to S triangle fcns

Water Wave Equations

Classical equations: Define the domain D by
$D = \{-\infty < x_1, x_2 < \infty, \quad -h < y < \eta(x,t), \ \ x = (x_1, x_2), \ \ t > 0\}$

The water wave equations satisfy the following system for $\phi(x,y,t)$ and $\eta(x,t)$:

$$\Delta\phi = 0 \ \text{ in } \ D$$

$$\phi_y = 0 \ \text{ on } \ y = -h$$

$$\eta_t + \nabla\phi \cdot \nabla\eta = \phi_y \ \text{ on } \ y = \eta$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right) \ \text{ on } \ y = \eta$$

where g: gravity, $\sigma = \frac{T}{\rho}$: T surface tension, ρ: density.

WW-Nonlocal Spectral Eq

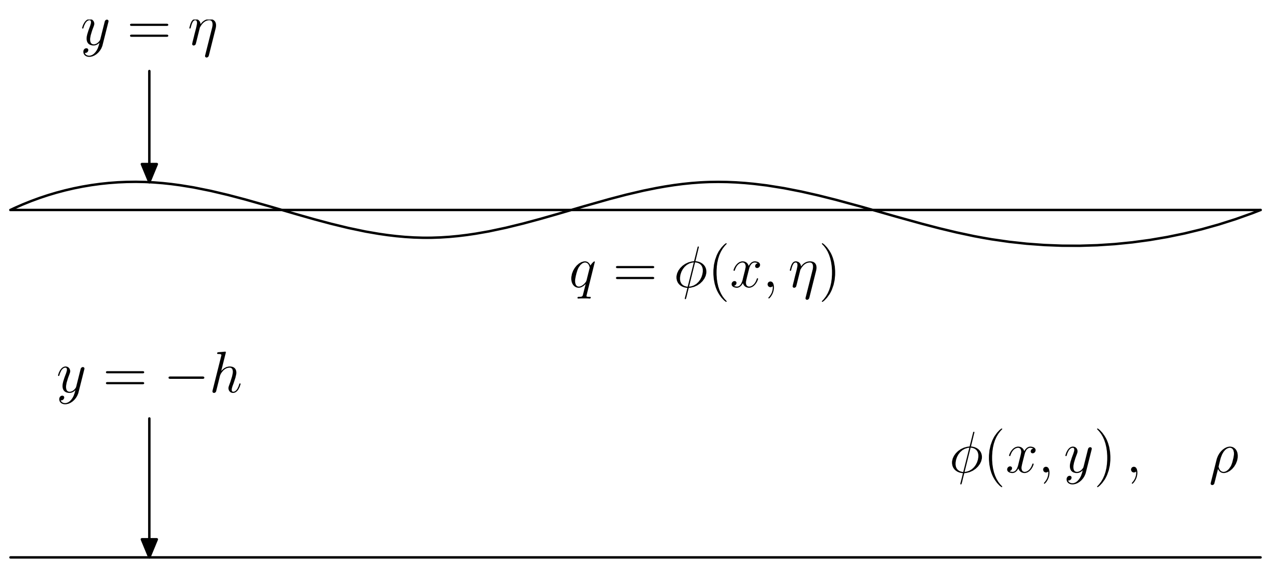
Work with A. Fokas, Z. Musslimani (JFM, 2006), reformulation: 2 eq., 2 unk: $\eta, q = \phi(x, \eta)$, rapid decay: 1 nonlocal spectral eq. and 1 PDE; fixed domain

$$\int dx e^{ik\cdot x}(i\eta_t \cosh[\kappa(\eta + h)] + \sinh[\kappa(\eta + h)] \frac{k \cdot \nabla q}{\kappa}) = 0 \qquad (I)$$

$$q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2(1 + |\nabla \eta|^2)} = \sigma \nabla \cdot (\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}) \qquad (II)$$

$$x = (x_1, x_2),\ k = (k_1, k_2),\ \kappa^2 = k_1^2 + k_2^2,\ q(x,t) = \phi(x, t, \eta(x,t))$$

WW: figure



Water wave configuration

Remarks

- Variables: η, q used by Zakharov ('68) in Hamiltonian formulation of WW
- Craig & Sulem ('93) derive Dirichlet-Neumann (DN) series in terms of η, q. Craig et al also investigate WW and interfacial waves via DN series

Remarks–con't

- (MJA,AF ZM '06) Derived nonlocal formulation and found:
  - Conserved quantities and new integral relations
  - Asymptotic reductions:
    - 1+1: KdV, Nonlinear Schrodinger (NLS) eq; i.e. find both shallow and deep water reductions
    - 2+1 Benny-Luke (BL) and Kadomtsev-Petvashvili (KP) eq
- MJA and Haut ('08-'10):
  - nonlocal eqs for waves with 1 and 2 free interfaces
  - connect to DN series/operators
  - asymptotic reductions: 2+1 ILW-BL, ILW-KP
  - high order asympt. expn's of 1-d and 2-d solitary waves

WW- Linearized System

If $|\eta|, |\nabla q|$ are small then eq. (I,II) simplify.

$$\int dx e^{ikx} \left(i\eta_t \cosh \kappa h + \frac{k \cdot \nabla q}{\kappa} \sinh \kappa h \right) = 0 \quad (1L)$$

recall $\kappa^2 = k_1^2 + k_2^2$. Use Fourier transform: $\hat{\eta} = \int dx e^{ikx} \eta$

$$i\hat{\eta}_t \cosh \kappa h + \frac{k \cdot \widehat{\nabla q}}{\kappa} \sinh \kappa h = 0 \quad (1L)$$

$$\widehat{q_t} + (g + \sigma \kappa^2)\hat{\eta} = 0 \quad (2L)$$

Then from eq. (1L), (2L)find:

$$\hat{\eta}_{tt} = -(g\kappa + \sigma\kappa^3) \tanh \kappa h \; \hat{\eta}$$

WW-Nonlocal System-Remarks

- Can find integral relations by taking $k \to 0$; first two (recall: $x \to (x_1, x_2)$):

$$\frac{\partial}{\partial t}\int dx\ \eta(x,t) = 0 \quad (Mass)$$

$$\frac{\partial}{\partial t}\int dx(x_j\eta) = \int dx\ q_{x_j}(\eta + h) \quad j = 1,2$$

LHS: COM in x_j direction -RHS related to x_j momentum: conserved; Higher order virial identities can also be found; e.g.

$$\frac{\partial}{\partial t}\int dx\ (\frac{x_j^2\eta}{2} - (\frac{\eta^3}{6} + \frac{\eta^2 h}{2})) = \int dx\ (x_j q_{x_j}(\eta + h)) \quad j = 1,2$$

Nondimensional Variables

We can make all variables nondimensional (nd):

$$x_1' = \frac{x_1}{l}\,,\ x_2' = \gamma\frac{x_2}{l}\,,\ a\eta' = \eta\,,\ \ t' = \frac{c_0}{l}t\,,\ q' = \frac{alg}{c_0}q\,,\ \sigma' = \frac{\sigma}{gh^2}$$

l, a are characteristic horiz. length, amplitude, and γ is a nd transverse length parameter; $c_0 = \sqrt{gh}$; hereafter drop $'$

Eq are written in terms of nd variables

$\epsilon = \frac{a}{h} << 1$: small amplitude

$\mu = \frac{h}{l} << 1$: long waves

$\gamma << 1$: slow transverse variation

WW-Asymptotic Systems

Expd cosh, sinh use nd paramters: $\epsilon = \frac{a}{h}, \mu = \frac{h}{l}$
Find: Benney-Luke (BL, ’64) eq. (nmlz’d surface tension, $\tilde{\sigma} = \sigma - 1/3$):

$$q_{tt} - \tilde{\Delta}q + \tilde{\sigma}\mu^2\tilde{\Delta}^2 q + \epsilon(\partial_t|\tilde{\nabla}q|^2 + q_t\tilde{\Delta}q) = 0 \ (BL)$$

$$\tilde{\Delta} = \partial_{x_1}^2 + \gamma^2\partial_{x_2}^2 \qquad |\tilde{\nabla}q|^2 = \left(q_{x_1}^2 + \gamma^2 q_{x_2}^2\right).$$

If $\epsilon = \mu^2 = \gamma^2 \ll 1$ then BL yields KP equation; after rescaling KP eq in std form ($x = (x_1, x_2) \to (x, y)$):

$$\partial_x(u_t + 6uu_x + u_{xxx}) - 3\,\mathrm{sgn}(\tilde{\sigma})u_{yy} = 0$$

Note: $\tilde{\sigma} > 0$ ‘strong’ surface tension: KPI Eq.

$\tilde{\sigma} < 0$ ‘weak’ surface tension: KPII Eq.

KP Equation

KP eq in standard form:

$$\partial_x(u_t + 6uu_x + u_{xxx}) - 3\,\mathrm{sgn}(\tilde{\sigma})u_{yy} = 0$$

Note: $\tilde{\sigma} > 0$ ‘strong’ surface tension: KPI Eq.

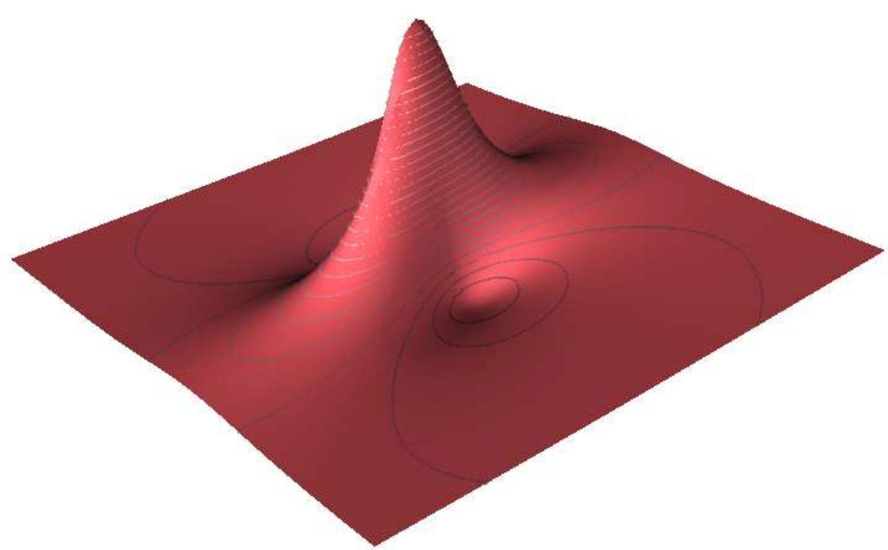
$\tilde{\sigma} < 0$ ‘weak’ surface tension: KPII Eq.

Lump Solution of KP

For $\tilde{\sigma} > 0$ strong ST, KPI has lump solutions

$$u = 16\frac{-4(x' - 2k_R y')^2 + 16k_I^2 y'^2 + \frac{1}{k_I^2}}{[4(x' - 2k_R y')^2 + 16k_I^2 y'^2 + \frac{1}{k_I^2}]^2}$$

where $x' = x - c_x t,\ y' = y - c_y t,\ c_x = 12(k_R^2 + k_I^2),\ c_y = 12k_R$



KP Equation: Line Solitons

KP equation in standard form with small surface tension ('KP II')

$$\partial_x(u_t + 6uu_x + u_{xxx}) + 3u_{yy} = 0$$

KP equation has line soliton solutions; simplest ones:

$$u = u_N = 2\frac{\partial^2 logF_N}{\partial x^2}$$

Where F_N is a polynomial in terms of sum of exponentials:

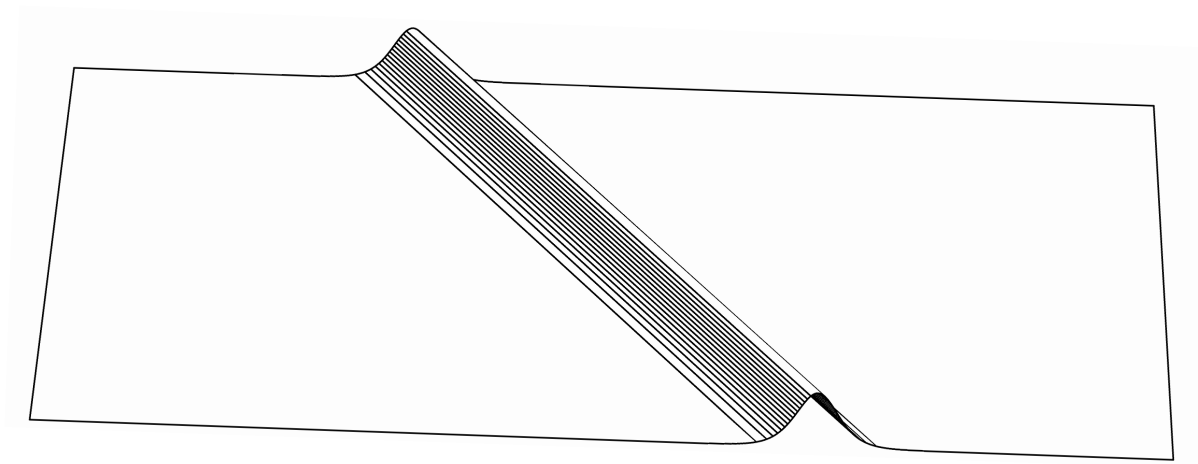
$$F_1 = 1 + e^{\eta_1}, \quad F_2 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}$$

where $\eta_j = k_j(x + P_j y - (k_j^2 + 3P_j^2)t + \eta_j^{(0)}), \quad e^{A_{12}} = \frac{(k_1-k_2)^2-(P_1-P_2)^2)}{(k_1+k_2)^2-(P_1-P_2)^2}$

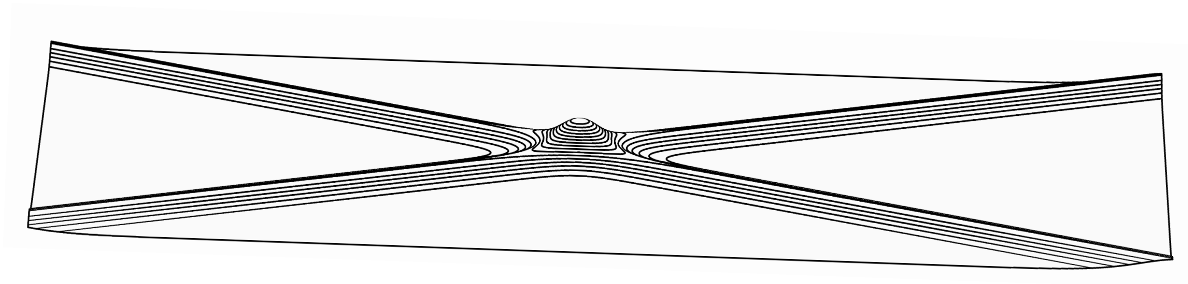
$k_j, P_j, \eta_j^{(0)}$ are constants

KP Equation–line solitons

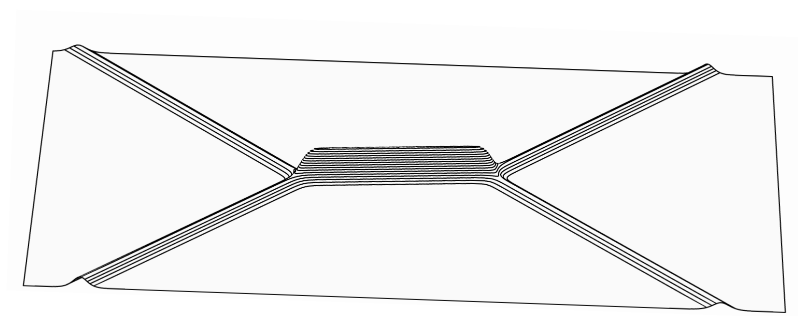
Basic line solitons are solutions of KP (KdV) eq; they are observed routinely: $F_1 = 1 + e^{\eta_1}$



KP Eq: Basic line soliton solutions–IIA

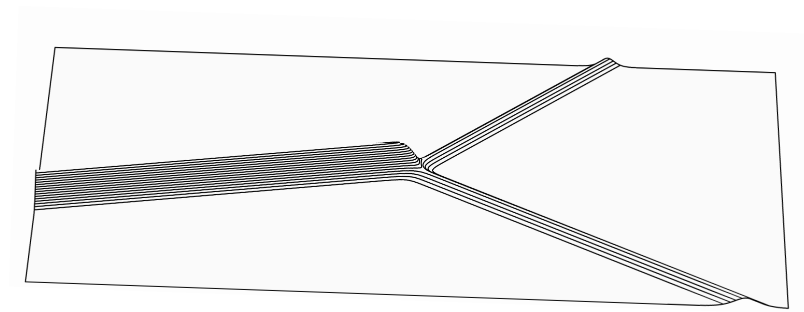


Typical KP two-soliton 'X-type' interaction with 'short stem': F_2, $e^{A_{12}} = O(1)$



Typical KP two-soliton interaction 'X-type' with 'long stem'; F_2, $e^{A_{12}} << 1$

KP Eq: Basic line soliton solutions–IIB



Typical KP 'Y-type' interaction; $F_2,\ e^{A_{12}} \to 0$

Beaches and Line Solitons

Planar waves seen frequently. But what about ‘X’ and ‘Y’ type waves? There was one photo of ‘X-type’ with ‘long stem’: Shallow water waves off the coast of Oregon (MJA & Segur 1981)

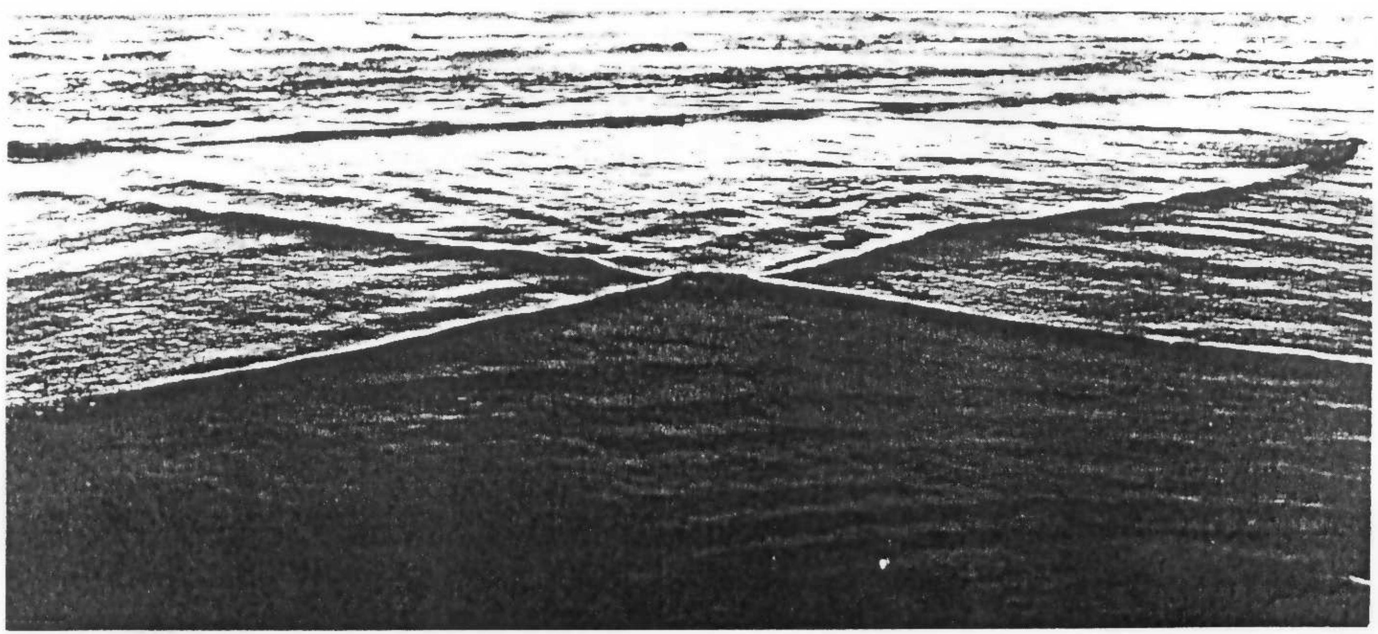


FIG. 4.7b. *Oblique interaction of two shallow water waves.* (*Photograph courtesy of T. Toedtemeier*)

Recent Beach Photos–I

X wave with short stem

Recent Beach Photos–IA

Another X wave with short stem

Recent Beach Photos–II

Depth of the shallow water waves can be understood by noting the person walking on the beach–not noticing a nearby an X interaction!

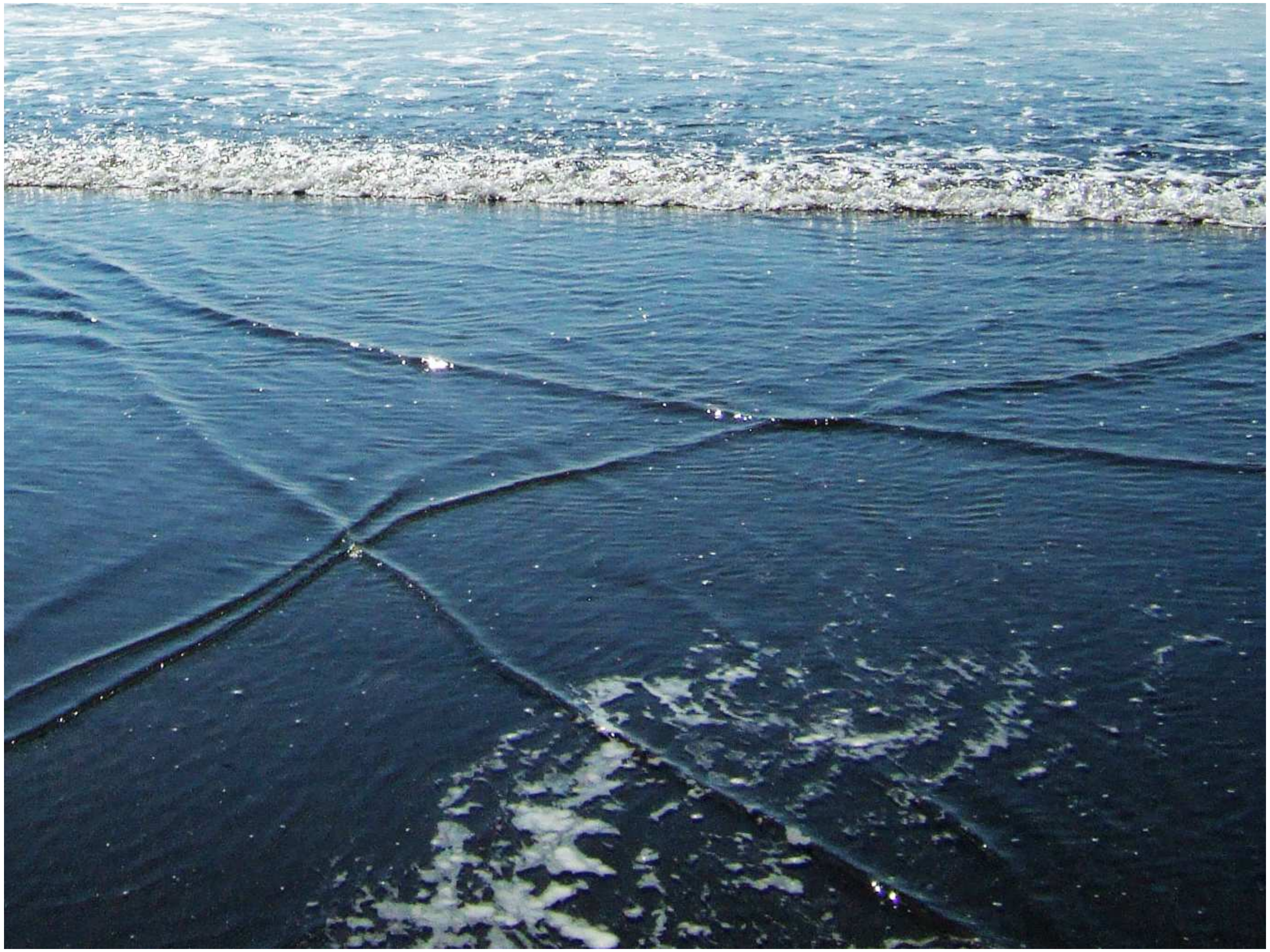
Recent Beach Photos–IIA

Double X short stem

Recent Beach Photos–III-A

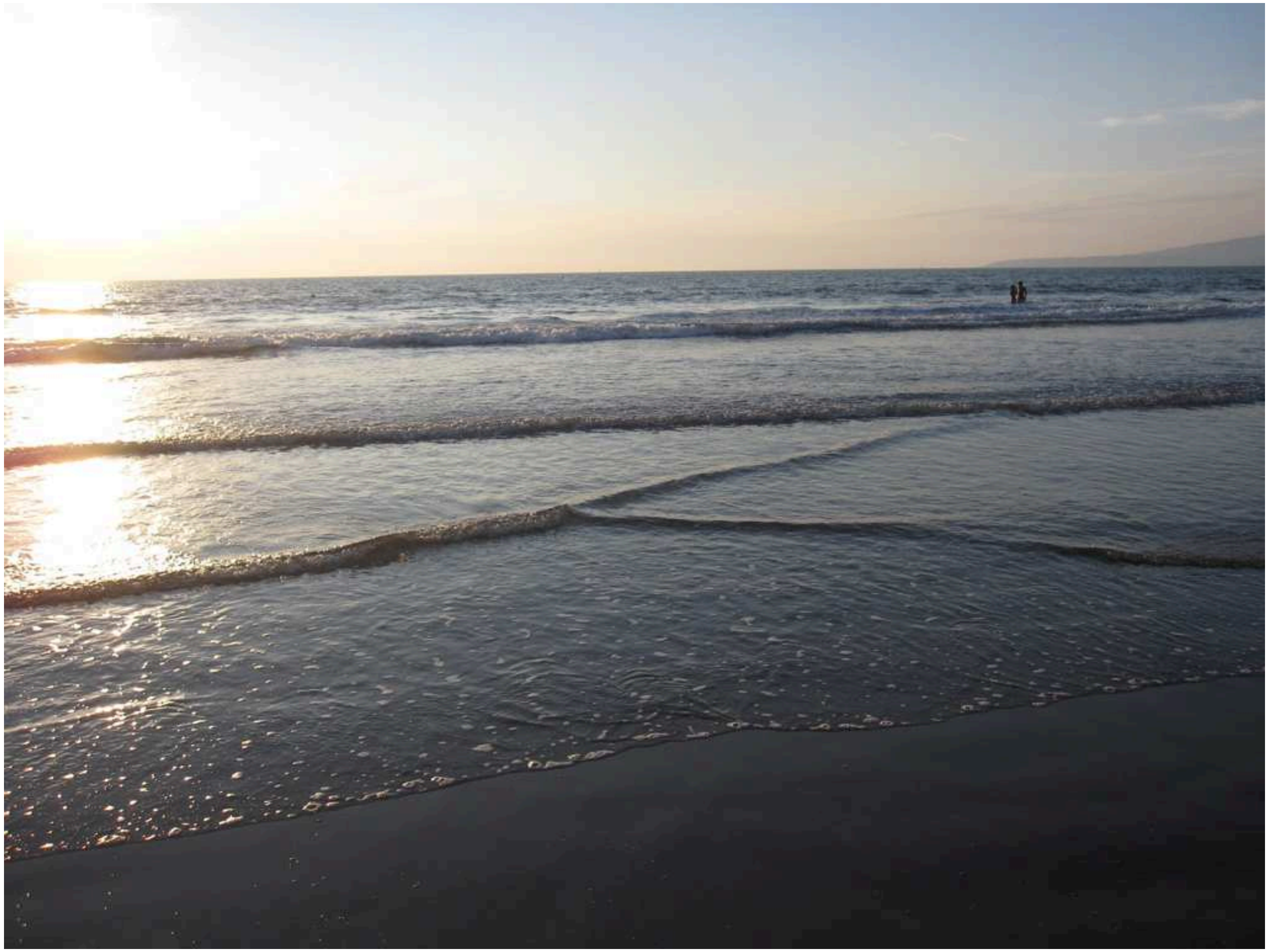
Long stem X

Recent Beach Photos–III-B



Long stem connected to a nearby interaction

Recent Beach Photos–IV



Strong Y interaction

Recent Beach Photos–V

Mutli-interaction

Recent Beach Photos–VI

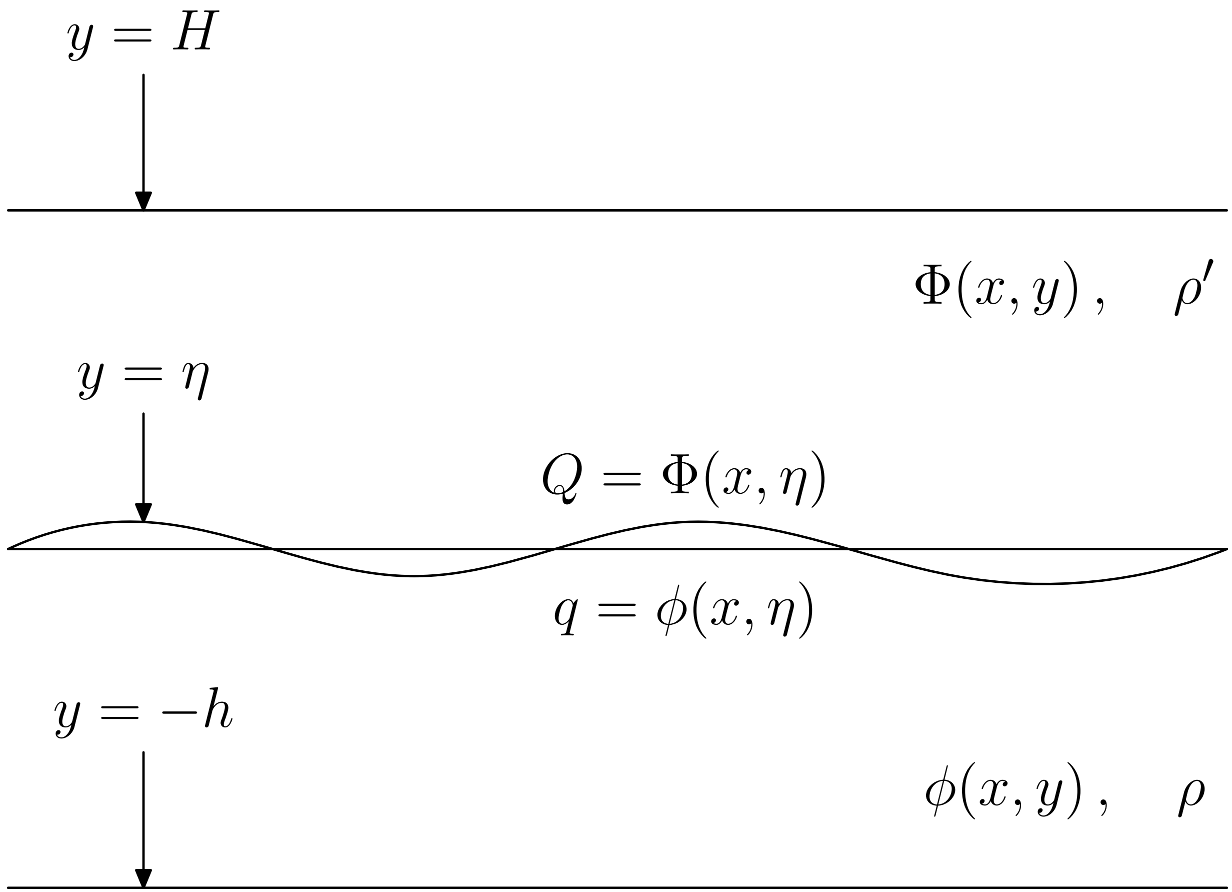
Mutli-interaction: ‘Triangle’

Recent Beach Photos–VI

Mutli-interaction: ‘Triangle’ — with color

Recent research on KP line ‘web’ structures...

Interfacial wave–rigid top (RT)



Interfacial wave-RT–nonlocal form

$$\int_{\mathbf{R}^2} e^{ikx} \cosh(\kappa(\eta + h))\eta_t \, dx = i \int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa(\eta + h)) \left(\frac{k}{\kappa} \cdot \nabla q \right) dx$$

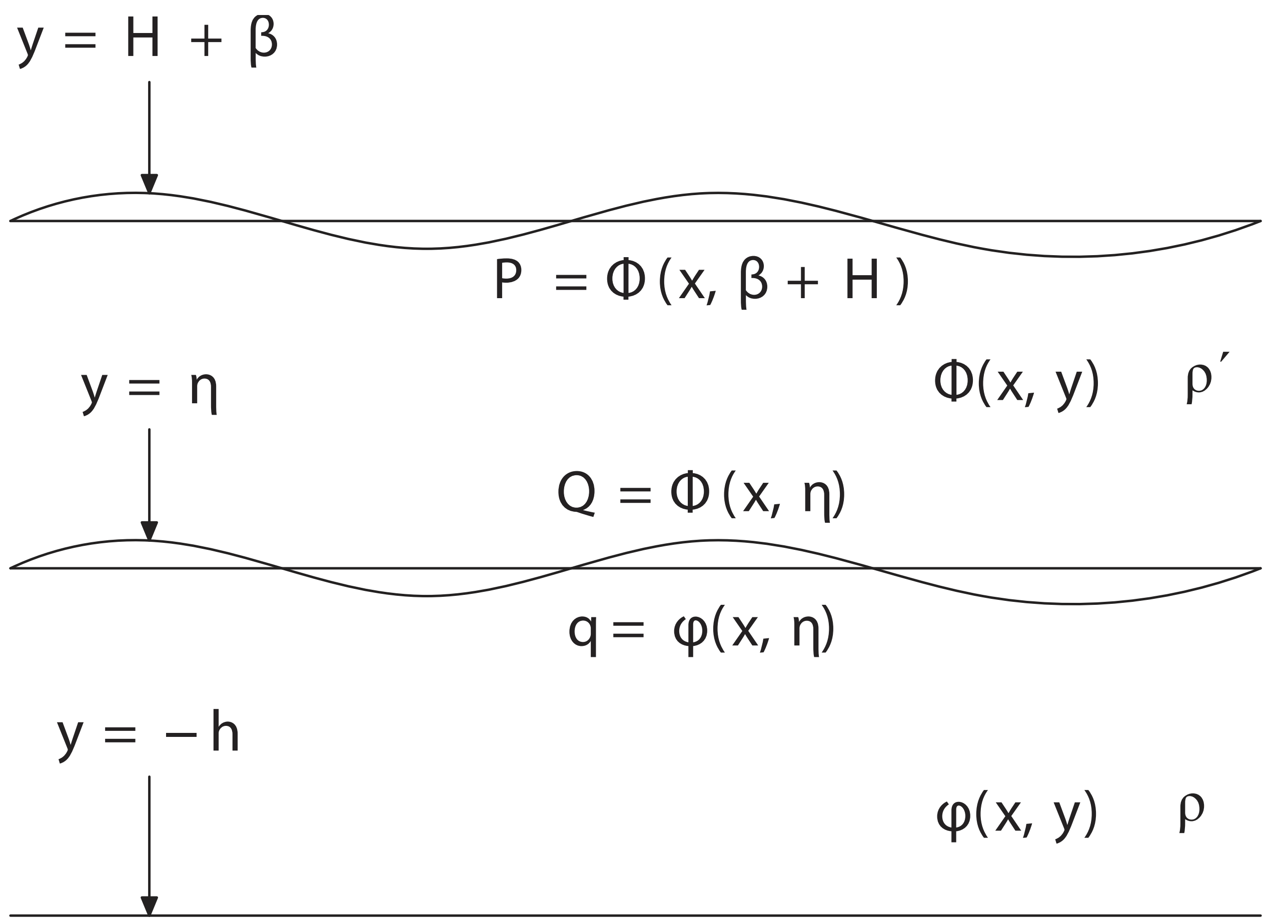
$$\int_{\mathbf{R}^2} e^{ikx} \cosh(\kappa(\eta - H))\eta_t \, dx = i \int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa(\eta - H)) \left(\frac{k}{\kappa} \cdot \nabla Q \right) dx$$

$$\rho \left(q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2\,(1 + |\nabla \eta|^2)} \right) -$$

$$\rho' \left(Q_t + \frac{1}{2}|\nabla Q|^2 + g\eta - \frac{(\eta_t + \nabla Q \cdot \nabla \eta)^2}{2\,(1 + |\nabla \eta|^2)} \right) = \sigma \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right)$$

3 eq., 3 unkowns η, q, Q: fixed domain!
May derive DN operator, and asymptotic reductions AND system with free top surface and free interface

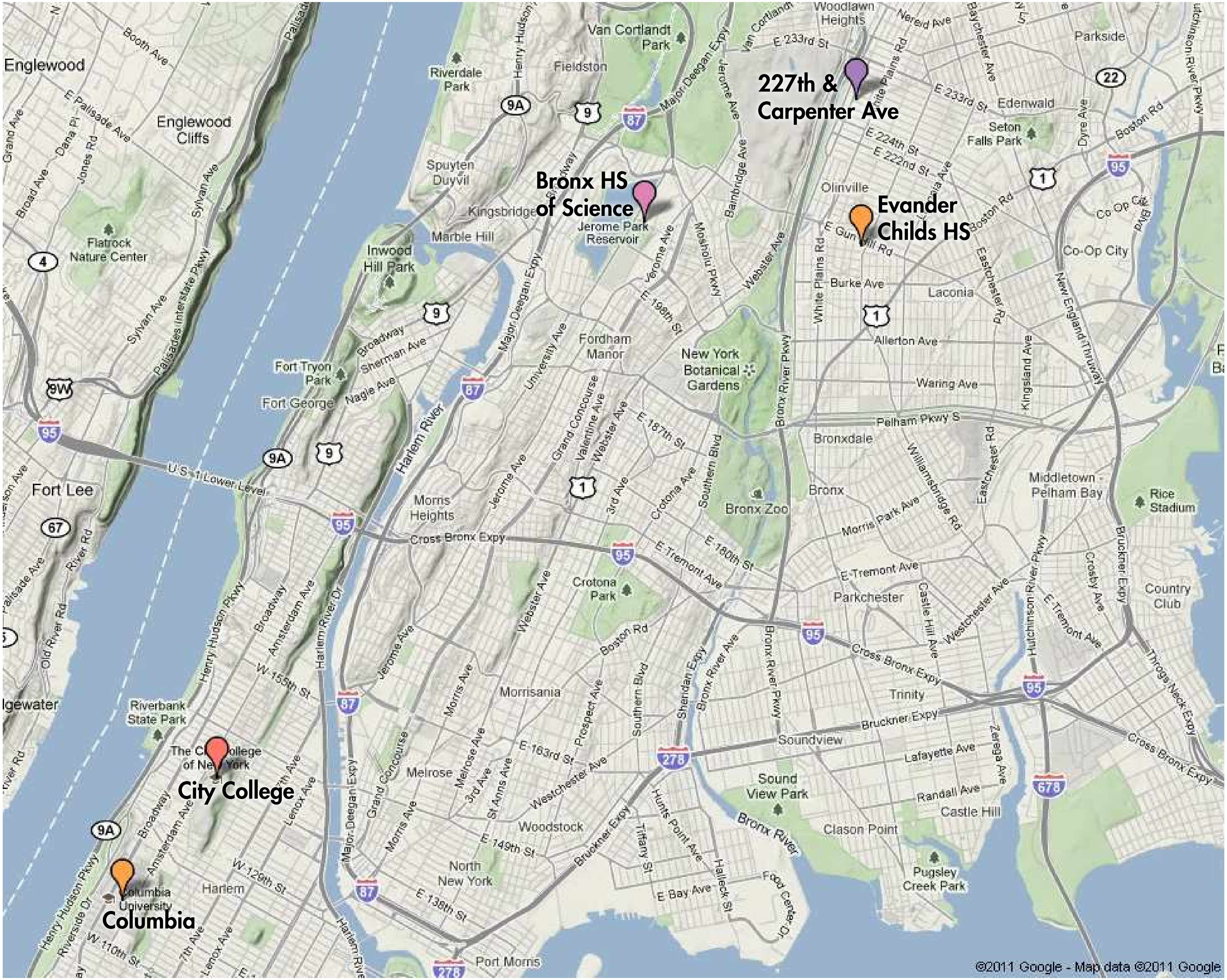
Interfacial wave and free surface(FS)



Two-fluids with two free interfaces: 5 Eq. 5 Unk

Conclusion-WW

- May reformulate water wave equations as a nonlocal spectral system
- Asymptotic systems: shallow water find: BL,KP eq.; deep water: NLS...
- KPI has lump sol’ns; KPII has line soliton sol’ns; Physical realization –long flat beaches..
- Can extend theory to interfacial flows in multiple fluids


227th &
Carpenter Ave
Bronx HS
of Science
Evander
Childs HS
City College
Columbia
Englewood
Englewood Cliffs
Van Cortlandt Park
Riverdale Park
Fieldston
Spuyten Duyvil
Kingsbridge
Marble Hill
Jerome Park Reservoir
Woodlawn Heights
Parkside
Edenwald
Seton Falls Park
Olinville
Co-Op City
Laconia
Flatrock Nature Center
Inwood Hill Park
Fort Tryon Park
Fort George
Fordham Manor
New York Botanical Gardens
Bronxdale
Fort Lee
Morris Heights
Bronx
Bronx Zoo
Middletown - Pelham Bay
Rice Stadium
Crotona Park
Parkchester
Country Club
Trinity
Morrisania
Riverbank State Park
The City College of New York
Soundview
Melrose
Sound View Park
Castle Hill
Woodstock
Clason Point
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Columbia University
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