Moment map and Bethe Ansatz in the Jaynes-Cummings Model.

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Plan of the talk

- 1– The Jaynes-Cummings n-spins Model, Lax matrix, spectral curve.
- 2– Moment map.
- 3– Rank zero, Normal forms, Classical Bethe Ansatz.
- 4– Rank > 0, Spectral curve.
- 5– Examples: One spin, Two spins.
- 6– Quantum model, monodromy, Bethe roots.
- 7– Conclusions.

The Jaynes-Cummings n-spin model.

We consider a system of n spins and a harmonic oscillator with Hamiltonian [Jaynes-Cummings (1963), Gaudin (1982), Yurbashyan, Kuznetsov, Altshuler (2005)....]:

$$H = \sum_{j=1}^{n} 2\epsilon_j s_j^z + \omega \bar{b} b + g \sum_{j=1}^{n} \left(\bar{b} s_j^- + b s_j^+ \right)$$

with Poisson brackets

$$\{b, \bar{b}\} = i,$$

$$\{s_j^a, s_j^b\} = -\epsilon_{abc} s_j^c, \quad \vec{s_j}^2 = s^2$$

This is a celebrated model in quantum optics, cold atoms....

Phase space has dimension $2(n+1)$.

Lax matrix.

We can write these equations in the Lax form $\dot{L} = [L, M]$.

$$\begin{aligned} L(\lambda) &= 2\lambda\sigma^z + 2(b\sigma^+ + \bar{b}\sigma^-) + \sum_{j=1}^{n} \frac{s_j}{\lambda - \epsilon_j} \\ M(\lambda) &= -i(\lambda + \tfrac{\omega}{2})\sigma^z - i(b\sigma^+ + \bar{b}\sigma^-) \end{aligned}$$

Letting

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$$

we have

$$A(\lambda) = 2\lambda + \sum_{j=1}^{n} \frac{s_j^z}{\lambda - \epsilon_j}$$

$$B(\lambda) = 2b + \sum_{j=1}^{n} \frac{s_j^-}{\lambda - \epsilon_j}, \quad C(\lambda) = 2\bar{b} + \sum_{j=1}^{n} \frac{s_j^+}{\lambda - \epsilon_j}$$

Spectral curve

The consequence of this equation is that the spectral curve

$$\Gamma(\lambda, \mu) \equiv \det(L(\lambda) - \mu) = 0$$

is independent of time. Since $L(\lambda)$ is traceless, it reads

$$\mu^2 = A^2(\lambda) + B(\lambda)C(\lambda) \equiv \frac{Q_{2n+2}(\lambda)}{\prod_{j=1}^{n}(\lambda - \epsilon_j)^2}$$

$$\frac{Q_{2n+2}(\lambda)}{\prod_{j=1}^{n}(\lambda - \epsilon_j)^2} = 4\lambda^2 + 4H_{n+1} + 2\sum_{j=1}^{n} \frac{H_j}{\lambda - \epsilon_j} + \sum_{j=1}^{n} \frac{s^2}{(\lambda - \epsilon_j)^2}$$

The genus of the curve is $g = n$. The $(n+1)$ Poisson commuting Hamiltonians are

$$H_{n+1} = b\bar{b} + \sum_{j=1}^{n} s_j^z$$

$$H_j = 2\epsilon_j s_j^z + (b s_j^+ + \bar{b} s_j^-) + \sum_{k \neq j} \frac{s_j \cdot s_k}{\epsilon_j - \epsilon_k}, \quad j = 1, \cdots, n$$

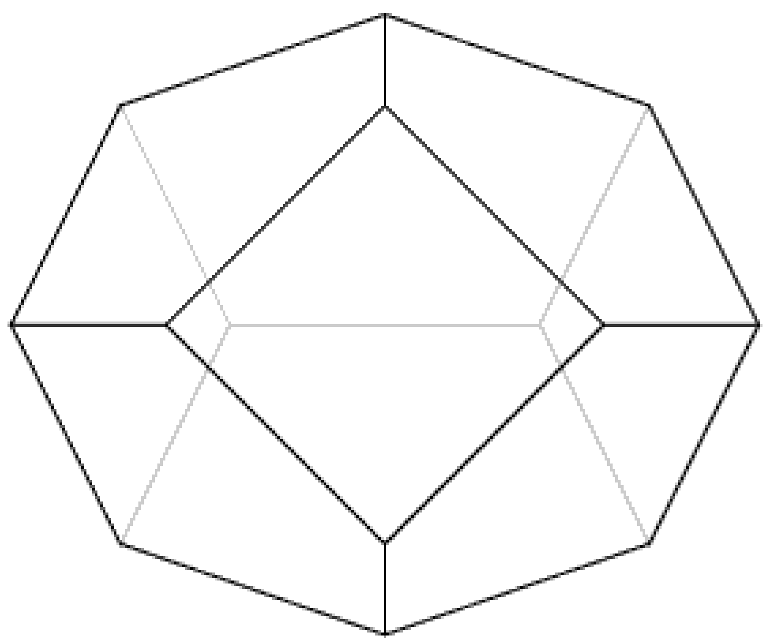
The Hamiltonian of the system is $H = \omega H_n + \sum_j H_j$

The moment map.

The Hamiltonians $H_j, j = 1 \cdots n+1$ define an application from phase space M to R^{n+1}. It is called the moment map.

$$\mathcal{F} : M \to R^{n+1}, \quad x \in M \to (H_1(x), H_2(x), \cdots H_n(x)) \in R^{n+1}$$

Its image is a domain of R^{n+1} which is a very important object. For instance when the H_j define a toric action (all flows are 2π-periodic) on a compact phase space, there is the famous theorem of Atiyah (1982), and Guillemin and Sternberg (1982) stating that the image of the moment map is a convex polytope.



Rank Zero, Normal Forms.

We look for the points such that $\partial_{x_i} H_j = 0$. In our case

$$\partial_b H_j = s_j^+, \quad \partial_{\bar{b}} H_n = b$$

so the points of rank zero are the points such that

$$P_i(e_1, \cdots, e_n) : s_j^z = s e_j, \quad e_j = \pm 1, \quad s_j^\pm = 0, \quad b = \bar{b} = 0$$

Hence we have 2^n points of rank zero.
To analyse the system around these points we have to expand the Hamiltonians H_j to second order.

$$H_j = H_j(0) + \sum \frac{\partial H_j}{\partial x_k \partial x_l}(0) x^k x^l + \cdots$$

Normal forms are obtained by the simultaneous “diagonalisation” of these quadratic forms. This is a non trivial problem because the “diagonalisation” has to be done using real symplectic transformations.

The result is Williamson theorem (1936): There exist canonical coordinates $q_1, \cdots q_n, p_1, \cdots, p_n$ such that the above quadratic forms can be decomposed on the following quadratic polynomials

$$
\begin{aligned}
P_i^{elliptic} &= p_i^2 + q_i^2, \quad i = 1, 2, \cdots, m_1 \\
P_i^{hyperbolic} &= p_i q_i, \quad i = m_1 + 1, \cdots m_1 + m_2 \\
P_i^{(1) focus-focus} &= p_i q_i + p_{i+1} q_{i+1}, \quad i = m_1 + m_2 + 1 \cdots \\
P_i^{(2) focus-focus} &= p_i q_{i+1} - p_{i+1} q_i, \quad \cdots m_1 + m_2 + m_3
\end{aligned}
$$

where $m_1 + m_2 + 2m_3 = n$. The triple (m_1, m_2, m_3) is the type of the singular point.

How to achieve this decomposition ? Clearly, theses coordinates also depict the spectrum of the quantum system around the critical points. But to analyse the quantum spectrum, the tool is well known : Bethe Ansatz.

Can we adapt the algebraic Bethe Ansatz technique to compute the normal forms ?

Classical Bethe Ansatz.

Recall the Lax matrix

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$$

We have the Poisson commutation relations

$$\{L_1(\lambda), L_2(\mu)\} = -i \left[\frac{P_{12}}{\lambda - \mu}, L_1(\lambda) + L_2(\mu) \right]$$

or explicitely

$$\begin{aligned} \{A(\lambda), B(\mu)\} &= \frac{i}{\lambda - \mu}(B(\lambda) - B(\mu)) \\ \{A(\lambda), C(\mu)\} &= -\frac{i}{\lambda - \mu}(C(\lambda) - C(\mu)) \\ \{B(\lambda), C(\mu)\} &= \frac{2i}{\lambda - \mu}(A(\lambda) - A(\mu)) \end{aligned}$$

It follows that $\frac{1}{2}\mathrm{Tr}\left(\mathrm{L}^2(\lambda)\right) = \mathrm{A}^2(\lambda) + \mathrm{B}(\lambda)\mathrm{C}(\lambda)$ has the nice commutaion relation

$$\left\{\frac{1}{2}\mathrm{Tr}\; L^2(\lambda), C(\mu)\right\} = \frac{2i}{\lambda-\mu}\Big(A(\lambda)C(\mu) - A(\mu)C(\lambda)\Big)$$

When we expand around a critical configuration, the quantities (b, $\bar{b}$, $s_j^{\pm}$) are first order, but s_j^z is second order because

$$s_j^z = e_j\sqrt{s^2 - s_j^+ s_j^-} = se_j - \frac{e_j}{2s}s_j^+ s_j^- + \cdots, \quad e_j = \pm 1$$

Notice that $C(\mu) = \frac{2\bar{b}}{g} + \sum_{j=0}^{n-1}\frac{s_j^+}{\mu-\epsilon_j}$ is first order while $A(\lambda) = \frac{2\lambda}{g^2} - \frac{\omega}{g^2} + \sum_{j=0}^{n-1}\frac{s_j^z}{\lambda-\epsilon_j}$ is constant plus second order. So in the right-hand side we can replace $A(\lambda)$ and $A(\mu)$ by their zeroth order expression :

$$A(\lambda) \simeq a(\lambda) = 2\lambda + \sum_{j=1}^{n}\frac{se_j}{\lambda-\epsilon_j}$$

We arrive at

$$\left\{\frac{1}{2}\mathrm{Tr}\, L^2(\lambda), C(\mu)\right\} = \frac{2i}{\lambda-\mu}\Big(a(\lambda)C(\mu) - a(\mu)C(\lambda)\Big)$$

This will be precisely of the wanted form if we can kill the unwanted term $C(\lambda)$. This is achieved by imposing the condition ("classical Bethe equation")

$$a(\mu) = 0 \qquad (\star)$$

This is an equation of degree $n+1$ for μ. Let us call μ_i its solutions. Hence we construct in this way $n+1$ variables $C(\mu_i)$. To construct the conjugate variables, we consider commutation relation of $B(\lambda)$ and $C(\mu)$. In our linear approximation it reads

$$\{B(\mu_i), C(\mu_j)\} = \frac{2i}{\mu_i - \mu_j}(a(\mu_i) - a(\mu_j))$$

If μ_i and μ_j are *different* solutions of eq.($\star$), then obviously

$$\{B(\mu_i), C(\mu_j)\} = 0, \quad \mu_i \neq \mu_j$$

If however $\mu_j = \mu_i$ then

$$\{B(\mu_i), C(\mu_i)\} = 2ia'(\mu_i)$$

We have indeed constructed canonical coordinates !

It is simple to express the quadratic Hamiltonians in theses coordinates:

$$\frac{1}{2}\mathrm{Tr}\, L^2(\lambda) = a^2(\lambda) + \sum_i \frac{a(\lambda)}{a'(\mu_i)(\lambda-\mu_i)} B(\mu_i)C(\mu_i)$$

Note that there is no pole at $\lambda = \mu_i$ because $a(\mu_i) = 0$.

If μ_i is real we have $C(\mu_i) = \overline{B(\mu_i)}$. We have an elliptic term.

$$C(\mu_i) \simeq \sqrt{|a'(\mu_i)|}(p_i + iq_i), \quad B(\mu_i) \simeq \sqrt{|a'(\mu_i)|}(p_i - iq_i),$$

$$B(\mu_i)C(\mu_i) \simeq |a'(\mu_i)|(p_i^2 + q_i^2)$$

If $\mu_{i+1} = \bar{\mu}_i$ is a pair of complex conjugate solutions, we have $C(\mu_{i+1}) = \overline{B(\mu_i)}$.

$$C(\mu_i) \simeq (p_i + ip_{i+1}), \quad B(\mu_i) \simeq -ia'(\mu_i)(q_i - iq_{i+1}),$$

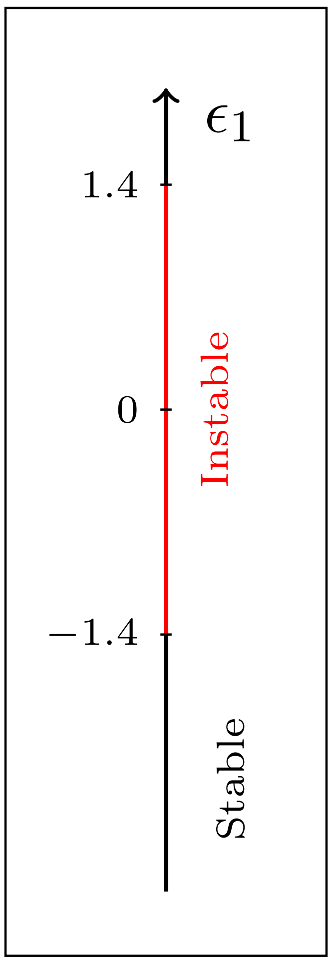
$$C(\mu_{i+1}) \simeq ia'(\mu_{i+1})(q_i + iq_{i+1}), \quad B(\mu_{i+1}) \simeq (p_i - ip_{i+1})$$

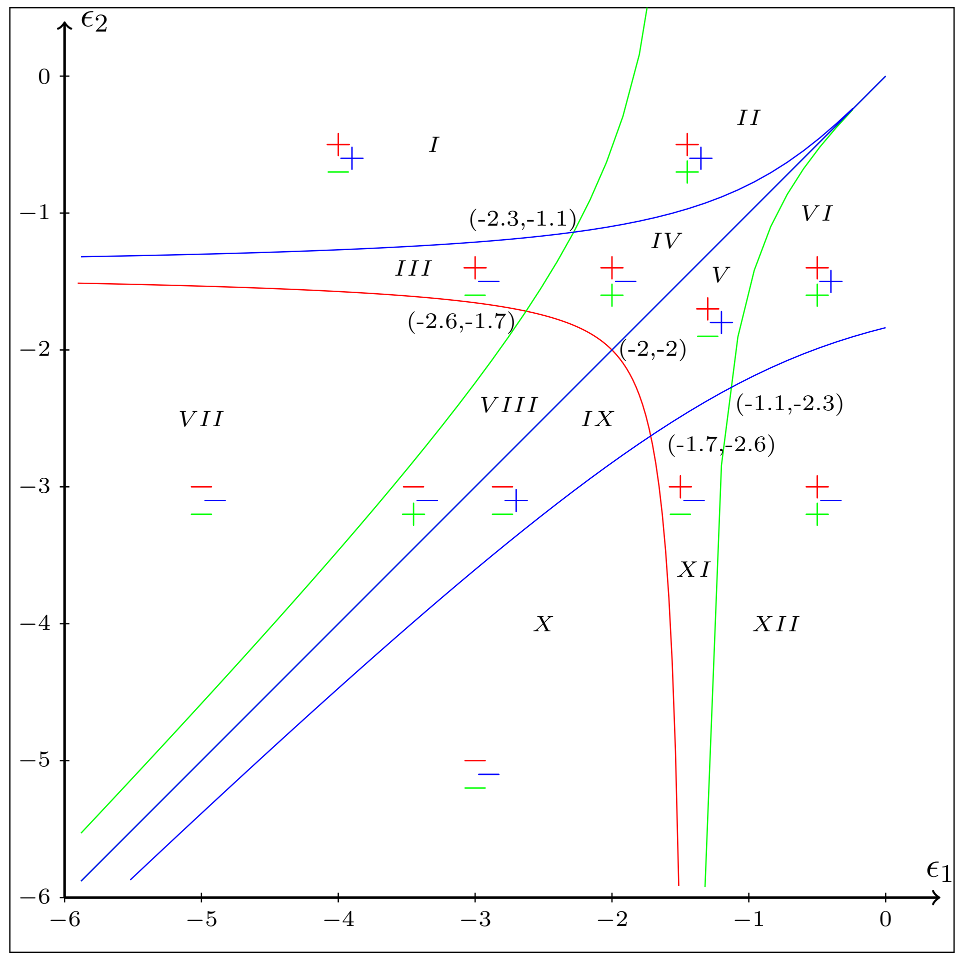
$$\mathrm{Re}\left(\frac{C(\mu_i)B(\mu_i)}{ia'(\mu_i)}\right) \simeq p_i q_i + p_{i+1} q_{i+1}, \quad \mathrm{Im}\left(\frac{C(\mu_i)B(\mu_i)}{ia'(\mu_i)}\right) \simeq p_i q_{i+1} - p_{i+1} q_i$$

This is a focus-focus term.

So in order to compute the type of the singularity we have to study the Classical Bethe equation

$$\mu_i = -\frac{s}{2}\sum_{j=1}^{n}\frac{e_j}{\mu_i - \epsilon_j}, \qquad (\star\star)$$





Rank > 0, Spectral curve.

The analysis of the other strata of the bifurcation diagram become rapidly very cumbersome. However it was remarked by Michèle Audin (1996) that all this was encoded into the degeneracies of the spectral curve. Let me explain why.

The spectral curve reads $\det(L(\lambda) - \mu) = 0$ or

$$\mu^2 = \frac{Q_{2n+2}(\lambda)}{\prod_{j=1}^n (\lambda - \epsilon_j)^2} = 4\lambda^2 + 0\lambda + 4H_{n+1} + 2\sum_{j=1}^n \frac{H_j}{\lambda - \epsilon_j} + \sum_{j=1}^n \frac{s^2}{(\lambda - \epsilon_j)^2}$$

hence $Q_{2n+2}(\lambda)$ is a polynomial of degree $2n+2$ subjected to $n+2$ constraints

Defining $y = \mu \prod_j (\lambda - \epsilon_j)$, the equation of the curve becomes

$$y^2 = Q_{2n+2}(\lambda)$$

It is a fundamental fact that one can reconstruct everything from the data of the spectral curve and g points on it (separated variables). If we call λ_k the coordinates of these points the equations of motion for the flow generated by H_j take the form (in our case)

$$\sum_k \partial_{t_j} \lambda_k \, \omega_j(\lambda_k) = -i\delta_{ij}$$

where $\omega_j(\lambda)$ are the g holomorphic differentials on Γ. For generic points λ_k the matrix $\omega_j(\lambda_k)$ is invertible. Hence so is the matrix $\partial_{t_j} \lambda_k$. This means that the flows ∂_{t_j} are independent and therefore the moment map has maximal rank as long as the curve is non degenerate. The curve degenerates when $Q_{2n+2}(\lambda)$ has a double zero

$$Q_{2n+2}(\lambda) = (\lambda - E)^2 \tilde{Q}_{2n}(\lambda)$$

Repeating the process of adding a double zero we construct the different strata of the bifurcation diagram.

For instance, we can repeat the process until $Q_{2n+2}(\lambda)$ is a perfect square

$$Q_{2n+2}(\lambda) = \left(\sum_{j=0}^{n+1} a_j \lambda^j \right)^2$$

we have $n+2$ coefficients a_j but we have $n+2$ constraints on $Q_{2n+2}(\lambda)$ hence they are completely determined. This is an easy calculation. We find

$$\frac{Q_{2n+2}(\lambda)}{\prod_j (\lambda - \epsilon_j)^2} = \left(2\lambda + \sum_{j=1}^{n} \frac{s e_j}{\lambda - \epsilon_j} \right)^2 = a^2(\lambda)$$

We recover the rank zero critical points. More generally the degeneracies we look at are of the form

$$Q_{2n+2}(\lambda) = \left(\sum_{i=0}^{n+1-r} a_i \lambda^i \right)^2 \left(\sum_{j=0}^{2r} b_j \lambda^j \right), \quad a_{n+1-r} = 1$$

We have $(n+1-r) + 2r + 1 = n + r + 2$ coefficients on which we impose $n+2$ constraints. Hence the leaf of rank r is of dimension r. We remark that the conditions are linear equations on the b_j so that we always start by solving them. If $2r > n+1$ it remains $2r - n - 1$ free coefficients b_j

Examples: One spin...

In this case the polynomial $Q_{2n+2}(\lambda)$ reads :

$$\frac{Q_4(\lambda)}{(\lambda-\epsilon_1)^2} = 4\lambda^2 + 4H_2 + \frac{2H_1}{\lambda-\epsilon_1} + \frac{s^2}{(\lambda-\epsilon_1)^2}$$

The most degenerate case is when $Q_4(\lambda)$ is a perfect square. Next we assume that

$$Q_4(\lambda) = (\lambda - \epsilon_1 + \tfrac{x}{2})^2(b_2\lambda^2 + b_1\lambda + b_0), \quad b_2 \neq 0$$

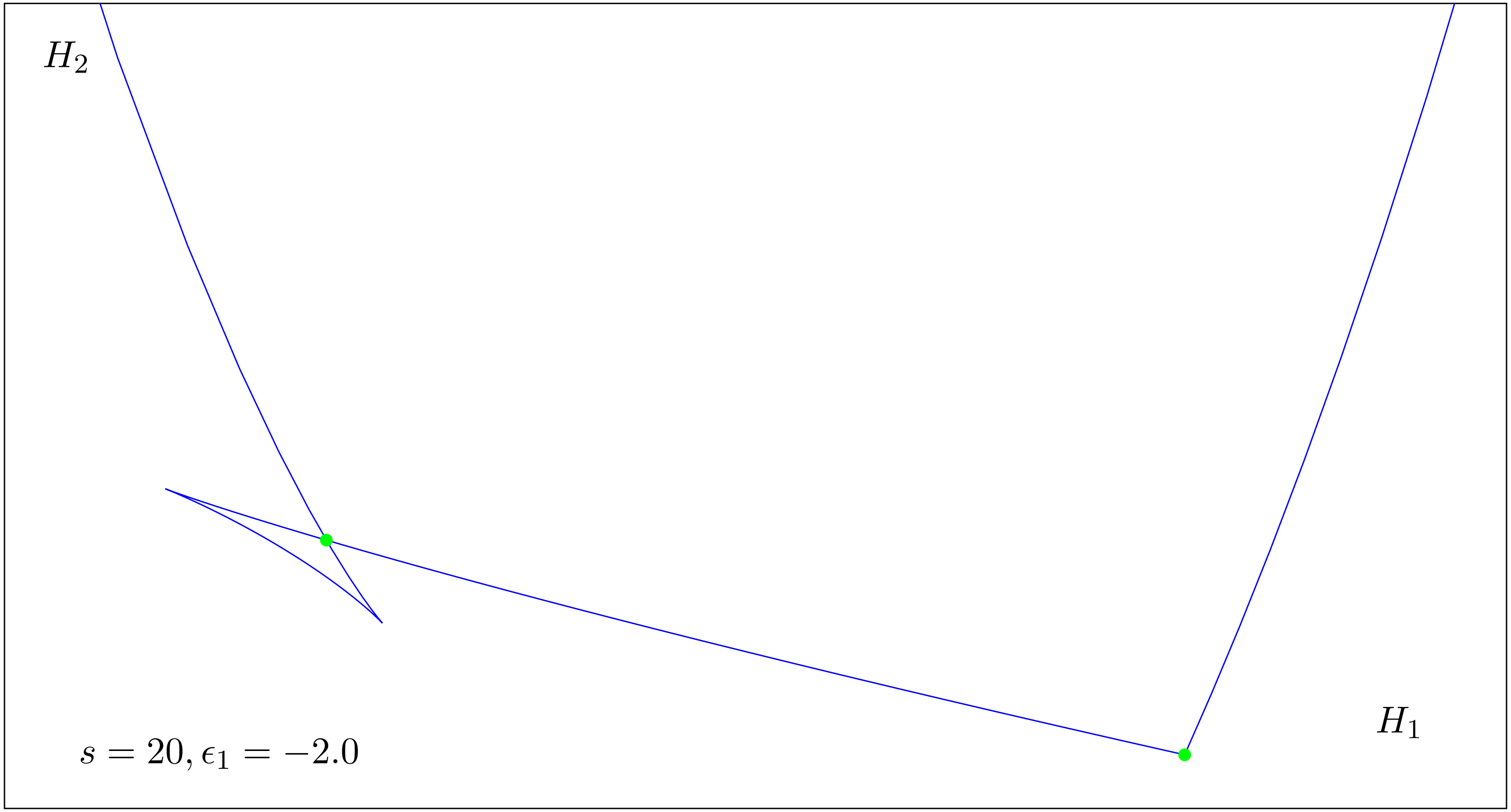
We impose the three constraints on $Q_4(\lambda)$ and we find

$$b_2 = 4, \quad b_1 = -4x, \quad b_0 = -4\left(\epsilon_1^2 - x\epsilon_1 - \frac{s^2}{x^2}\right)$$

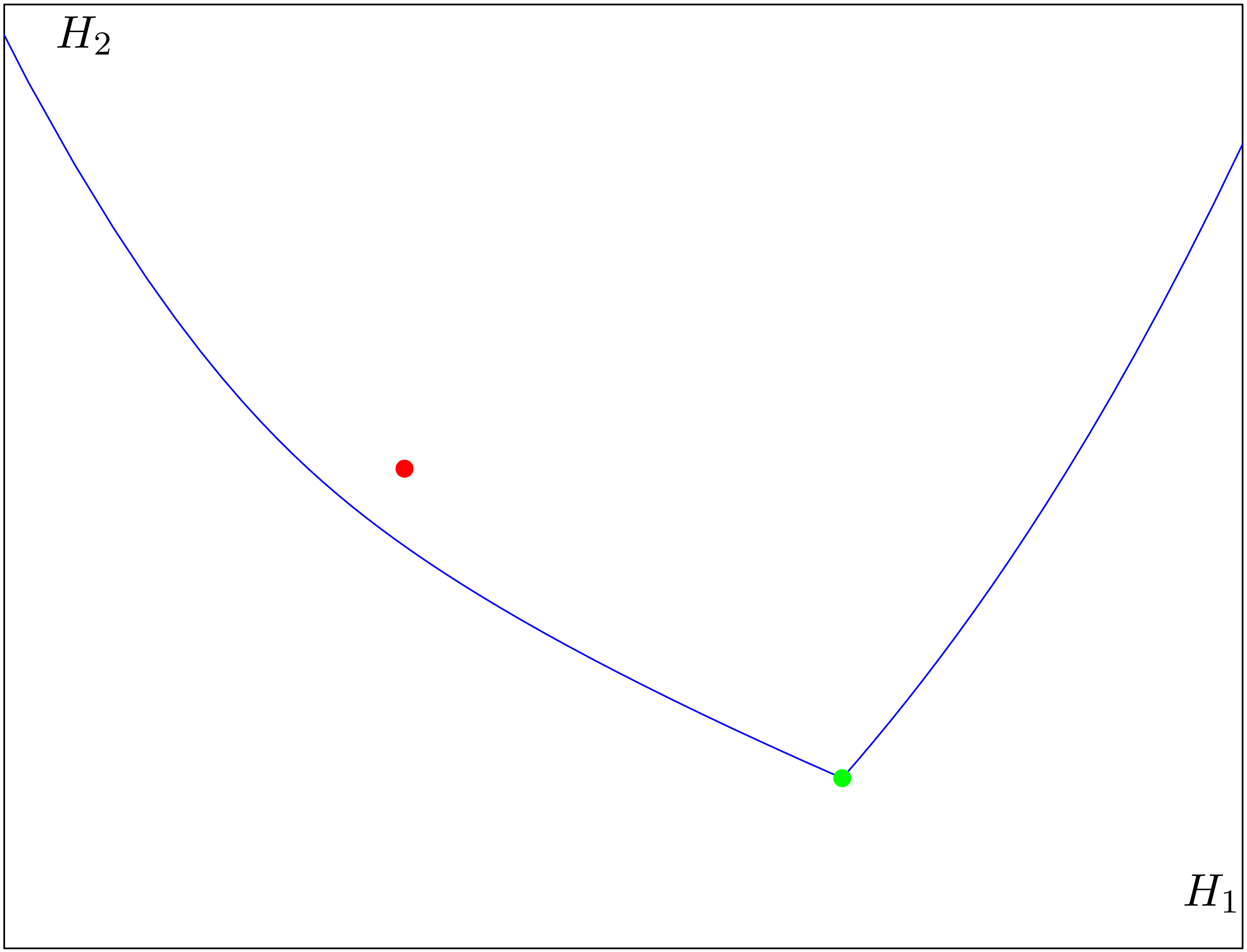
The values of the Hamiltonians are

$$H_1 = -\frac{x^4 - 2\,\epsilon_1\,x^3 - 4\,s^2}{2\,x}, \quad H_2 = -\frac{3\,x^4 - 8\,\epsilon_1\,x^3 + 4\,\epsilon_1^2\,x^2 - 4\,s^2}{4\,x^2}$$

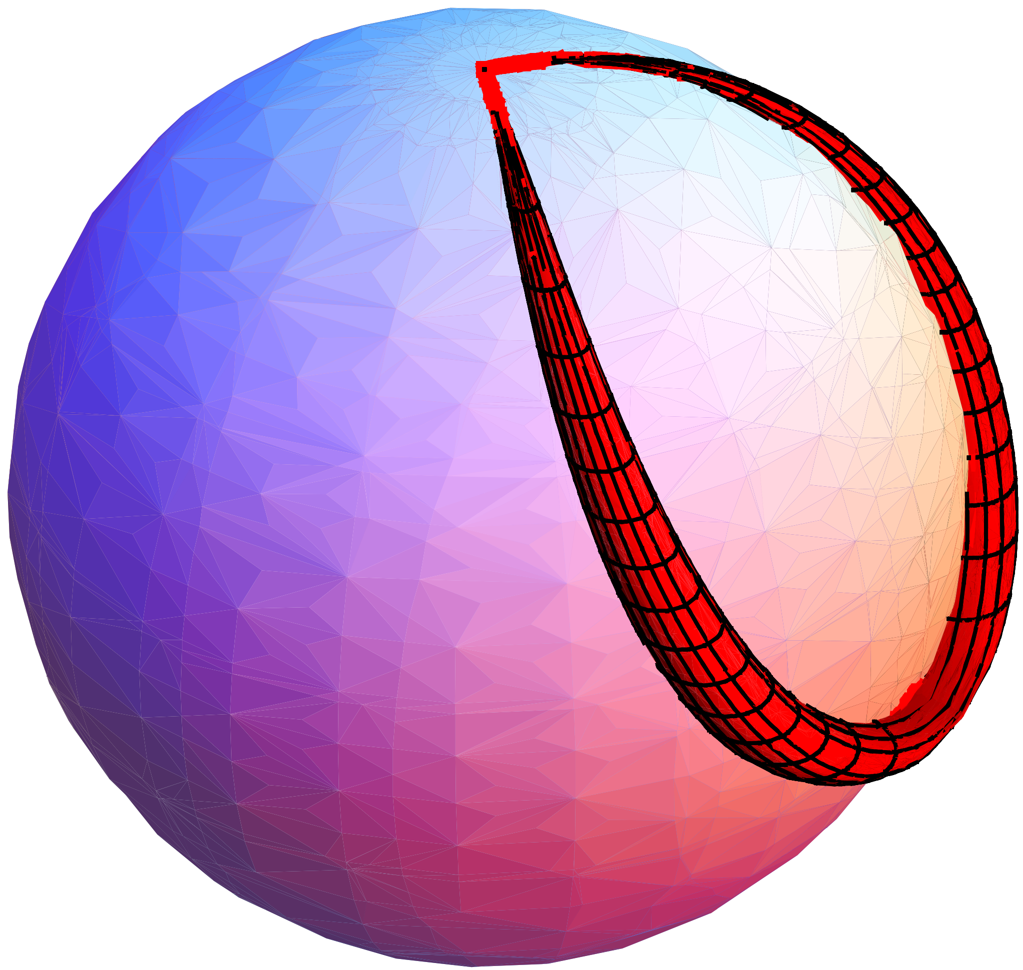
One spin. Two stable points (●)



One spin. One stable point (●) and one unstable point (●)

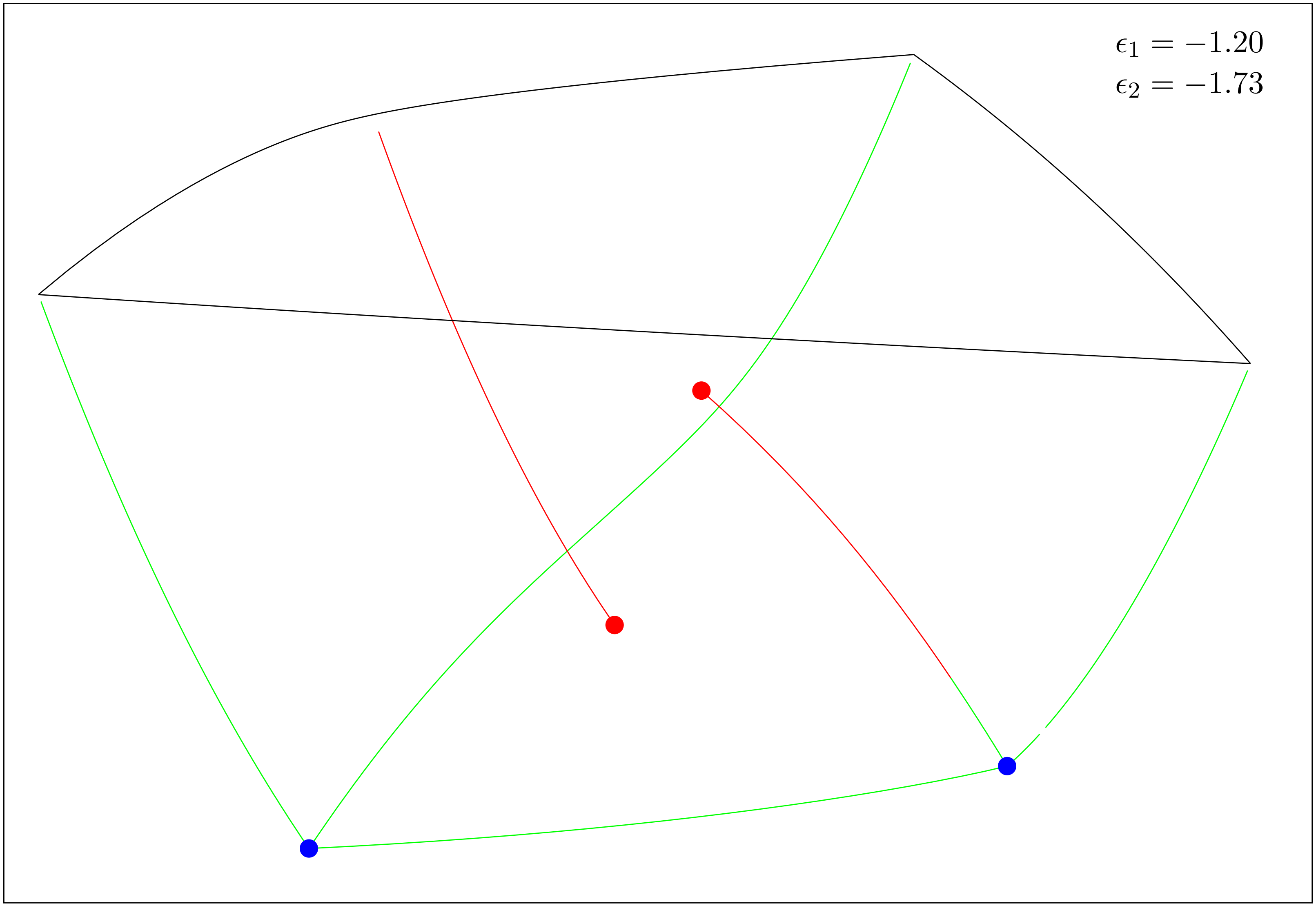


The preimage of the unstable (focus-focus) point is a pinched torus.



These pinched tori are obstructions to the existence of global action-angle variables Duistermaat (1980).

Two spins. Two stable points (●) and two unstable points (●).



The Quantum model.

As before the Hamiltonian reads

$$H = \sum_{j=1}^{n} 2\epsilon_j s_j^z + \omega \bar{b} b + g \sum_{j=1}^{n} \left(\bar{b} s_j^- + b s_j^+ \right)$$

with commutation relations $[b, \bar{b}] = \hbar, \quad [s_j^a, s_j^b] = i\hbar \, \epsilon_{abc} \, s_j^c$ We consider spin s representations. The semi-classical limit is defined as

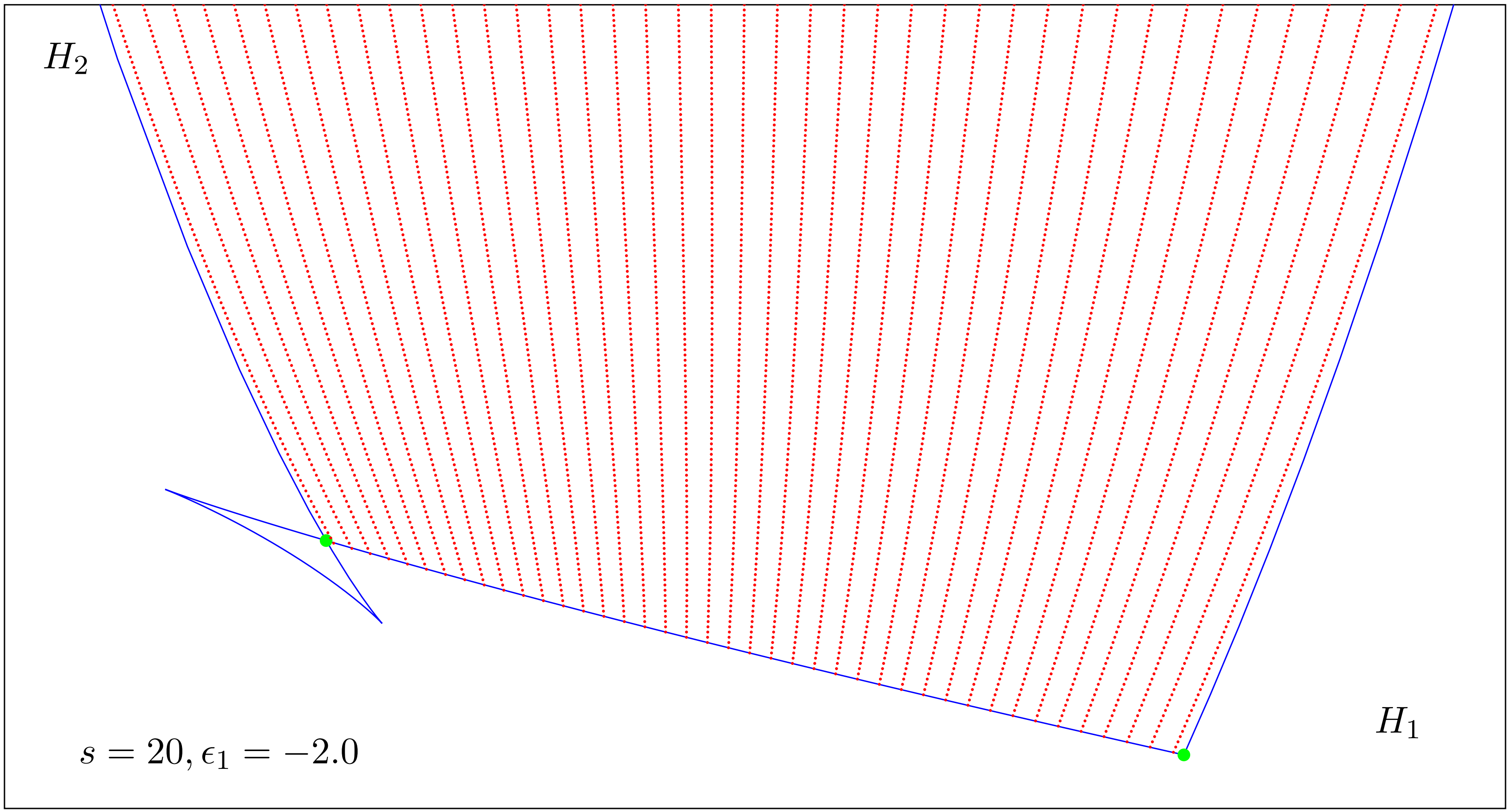
$$\hbar^2 s(s+1) = 1, \quad \hbar \to 0$$

In the one spin case we have two commuting Hamiltonians, H_1, H_2

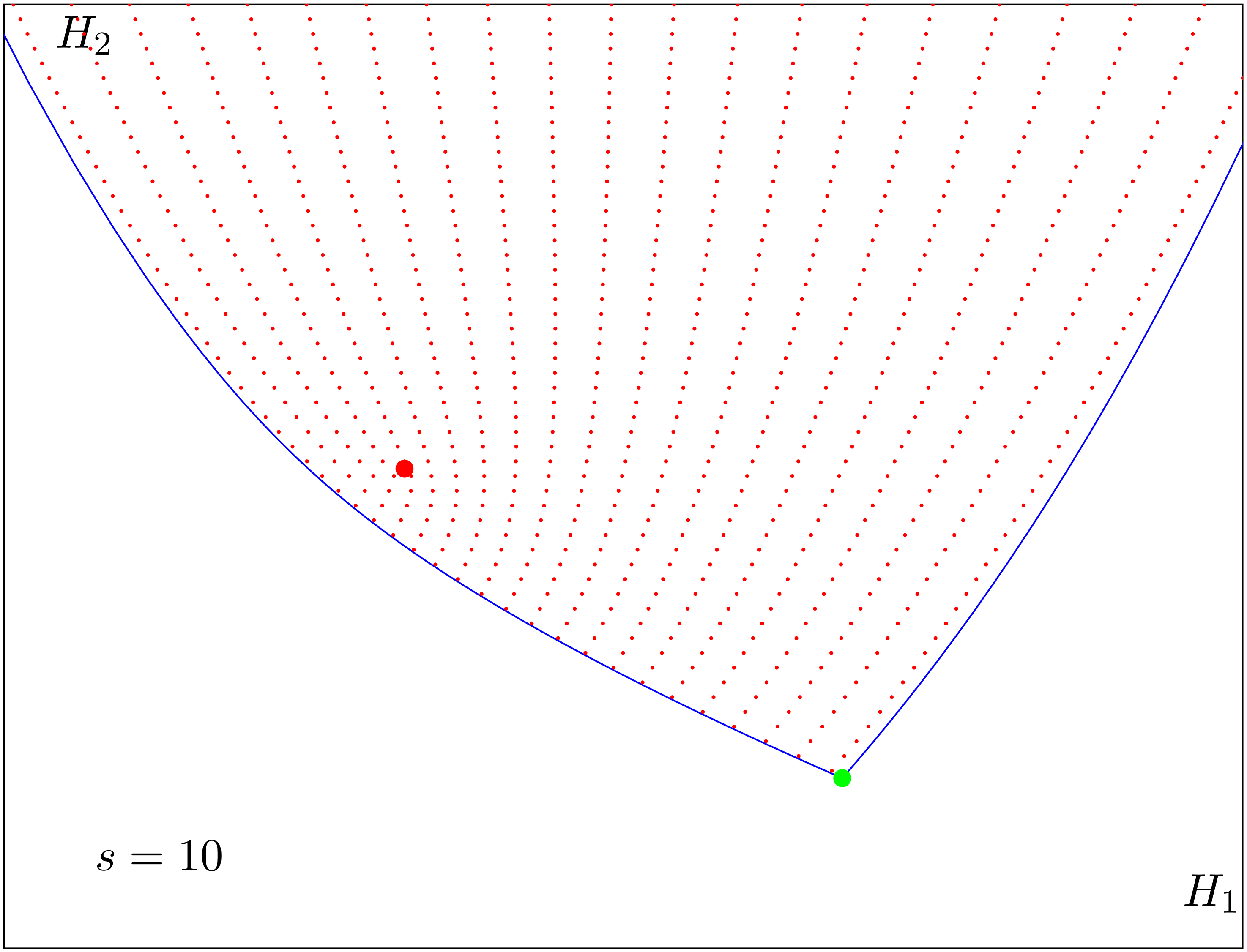
$$H_2 = s_1^z + b^\dagger b = -s + \hbar M, \quad M \quad integer$$

On the subspace M fixed, H_1 can be written as a Jacobi matrix and is easy to diagonalize numerically.

One spin. Two stable points (●)



One spin. One stable point (●) and one unstable point (●)



The moment map: $(p_i, q_i) \to (H_1, \cdots, H_n)$ induces a fibration of phase space by tori. Take a cycle basis $\gamma_1, \cdots, \gamma_n$ for the torus above $(H_1, \cdots, H_n)$. After a closed loop in the $(H_1, \cdots, H_n)$ space

$$\gamma_i \to \gamma_i' = \sum_{j=1}^{n} \mathcal{M}_{ji} \gamma_i$$

The local action variables $J_i = \frac{1}{2\pi} \oint_{\gamma_i} \sum_\alpha p_\alpha dq_\alpha$ transform as

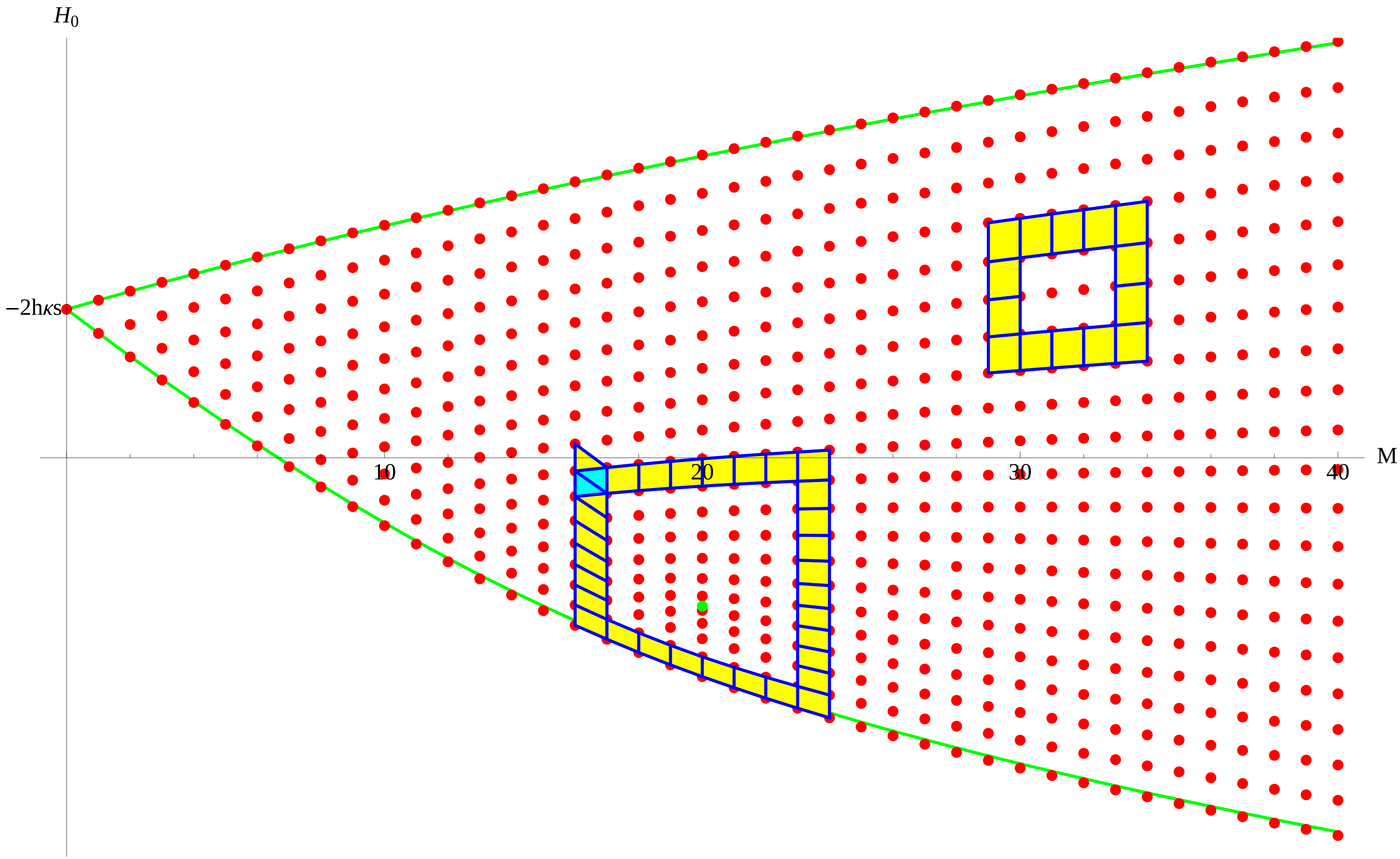
$$J_i \to J_i' = \sum_{j=1}^{n} \mathcal{M}_{ji} J_i$$

Bohr Sommerfeld quantization condition: $J_i = \hbar n_i$. $\delta H_j = \frac{\partial H_j}{\partial J_i} \delta(\hbar n_i)$

$$\vec{e}_i = \frac{\partial \vec{H}}{\partial J_i}, \quad \vec{e}_i\,' = \frac{\partial \vec{H}}{\partial J_i'} = (\mathcal{M}^{-1})_{ij} \vec{e}_j$$

So we can read the monodromy matrix on the lattice of joint spectrum. San Vu Ngoc (1999).

The result is summarized on the following picture :



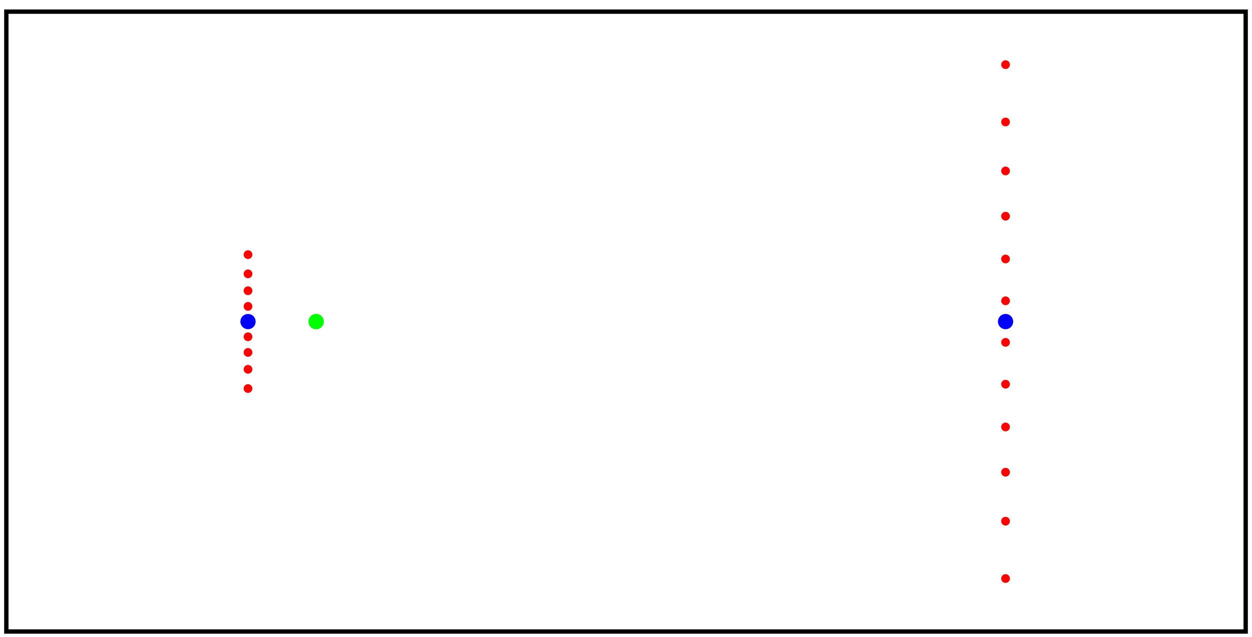
Bethe Ansatz.

Classical Bethe Ansatz suggets Fock space quantization (elliptic case)

$$|\Psi\rangle = C(\mu_1^{cl})^{m_1} \cdots C(\mu_{n+1}^{cl})^{m_{n+1}}|0\rangle, \quad a(\mu_i^{cl}) = 0$$

Quantum Bethe Ansatz

$$|\Psi\rangle = C(\mu_1) \cdots C(\mu_M)|0\rangle, \quad a(\mu_i) = \sum_{j=1}^{M} \frac{\hbar}{\mu_i - \mu_j}$$

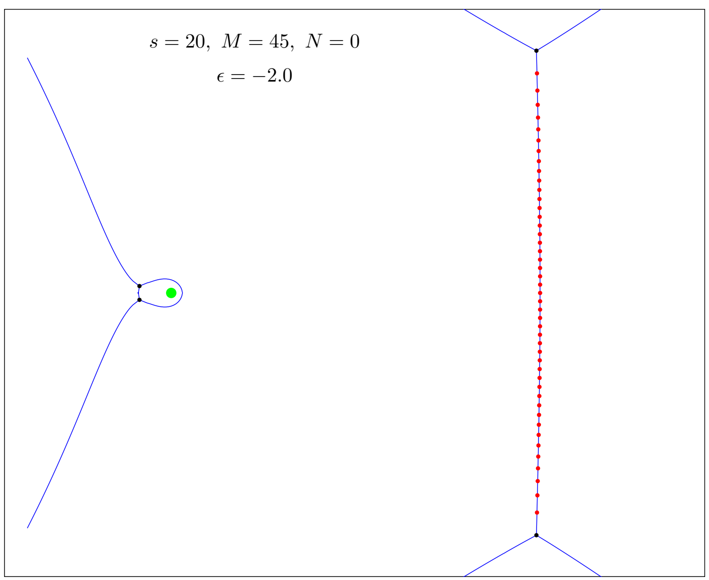


Bethe roots.

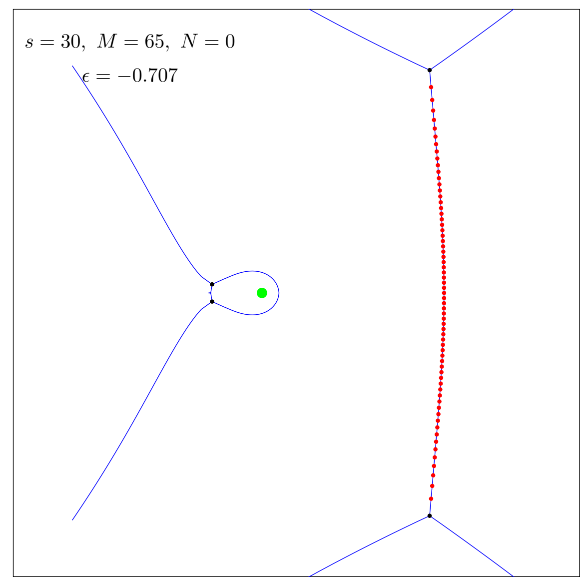
Bethe roots are located (semiclassically) on the curves

$$\frac{d\mu}{dt} = -\frac{i\pi}{\sqrt{\Lambda(\mu)}}, \quad \Lambda(\mu) = \frac{Q_{2n+2}(\mu)}{\prod(\mu - \epsilon_j)^2}$$

Notice that around a branch point $\mu - \mu_b \simeq at^{2/3}$ so that we have three branches.

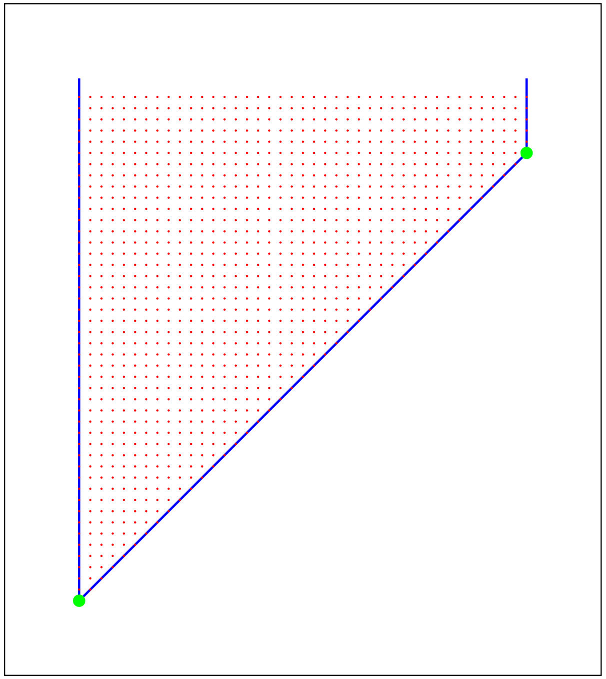


Stable case

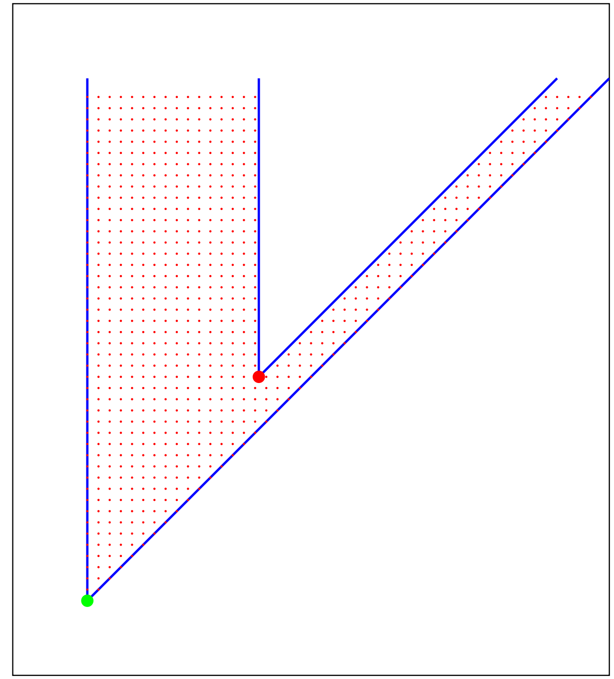


Unstable case

Polytopes.



Stable case



Unstable case

The quantization of the system around such a singularity is a non trivial problem. In particular, in the semi classical regime, the Bohr-Sommerfeld quantisation relations have to be modified (Colin de Verdière, San Vu Ngoc). By studying the Schroedinger equation around the singularity and gluing this “small x” analysis to the WKB wave function, we find the quantization condition:

$$\Phi_{Sing}(\epsilon_n) = 2\pi\hbar\left(n + \frac{1}{2}\right), \quad n \in Z, \quad E_n = 2\kappa s_{cl} + \hbar\epsilon_n$$

where: ($\Omega = \sqrt{2s_{cl} - \kappa^2}$)

$$\Phi_{Sing}(\epsilon) = 2(2s+1)\hbar\nu + 2\kappa\Omega - i\hbar \log \frac{\Gamma\left(\frac{1}{2} - i\frac{\epsilon-\kappa}{2\Omega}\right)}{\Gamma\left(\frac{1}{2} + i\frac{\epsilon-\kappa}{2\Omega}\right)} + \hbar\frac{\epsilon-\kappa}{\Omega}\log\left(\frac{8\Omega^3}{\hbar\sqrt{2s_{cl}}}\right)$$

and

$$4s\hbar\nu + \frac{2\kappa\Omega}{\hbar} = \oint pdq$$

Conclusions

- The Jaynes-Cummings-Gaudin model, one of the simplest integrable models, has an extremely rich physical and mathematical containt.
- Lax pair techniques, are very powerful:
  - Classical Bethe Ansatz allow a very easy computation of the normal forms.
  - The spectral curve and its degeneracies encode very efficiently the bifurcation diagram.
- Bethe equations "know" the geometry of the bifurcation diagram.
- Much more work to do.

HAPPY BIRTHDAY, IGOR