Krichever formal group law and deformed Baker-Akhiezer function

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Celebrating Igor Krichever's 60th birthday

Columbia University

May 4–7 2011.

We describe explicitly the formal group law over $\mathbb{Z}[\mu]$ corresponding to the Tate uniformization of the general Weierstrass model of the cubic curve with parameters $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_6)$. This law is called the general elliptic formal group law.

We introduce the universal Krichever formal group law with the ring of coefficients $\mathcal{A}_{Kr}$. Its exponential is defined by the Baker-Akhiezer function $\Phi(t) = \Phi(t; \tau, g_2, g_3)$, where τ is a point on the elliptic curve with Weierstrass parameters (g_2, g_3). Results on the ring $\mathcal{A}_{Kr}$ are obtained.

We find the conditions necessary and sufficient for the elliptic formal group law to be a Krichever formal group law, and thus to define a rigid elliptic genus on S^1-equivariant SU-manifolds.

We introduce the deformed Baker-Akhiezer function $\Psi(t) = \Psi(t; v, w, \mu)$ where v and w are points of the curve with parameters μ. It is a quasiperiodic function with logarithmic derivative determined by the exponential of the general elliptic formal group law.

The deformation parameter is $\alpha = \wp'(w)/\wp'(v)$. The function $\Psi(t)$ coincides with the Baker-Akhiezer function $\Phi(t)$ if $\alpha = \pm 1$.

We obtain the addition theorem for $\Psi(t)$. Using this theorem we prove that the deformed Baker-Akhiezer function is the common eigenfunction of two double periodic differential operators of degree 2 and 3. Their commutator has $(1 - \alpha^2)$ as a factor.

Main definitions will be introduced during the talk.
New results presented in the talk have been obtained in recent joint works with E. Yu. Bunkova.

New results are published in

[1] V. M. Buchstaber,
The general Krichever genus, Russian Mathematical Surveys, 2010, 65:5, 979–981.

[2] Victor M. Buchstaber, Elena Yu. Bunkova,
Elliptic formal group laws, integral Hirzebruch genera and Krichever genera, arXiv:1010.0944.

[3] E. Yu. Bunkova,
The addition theorem for the deformed Baker-Akhiezer function, Russian Mathematical Surveys, 2010, 65:6.

[4] V. M. Buchstaber and E. Yu. Bunkova,
Krichever formal group law, Functional. Anal. Appl., 45:2 (2011).

In his first works (in the 70-s) I. M. Krichever obtained important results on Hirzebrugh genera of manifolds, using the formal group of geometric cobordisms.

In his works in the 80-s the Baker-Akhiezer function became a powerful tool for solving problems of the theory of integrable systems.

In 1990 I. M. Krichever introduced the Hirzebrugh genus defined by the Baker-Akhiezer function. Using the theory of elliptic functions he proved that his genus has the fundamental rigidity property on manifolds with an S^1-equivariant SU-structure.

The formal group law.

A commutative one-dim formal group law over a commutative ring A is the formal series

$$F(u,v) = u + v + \sum a_{i,j} u^i v^j, \quad a_{i,j} \in A, \quad i > 0,\ j > 0,$$

which satisfies the conditions

$$F(u,v) = F(v,u),$$

$$F\Big(u, F(v,w)\Big) = F\Big(F(u,v), w\Big).$$

A *homomorphism* of formal group laws $h : F_1 \to\ F_2$ over the ring A is a series $h(u) \in A[[u]]$, $h(0) = 0$, such that

$$h\Big(F_1(u,v)\Big) = F_2\Big(h(u), h(v)\Big).$$

A homomorphism h is an *isomorphism* if $h'(0)$ is a unit in A, and it is a *strong isomorphism* if $h'(0) = 1$.

The exponential and the logarithm of the formal group law.

Let $F_a(u, v) = u + v$. For each formal group law $F(u, v) \in A[[u, v]]$ there exists an isomorphism $h : F_a \to F$ over the ring $A \otimes \mathbb{Q}$.

The corresponding series $f(t) \in A \otimes \mathbb{Q}[[t]]$ is uniquely defined by the conditions

$$f(t_1 + t_2) = F\Big(f(t_1), f(t_2)\Big), \quad f(0) = 0, \quad f'(0) = 1.$$

It is called the *exponential* of the formal group law $F(u, v)$. The *logarithm* of the formal group law F is the formal series $g(u)$ such that $g\Big(f(t)\Big) = t$. We have

$$\frac{\partial F(u, v)}{\partial v}\bigg|_{v=0} = \frac{1}{g'(u)}.$$

The universal formal group law.

The formal group law

$$\mathcal{F}(u,v) = u + v + \sum \alpha_{i,j} u^i v^j$$

over the ring $\mathcal{A}$ is called *the universal formal group law*, if for any formal group $F(u,v)$ over a ring A there exists a unique homomorphism $r : \mathcal{A} \to A$ such that

$$F(u,v) = u + v + \sum r(\alpha_{i,j}) u^i v^j.$$

Construction of the universal formal group.

Over the ring $\mathcal{U} = \mathbb{Z}[\beta_{i,j} : i,j > 0]$, $\deg(\beta_{i,j}) = -2(i+j-1)$, define

$$\widehat{F}(u,v) = u + v + \sum \beta_{i,j} u^i v^j.$$

We have

$$\widehat{F}\Big(\widehat{F}(u,v), w\Big) = u + v + w + \sum \beta^l_{i,j,k} u^i v^j w^k,$$
$$\widehat{F}\Big(u, \widehat{F}(v,w)\Big) = u + v + w + \sum \beta^r_{i,j,k} u^i v^j w^k.$$

where $\beta^l_{i,j,k}$ and $\beta^r_{i,j,k}$ are homogeneous polynomials of $\beta_{i,j}$ and $\deg \beta^l_{i,j,k} = \deg \beta^r_{i,j,k} = -2(i+j+k-1)$.

Let $J \subset \mathcal{U}$ be the associativity ideal with generators $\beta^l_{i,j,k} - \beta^r_{i,j,k}$.

Consider the ring $\mathcal{A} = \mathcal{U}/J$ and the canonical projection

$$\pi : \mathcal{U} \to \mathcal{A}.$$

Set $\mathcal{F}(u,v) = u + v + \sum \alpha_{i,j} u^i v^j$, where $\alpha_{i,j} = \pi(\beta_{i,j})$.

By the construction the series $\mathcal{F}(u,v)$ over the graded ring $\mathcal{A}$ gives the universal formal group.

Theorem. (M. Lazard, 1955)

$$\mathcal{A} \simeq \mathbb{Z}[a_n : n = 1, 2, \ldots]$$

where $\deg a_n = -2n$.

Well-known formal groups and their exponentials.

$$F(u,v) = u + v - \mu_1 uv, \qquad f(t) = \tfrac{1}{\mu_1}\Big(1 - \exp(-\mu_1 t)\Big).$$

$$F(u,v) = \tfrac{u+v}{1+\mu_2 uv}, \qquad f(t) = \tfrac{1}{\sqrt{\mu_2}}\, th\Big(\sqrt{\mu_2}\, t\Big).$$

$$F(u,v) = u\sqrt{1 + \tfrac{1}{4}\mu_2 v^2} + v\sqrt{1 + \tfrac{1}{4}\mu_2 u^2}, \qquad f(t) = \tfrac{2}{\sqrt{\mu_2}}\, sh\Big(\tfrac{1}{2}\sqrt{\mu_2}\, t\Big).$$

$$F(u,v) = \tfrac{u\sqrt{1-2\delta v^2+\varepsilon v^4} + v\sqrt{1-2\delta u^2+\varepsilon u^4}}{1-\varepsilon u^2 v^2}, \qquad f(t) = sn(t),$$

where $sn(t)$ is the elliptic sine (Jacobi sine) such that

$$(f')^2 = 1 - 2\delta f^2 + \varepsilon f^4.$$

The general Weierstrass model of the elliptic curve

$$Y^2Z + \mu_1 XYZ + \mu_3 YZ^2 = X^3 + \mu_2 X^2 Z + \mu_4 XZ^2 + \mu_6 Z^3$$

depends on 5 parameters $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_6)$.

The geometric group structure on the elliptic curve is defined in the following way:
the points P, Q, R of the curve lie on a straight line if and only if $P + Q + R = 0$.

Let the zero of the geometric group structure be $O = (0 : 1 : 0)$.

For $P + Q + R = 0$ and $R + \overline{R} + O = 0$ we have $P + Q = \overline{R}$.

The Tate coordinates of the elliptic curve.

In the coordinate map $\{Y \neq 0\}$ with $u = -X/Y$ and $s = -Z/Y$:

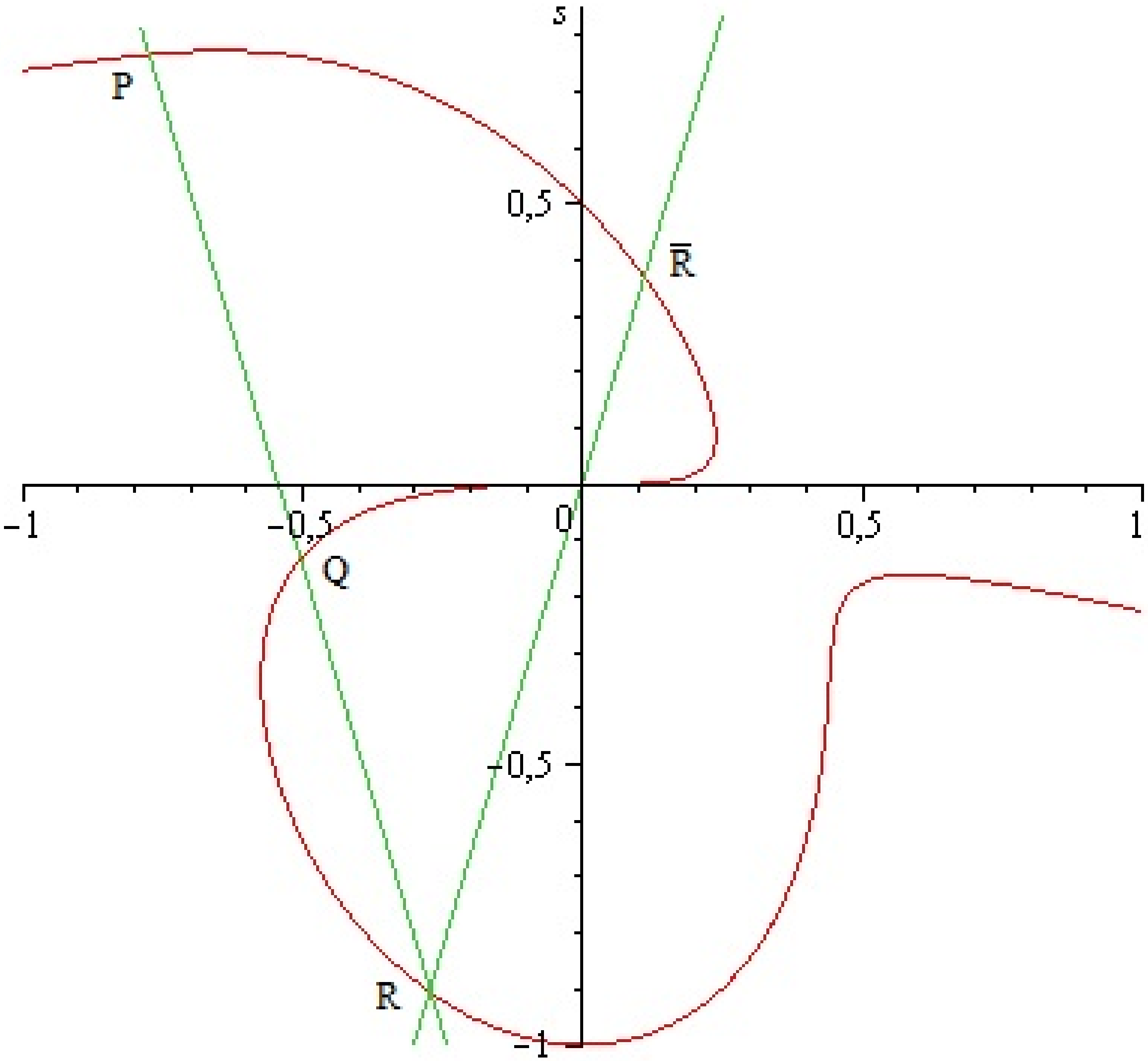
$$s = u^3 + \mu_1 us + \mu_2 u^2 s + \mu_3 s^2 + \mu_4 us^2 + \mu_6 s^3.$$

Thus we obtain the formal series $s(u) \in \mathbb{Z}[\mu][[u]]$:

$$s = u^3 + \mu_1 u^4 + (\mu_1^2 + \mu_2)u^5 + (\mu_1^3 + 2\mu_1\mu_2 + \mu_3)u^6 + \\ + (\mu_1^4 + 3\mu_1^2\mu_2 + \mu_2^2 + 3\mu_1\mu_3 + \mu_4)u^7 + ...$$

The coordinates $(u, s(u))$ of the point P are called the arithmetic Tate coordinates of P and give the Tate uniformization of the elliptic curve.

$$s = u^3 + 2\,u\,s + 4\,u^2 s + s^2 + 2\,u\,s^2 + 2\,s^3$$



The elliptic formal group law.
Let $P=\big(u,s(u)\big)$, $Q=\big(v,s(v)\big)$, $R=\big(w,s(w)\big)$ and $\overline{R}=\big(\bar{w},s(\bar{w})\big)$. The geometric group structure on the elliptic curve V_μ defines the series $\mathcal{F}_{El}(u,v)$ over $\mathbb{Z}[\mu]$ by the equation $\mathcal{F}_{El}(u,v)=\bar{w}$.

Theorem. (General elliptic formal group law)

$$\mathcal{F}_{El}(u,v)=\left(u+v-uv\frac{(\mu_1+\mu_3 m)+(\mu_4+2\mu_6 m)k}{(1-\mu_3 k-\mu_6 k^2)}\right)\times$$
$$\times\frac{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}{(1+\mu_2 n+\mu_4 n^2+\mu_6 n^3)(1-\mu_3 k-\mu_6 k^2)},$$

where $\quad \big(u,s(u)\big)\in V_\mu \quad$ and $\quad m=\dfrac{s(u)-s(v)}{u-v},$

$$k=\frac{us(v)-vs(u)}{u-v},\qquad n=m+uv\frac{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}{(1-\mu_3 k-\mu_6 k^2)}.$$

Definition. The formal group over A is called the *elliptic formal group* $F_{El}(u,v)$ if its classifying ring homomorphism $\mathcal{A} \to A$ can be presented as $\mathcal{A} \to \mathbb{Z}[\mu] \to A$.

Remark. We will also denote by $\mu_i \in A$ the images of $\mu_i \in \mathbb{Z}[\mu]$.

Corollary. For $\mu = (\mu_1, \mu_2, \mu_3, 0, 0)$ we have

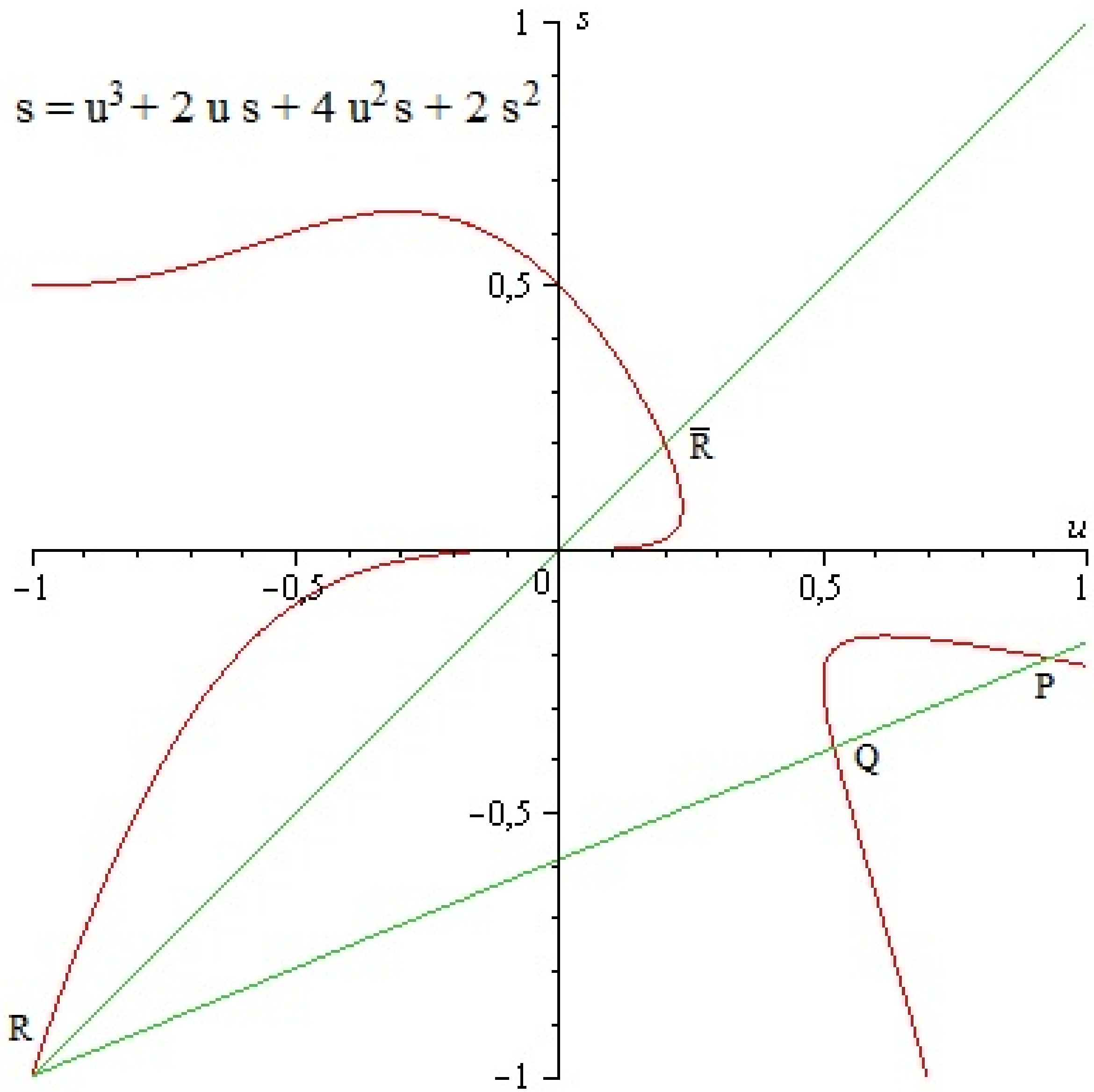
$$F_{El}(u,v) = \frac{(u+v)(1-\mu_3 k) - \mu_1 uv - \mu_3 uvm}{(1-\mu_3 k)(1+\mu_2 uv - \mu_3 k)}$$

and the epimorphism $\mathcal{A} \to \mathbb{Z}[\mu_1, \mu_2, \mu_3]$.

Corollary. For $\mu = (0, 0, 0, \mu_4, \mu_6)$ we have

$$F_{El}(u,v) = u + v + k\,\frac{2\mu_4 m + 3\mu_6 m^2}{1 + \mu_4 m^2 + \mu_6 m^3}$$

and the epimorphism $\mathcal{A}[\frac{1}{2}, \frac{1}{3}] \to \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \mu_4, \mu_6]$.


$s = u^3 + 2\,u\,s + 4\,u^2 s + 2\,s^2$
s
u
1
0,5
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R̄
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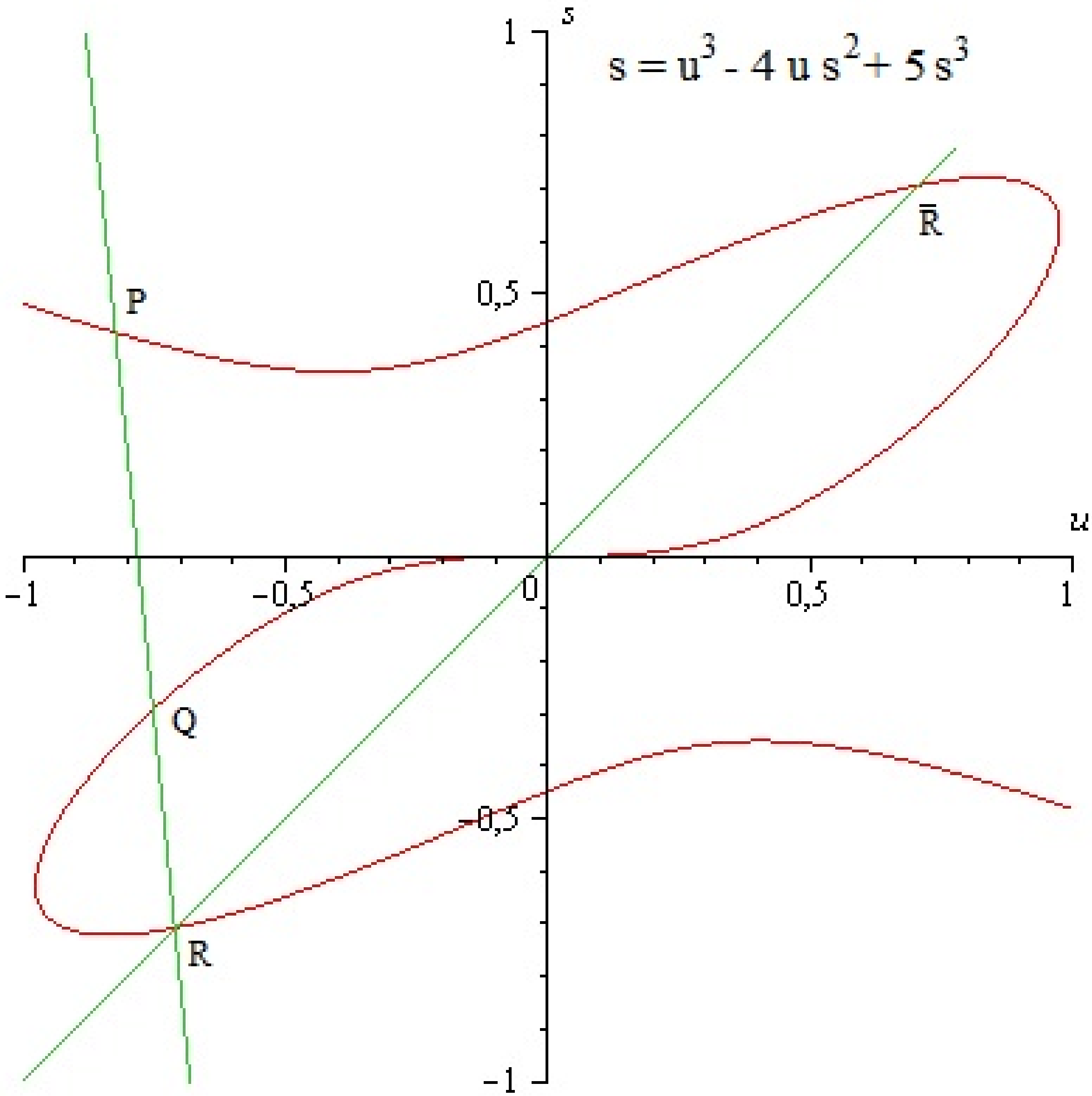

s
1
$s = u^3 - 4us^2 + 5s^3$
R̄
P
0,5
u
−1
−0,5
0
0,5
1
Q
−0,5
R
−1

The Weierstrass functions.

Let $\wp(t)$ such that $\lim_{t\to 0}\left(\wp(t) - \frac{1}{t^2}\right) = 0$

be the unique doubly periodic even meromorphic function on $\mathbb{C}$ with periods $2\omega_1$, $2\omega_2$ and double poles in lattice points only. Here $Im(\omega_2/\omega_1) > 0$. It defines the odd function

$$\zeta(t) \text{ such that } \zeta'(t) = -\wp(t) \text{ and } \lim_{t\to 0}\left(\zeta(t) - \frac{1}{t}\right) = 0$$

and the entire odd function

$$\sigma(t) \text{ such that } \Big(\ln\sigma(t)\Big)' = \zeta(t) \text{ and } \sigma'(0) = 1.$$

Periodic properties:

$$\wp(t+2\omega_k) = \wp(t), \qquad \zeta(t+2\omega_k) = \zeta(t) + 2\eta_k,$$

$$\sigma(t+2\omega_k) = -\sigma(t)\exp\Big(2(t+\omega_k)\eta_k\Big), \quad k = 1, 2.$$

The standard Weierstrass model of the elliptic curve.
In the coordinate map $\{Z \neq 0\}$ with $x = X/Z$ and $y = Y/Z$:

$$y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6.$$

Set $\quad b_2 = \mu_1^2 + 4\mu_2, \quad b_4 = \mu_1\mu_3 + 2\mu_4, \quad b_6 = \mu_3^2 + 4\mu_6.$

By the linear transform $(x, y) \mapsto (x + \frac{1}{12}b_2, 2y + \mu_1 x + \mu_3)$
we come to *the standard Weierstrass model*

$$y^2 = 4x^3 - g_2 x - g_3,$$

where $\quad g_2(\mu) = \dfrac{1}{12}b_2^2 - 2b_4, \quad g_3(\mu) = \dfrac{1}{6}b_2 b_4 - \left(\dfrac{1}{6}b_2\right)^3 - b_6.$

The standard Weierstrass model of the curve has the classical Weierstrass uniformization

$$\wp'(t)^2 = 4\wp(t)^3 - g_2\wp(t) - g_3,$$

where the periods of $\wp(t) = \wp(t; g_2, g_3)$ are

$$2\omega_k = \oint \frac{dx}{y}, \qquad k = 1, 2$$

The discriminant of the standard elliptic curve is

$$\Delta(\mathbf{g}) = g_2^3 - 27g_3^2.$$

The curve is degenerate if $\Delta(\mathbf{g}(\mu)) = 0$.

Theorem. The exponential of the general elliptic formal group is

$$f_{El}(t) = -2\frac{\wp(t) - \frac{1}{12}b_2}{\wp'(t) - \mu_1\wp(t) + \frac{1}{12}\mu_1 b_2 - \mu_3}$$

where $\wp(t) = \wp(t; g_2(\mu), g_3(\mu))$.

$f_{El}(t)$ is an elliptic function of order 3 for $\mu_6 \neq 0$
and of order 2 for $\mu_6 = 0$ (in the case of a non-degenerate curve).

Example of degenerate curve gives $\mu = (\mu_1, \mu_2, 0, 0, 0)$. Then

$$\wp(t) = \frac{(a-b)^2}{4}\left(\left(\frac{e^{at}+e^{bt}}{e^{at}-e^{bt}}\right)^2 - \frac{2}{3}\right) = \frac{1}{t^2} + \frac{(a-b)^4}{240}t^2 + \dots$$

where $\mu_1 = a + b,\ \ \mu_2 = -ab$. The formal group law is rational

$$F(u,v) = \frac{u + v - \mu_1 uv}{1 + \mu_2 uv} \quad \text{with} \quad f(t) = \frac{e^{at} - e^{bt}}{ae^{at} - be^{bt}}.$$

The Baker-Akhiezer function

$$\Phi(t) = \frac{\sigma(\tau + t)}{\sigma(t)\sigma(\tau)} e^{-\zeta(\tau)t}.$$

Its parameters are the parameters (g_2, g_3) of the curve in the standard Weierstrass form and a point τ on this curve.

Set
$$L_2 = \frac{d^2}{dt^2} - 2\wp(t).$$

$\Phi(t)$ is a quasiperiodic solution of the Lame equation

$$L_2\Phi(t) = \wp(\tau)\Phi(t)$$

such that $\lim\limits_{t\to 0}\left(\Phi(t) - \frac{1}{t}\right) = 0$. Set

$$L_3 = 2\frac{d^3}{dt^3} - 6\wp(t)\frac{d}{dt} - 3\wp'(t).$$

Then
$$L_3\Phi(t) = \wp'(\tau)\Phi(t).$$

We have

$$[L_2, L_3] = \wp'''(t) - 12\wp(t)\wp'(t) = 0.$$

Set

$$L_1 = \frac{d}{dt} + \phi(t), \quad \text{where} \quad \phi(t) = -\frac{1}{2}\,\frac{\wp'(t) - \wp'(\tau)}{\wp(t) - \wp(\tau)}$$

Then

$$L_1\Phi(t) = 0$$

and

$$L_2 = L_1^- L_1^+ + \wp(\tau) \quad \text{where} \quad L_1 = \frac{d}{dt} - \phi(t).$$

The addition theorem for the Baker-Akhiezer function.

$$\Phi(t+q) = \frac{\Phi(t)\Phi'(q) - \Phi'(t)\Phi(q)}{\wp(t) - \wp(q)}.$$

Corollary.

$$\frac{\Phi(t+q)}{\Phi(t)\Phi(q)} = -\frac{1}{2}\frac{\begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(\tau) \\ \wp'(t) & \wp'(q) & \wp'(\tau) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(\tau) \\ \wp(t)^2 & \wp(q)^2 & \wp(\tau)^2 \end{vmatrix}}.$$

The universal Krichever formal group law.

Let $\mathcal{B} = \mathbb{Z}[\chi_k : k = 1, 2, ...]$.

Consider the series of the following special form

$$\widehat{\mathcal{F}}(u,v) = ub(v) + vb(u) - b'(0)uv + \frac{b(u)\beta(u) - b(v)\beta(v)}{ub(v) - vb(u)}u^2v^2,$$

with $b(u) = 1 + \sum b_i u^i$, and $\beta(u) = \frac{b'(u)-b'(0)}{2u} = \sum_{k\geqslant 0} \beta_{k+2}u^k$.
Here $b_1 = \chi_1$, $b_{2i} = \chi_{2i}$ $b_{2i+1} = 2\chi_{2i+1}$ and
$\beta_{2k} = k\chi_{2k}$, $\beta_{2k+1} = (2k+\ 1)\chi_{2k+1}$.

This series gives the formal group law $\widehat{\mathcal{F}}(u,v) \in \widehat{\mathcal{A}}[[u,v]]$
where $\widehat{\mathcal{A}} = \mathcal{B}/\widehat{\mathcal{J}}$ and $\widehat{\mathcal{J}}$ is the associativity ideal.

Theorem. $\hat{\mathcal{A}} \otimes \mathbb{Q} \cong \mathbb{Q}[\chi_1, \chi_2, \chi_3, \chi_4]$.

Theorem. $\widehat{\mathcal{J}} = \sum_{k \geq 5} \widehat{\mathcal{J}}_{2k}$.

Lemma. $\widehat{\mathcal{J}}_{10} = \mathbb{Z}$ generated by

$$j_{10} = 5\chi_5 + 4\chi_1\chi_4 + 3\chi_1^2\chi_3 + 2\chi_2\chi_3.$$

Lemma. $\widehat{\mathcal{J}}_{12} = \mathbb{Z} \oplus \mathbb{Z}$ generated by $\chi_1 j_{10}$ and $j_{12} = 2\hat{j}_{12}$, where

$$\hat{j}_{12} = \chi_6 + \chi_3^2 + \chi_2\chi_4 - 2\chi_1^2(\chi_1\chi_3 + \chi_4).$$

Corollary. There is an element of order 2 in the group $\hat{\mathcal{A}}_{12}$.

I. M. Krichever introduced the Hirzebruch genus L_f that is determined by the function

$$f_{Kr}(t) = \frac{\exp \frac{1}{2}\lambda t}{\Phi(t;\tau, g_2, g_3)}$$

and proved the fundamental rigidity property for this genus on manifolds with an S^1-equivariant SU-structure.

Theorem. The isomorphism $\gamma : \widehat{\mathcal{A}} \otimes \mathbb{Q} \to \mathbb{Q}[\lambda, \wp(\tau), \wp'(\tau), g_2]$

where $\gamma(\chi_1) = \lambda, \qquad \gamma(\chi_2) = -\frac{1}{8}\lambda^2 + \frac{3}{2}\wp(\tau),$

$$\gamma(\chi_3) = \frac{1}{24}\lambda^3 - \frac{1}{2}\wp(\tau)\lambda + \frac{1}{3}\wp'(\tau),$$

$$\gamma(\chi_4) = -\frac{9}{128}\lambda^4 + \frac{15}{16}\wp(\tau)\lambda^2 - \frac{3}{4}\wp'(\tau)\lambda + \frac{3}{8}\wp(\tau)^2 - \frac{1}{8}g_2.$$

transforms the exponential of the formal group $\widehat{\mathcal{F}}(u,v)$ to $f_{Kr}(t)$.

The previous theorem leads to the following:

Definition. The formal group law $\widehat{\mathcal{F}}(u,v)$ will be called the *universal Krichever formal group law* and denoted $\mathcal{F}_{Kr}$. The ring $\widehat{\mathcal{A}}$ will be denoted $\mathcal{A}_{Kr}$.

Definition. A formal group over a ring A is called a *Krichever formal group* $F_{Kr}(u,v)$ if its classifying ring homomorphism $\mathcal{A} \to A$ can be presented as $\mathcal{A} \to \mathcal{A}_{Kr} \to A$.

The parameters (g_2, g_3) of a curve, a point τ on this curve and a parameter λ define the exponential of the Krichever formal group law. The set $(\lambda, \tau, g_2, g_3)$ we will call the parameters of the Krichever formal group law.

Definition. A formal group over a ring A is called an *elliptic Krichever formal group* if its classifying ring homomorphism $\mathcal{A} \to A$ can be presented simultaneously as $\mathcal{A} \to \mathcal{A}_{Kr} \to A$ and $\mathcal{A} \to \mathbb{Z}[\mu] \to A$.

Theorem.
An elliptic formal group law over a ring A without zero divisors is a Krichever formal group law if and only if in A we have:

$$\mu_2\mu_3 - \mu_1\mu_4 = 0, \qquad \mu_3^2 + 3\mu_6 = 0, \qquad \mu_3(\mu_1\mu_3 + \mu_4) = 0.$$

Corollary. The conditions of the Theorem are equivalent to

$$\begin{aligned}
\mu_6 &= 0 && \text{if } \mu_1 = 0,\ \mu_3 = 0;\\
\mu_2 = 0,\ \mu_3^2 = -3\mu_6, \quad \mu_4 &= 0 && \text{if } \mu_1 = 0,\ \mu_3 \neq 0;\\
\mu_4 = 0, \quad \mu_6 &= 0 && \text{if } \mu_1 \neq 0,\ \mu_3 = 0;\\
\mu_2 = -\mu_1^2,\ \mu_4 = -\mu_1\mu_3,\ -3\mu_6 &= \mu_3^2 && \text{if } \mu_1 \neq 0,\ \mu_3 \neq 0.
\end{aligned}$$

Examples of the elliptic Krichever formal groups.

Let $\mu_1 = \mu_3 = \mu_6 = 0$ and $\delta = \mu_2$, $\varepsilon = \mu_2^2 - 4\mu_4$ then

$$F_{El}(u,v) = F_{Kr}(u,v) = \frac{u\rho(v) + v\rho(u)}{1 - \varepsilon u^2 v^2}$$

for $\rho^2(u) = 1 - 2\delta u^2 + \varepsilon u^4$.

Let $\mu_1 = \mu_2 = \mu_4 = 0$ and $\mu_3^2 = -3\mu_6$ then

$$F_{El}(u,v) = F_{Kr}(u,v) = \frac{u^2 r(v) - v^2 r(u)}{u r^2(v) - v r^2(u)}$$

for $r^3(u) = 1 - 3\mu_3 u^3$.

Let $\mu_3 = \mu_4 = \mu_6 = 0$ then

$$F_{El}(u,v) = F_{Kr}(u,v) = \frac{u + v - \mu_1 uv}{1 + \mu_2 uv}\,.$$

Theorem.
The Krichever formal group law with parameters $(\lambda, \tau, g_2, g_3)$ over a ring without zero divisors is an elliptic formal group law if and only if the following conditions hold:

Set $a_1 = \frac{\lambda}{2}$, $a_2 = \wp(\tau)$, $a_3 = \wp'(\tau)$, $a_4 = \frac{g_2}{2}$ then

$$3a_1^5 - 10a_2a_1^3 + 15a_2^2a_1 - 2a_4a_1 - 4a_2a_3 = 0,$$

$$(a_1^3 - 3a_2a_1 + a_3)(9a_1^4 - 30a_2a_1^2 + 12a_3a_1 + 2a_4 - 3a_2^2) = 0.$$

Corollary. For $\lambda = 0$ the conditions of the theorem become

$$\wp'(\tau) = 0 \quad \text{or} \quad \wp(\tau) = 0, \quad g_2 = 0.$$

We have

$$f_{Kr}(t) = \frac{\exp \frac{1}{2}\lambda t}{\Phi(t;\tau, g_2, g_3)}.$$

Let $\lambda = 0$, $\wp'(\tau) = 0$. Then $\mu_1 = \mu_3 = \mu_6 = 0$.
In this case $f_{Kr}(t) = sn(t)$ is the solution of the equation

$$f'^2 = 1 + 3a_2 f^2 + (3a_2^2 - \frac{1}{2}a_4)f^4.$$

Let $\lambda = 0$, $\wp(\tau) = 0$, $g_2 = 0$. Then $\mu_1 = \mu_2 = \mu_4 = 0$ and $\mu_3^2 = -3\mu_6 = \frac{1}{9}a_3^2$. Set $\wp(t) = \wp(t; 0, \frac{1}{27}a_3^2)$.
Then $f_{Kr}(t) = \frac{-2\wp(t)}{\wp'(t)+\frac{1}{3}a_3}$ is the solution of the equation

$$f'^3 = (1 + a_3 f^3)^2.$$

The deformed Baker-Akhiezer function.

$$\text{Set} \quad \phi(t; v, w) = \frac{1}{f_{El}(t)} - \frac{\mu_1}{2} = -\frac{1}{2}\frac{\wp'(t) + \wp'(w)}{\wp(t) - \wp(v)},$$

$\wp(t) = \wp\Big(t; g_2(\mu), g_3(\mu)\Big)$, $\wp'(w) = -\mu_3$, $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$.

The deformed Baker-Akhiezer function $\Psi(t)$ is the solution of

$$L_1\Psi(t) = 0, \text{ where } L_1 = \frac{d}{dt} + \phi(t)$$

such that $\lim\limits_{t\to 0}\left(\Psi(t) - \frac{1}{t}\right) = 0$.

Lemma.

$$\Psi(t) = \frac{\sigma(v+t)^{\frac{1}{2}(1-\alpha)}\sigma(v-t)^{\frac{1}{2}(1+\alpha)}}{\sigma(t)\sigma(v)} \exp\left(\Big(-\frac{\mu_1}{2} + \alpha\zeta(v)\Big)t\right),$$

where $\alpha = \frac{\wp'(w)}{\wp'(v)}$ is the deformation parameter.

$\Psi(t; v, w) = \Phi(t; v)$ for $\alpha = -1$ and $\Psi(t; v, w) = \Phi(t; -v)$ for $\alpha = 1$.

Let $\wp'(w) = -\wp'(\tau)$, $\wp(v) = \wp(\tau)$. Then $\Psi(t; v, w)$ is the usual Baker-Akhiezer function $\Phi(t; \tau)$. Here τ is the solution of the system $\wp'(\tau) = \mu_3$, $\wp(\tau) = \frac{1}{12}(4\mu_2 + \mu_1^2)$, which is compatible only in if $\mu_6 = 0$.

The addition formula.

Theorem.

$$\Psi(t+q) =$$

$$= \frac{\begin{vmatrix} \Psi(t) & \Psi(q) \\ \Psi'(t) & \Psi'(q) \end{vmatrix}}{\wp(t) - \wp(q)} \times \frac{\begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(v) \\ \wp'(t) & \wp'(q) & \wp'(v) \end{vmatrix}^{\frac{1-\alpha}{2}} \begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(-v) \\ \wp'(t) & \wp'(q) & \wp'(-v) \end{vmatrix}^{\frac{1+\alpha}{2}}}{\frac{1-\alpha}{2} \begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(v) \\ \wp'(t) & \wp'(q) & \wp'(v) \end{vmatrix} + \frac{1+\alpha}{2} \begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(-v) \\ \wp'(t) & \wp'(q) & \wp'(-v) \end{vmatrix}}.$$

Properties of the deformed Baker-Akhiezer function for $\mu_1 = 0$.

1. In the vicinity of $t = 0$ we have

$$\Psi(t; v, w) = \frac{1}{t} - \frac{1}{2}\wp(v)t + \frac{1}{6}\wp'(w)t^2 + (t^3).$$

2. The functions $\Psi(t; v, w)$ and $\Psi(t; -v, w)$ give the solutions of the deformed Lame equation

$$L_2\Psi(t) = \wp(v)\Psi(t), \qquad \text{where} \qquad L_2 = \frac{d^2}{dt^2} - U$$

$$\text{and} \quad U = 2\wp(t) - \frac{1-\alpha^2}{4}\left(\frac{\wp'(v)}{\wp(t) - \wp(v)}\right)^2.$$

3. Let $2\omega_k$, $k = 1, 2$ be the periods of the $\wp$-function. Then

$$\Psi(t + 2\omega_k; v, w) = \Psi(t; v, w) \exp\Big(2\alpha(\zeta(v)\omega_k - \eta_k v)\Big);$$

$$\Psi(t; v + 2\omega_k, w) = \Psi(t; v, w).$$

The function $\Psi(t; \omega_k, w) = \Psi(t; \omega_k)$ does not depend on w and $\Psi(t + 2\omega_k; \omega_k) = \Psi(t; \omega_k)$.

4. $\Psi(t; v, \omega_k) = \sqrt{\wp(t) - \wp(v)}$. In this case $\alpha = 0$.

5. We have

$$\Psi(t; v, w) = \Psi(t; -v, -w) = -\Psi(-t; v, -w).$$

Let $L_1^+ = L_1 = \partial + \phi(t)$ and $L_1^- = \partial - \phi(t)$ where

$$\phi(t) = -\frac{1}{2}\frac{\wp'(t)+\wp'(v)}{\wp(t)-\wp(v)} + \frac{1-\alpha}{2}\frac{\wp'(v)}{\wp(t)-\wp(v)}.$$

We have

$$L_2 - \wp(v) = L_1^- L_1^+.$$

Set $V = \frac{(1-\alpha^2)}{16}\wp'(v)^2\mathcal{T}$, where $\mathcal{T} = \frac{(3\wp'(t)+\alpha\wp'(v))}{(\wp(t)-\wp(v))^3}$.

The addition formula gives the operator

$$L_3 = 2\partial^3 - 3U\partial - U_0,$$

where $U_0 = \frac{3}{2}U' + 2V$, such that $L_3\Psi(t) = -\alpha\wp'(v)\Psi(t)$. We have

$$[L_2, L_3] = -\frac{1}{4}(1-\alpha^2)\wp'(v)^2\left(\frac{\partial}{\partial t}\mathcal{T}\right)L_1.$$

The Hirzebruch genera.
Let $f(t) = t + \sum f_k t^{k+1}$, where $f_k \in A \otimes \mathbb{Q}$.

$$\text{Set} \quad L_f(\sigma_1, ..., \sigma_n) = \prod_{i=1}^{n} \frac{t_i}{f(t_i)}, n = 1, 2, 3, ...$$

where σ_k is the k-th elementary symmetric polynomial of $t_1, ..., t_n$.

The Hirzebruch genus L_f of a stably complex manifold M^{2n} with tangent Chern classes $c_i = c_i(\tau(M^{2n}))$ and fundamental cycle $\langle M^{2n} \rangle$ is defined by the formula

$$L_f(M^{2n}) = (L_f(c_1, ..., c_n), \langle M^{2n} \rangle) \in A \otimes \mathbb{Q}.$$

The Hirzebruch genus L_f is called A-*integer* if $L_f(M^{2n}) \in A$ for any stably complex manifold M^{2n}.

A formal group law over A corresponds to a ring homomorphism $\Omega_U \to A$, that defines an A-integer Hirzebruch genus.