The rigidity problem, elliptic functions and Hurwitz series.

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For each Hirzebruch genus $L_f : \Omega_U \to \mathbb{Q}$
I. Krichever (1974) constructed an equivariant genus with values in $K(\mathbb{C}P^\infty) \otimes \mathbb{Q}$ and described this genus for almost complex S^1-equivariant manifolds M^{2n} in terms of fixed points of the S^1-action on M^{2n}.
In this case the signs of isolated fixed points are $+1$.

He introduced (1990) the genus L_f where f is the Baker-Akhiezer function of an elliptic curve.
I. Krichever proved that the corresponding equivariant genus has the fundamental rigidity property on S^1-equivariant SU-manifolds.

The universal S^1-equivariant genera for stably complex S^1-manifolds with values in $U^*(CP^\infty)$ was considered by V. Buchstaber and N. Ray in 2007.

The general theory of universal toric genera was constructed by V. Buchstaber, T. Panov and N. Ray (2008).

Applications of this theory to the homogeneous spaces of compact Lie groups is given by V. Buchstaber and S. Terzic (2008).

We will describe the general elliptic genus associated with the general Weierstrass model of the elliptic curve with 5 parameters and arithmetic Tate uniformization.
The main task of this talk is to present the results about the S^1-equivariant elliptic genera.

New results are published in

[1] V. M. Buchstaber,
The general Krichever genus, Russian Mathematical Surveys, 2010, 65:5, 979–981.

[2] Victor M. Buchstaber, Elena Yu. Bunkova,
Elliptic formal group laws, integral Hirzebruch genera and Krichever genera, arXiv: 1010.0944 v1 [math-ph] 5 Oct 2010.

[3] V. M. Buchstaber and E. Yu. Bunkova,
Krichever formal group law, Functional. Anal. Appl., 45:2 (2011).

Complex cobordisms.

Let $U^*(X)$ be the complex cobordism theory.
Then $U^*(pt) = \Omega_U = \sum \Omega_{-2n}$, $n \geqslant 0$, is the ring of cobordisms of stable-complex manifolds.
A representative of $a \in \Omega_{-2n}$ is $2n$-dim manifold M^{2n} with the complex normal bundle in the real Euclidian space $\mathbb{R}^{2N}$.

Theorem. (J. Milnor, S. Novikov, 1960)

$$\Omega_U \simeq \mathbb{Z}[a_n : n = 1, \ldots]$$

where $\deg a_n = -2n$.

The Hirzebruch genera.

Let $A = \sum A_{-2k}$ be a commutative associative graded torsion-free ring. Consider the series
$f(t) = t + \sum f_k t^{k+1}$, $k \geqslant 1$, where $f_k \in A_{-2k} \otimes \mathbb{Q}$.
The series

$$\prod_{i=1}^{n} \frac{t_i}{f(t_i)}$$

can be presented in the form $L_f(\sigma_1, ..., \sigma_n)$, where σ_k is the k-th elementary symmetric polynomial of $t_1, ..., t_n$.

We have

$$L_f(\sigma_1, ..., \sigma_n) = 1 - f_1\sigma_1 + (f_1^2 - f_2)\sigma_1^2 + (2f_2 - f_1^2)\sigma_2 + \dots$$

Definition.
The Hirzebruch genus L_f of a stably complex manifold M^{2n} with tangent Chern classes $c_i = c_i\Big(\tau(M^{2n})\Big)$ and fundamental cycle $\langle M^{2n}\rangle$ is defined by the formula

$$L_f(M^{2n}) = \Big(L_f(c_1, ..., c_n), \langle M^{2n}\rangle\Big) \in A_{-2n} \otimes \mathbb{Q}.$$

Theorem. (Corollary of F. Hirzebruch result, 1956.)
Any Hirzebruch genus L_f defines a ring homomorphism

$$L_f : \Omega_U \to A \otimes \mathbb{Q}.$$

The formal group law.

Let A be a commutative associative ring.
A commutative one-dim formal group law over A
(or shortly formal group) is the formal series

$$F(u,v) = u + v + \sum a_{i,j} u^i v^j, \quad a_{i,j} \in A, \quad i > 0,\ j > 0,$$

which satisfies the conditions

$$F(u,v) = F(v,u), \qquad F\Big(u, F(v,w)\Big) = F\Big(F(u,v), w\Big).$$

The *exponential* of a formal group $F(u,v) \in A[[u,v]]$ is a uniquely defined formal series $f(t) \in A \otimes \mathbb{Q}[[t]]$ such that

$$f(t_1 + t_2) = F\Big(f(t_1), f(t_2)\Big), \quad f(0) = 0, \quad f'(0) = 1.$$

The *logarithm* of the formal group law $F(u,v)$ is the formal series $g(u)$ such that $g\Big(f(t)\Big) = t$.

Construction. (S. Novikov, A. Mishchenko, 1967)

Let $c_1(\eta) \in U^*(\mathbb{C}P^\infty) = \Omega_U[[u]]$ be the first Chern class of the universal complex linear bundle $\eta \to \mathbb{C}P^\infty$.

The class $c_1(\eta_1 \otimes \eta_2) \in U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega_U[[u,v]]$ defines the formal group law of geometric cobordisms

$$\mathcal{F}_U(u,v) = u + v + \sum a_{i,j} u^i v^j.$$

Here $a_{i,j} \in \Omega_{-2n}$, where $n = i + j - 1$.

Theorem. (A. Mishchenko, 1967)

The logarithm $g_{\mathcal{F}_U}(u)$ of the formal group $\mathcal{F}_U(u,v)$ has the form

$$g_{\mathcal{F}_U}(u) = u + \sum [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}.$$

Theorem. (S. Novikov, 1968)
Let $L_f : \Omega_U \to A \otimes Q$ a Hirzebrugh genus. Set

$$g_f(u) = u + \sum L_f(\mathbb{C}P^n) \frac{u^{n+1}}{n+1}.$$

Then

$$g_f(f(t)) = t.$$

The universal formal group law.

The formal group law

$$\mathcal{F}(u,v) = u + v + \sum \alpha_{i,j} u^i v^j$$

over the ring $\mathcal{A}$ is called *the universal formal group law*, if for any formal group $F(u,v)$ over a ring A there exists a unique homomorphism $r : \mathcal{A} \to A$ such that

$$F(u,v) = u + v + \sum r(\alpha_{i,j}) u^i v^j.$$

Construction of the universal formal group.

Consider the graded ring

$$\mathcal{U} = \mathbb{Z}[\beta_{i,j} : i > 0,\ j > 0],\ \deg \beta_{i,j} = -2(i + j - 1).$$

and series

$$\widehat{F}(u, v) = u + v + \sum \beta_{i,j} u^i v^j.$$

We have

$$\widehat{F}\Big(\widehat{F}(u, v), w\Big) = u + v + w + \sum \beta^l_{i,j,k} u^i v^j w^k,$$
$$\widehat{F}\Big(u, \widehat{F}(v, w)\Big) = u + v + w + \sum \beta^r_{i,j,k} u^i v^j w^k,$$

where $\beta^l_{i,j,k}$ and $\beta^r_{i,j,k}$ are homogeneous polynomials of $\beta_{i,j}$ and $\deg \beta^l_{i,j,k} = \deg \beta^r_{i,j,k} = -2(i + j + k - 1)$.

Let $J \subset \mathcal{U}$ be the associativity ideal with generators $\beta^l_{i,j,k} - \beta^r_{i,j,k}$. Set $\mathcal{A} = \mathcal{U}/J$ and consider the canonical projection

$$\pi : \mathcal{U} \to \mathcal{A}.$$

Set $\mathcal{F}(u,v) = u + v + \sum \alpha_{i,j} u^i v^j$, where $\alpha_{i,j} = \pi(\beta_{i,j})$.

By the construction the series $\mathcal{F}(u,v)$ over the graded ring $\mathcal{A}$ gives the universal formal group.

Theorem. (M. Lazard, 1955)

$$\mathcal{A} \simeq \mathbb{Z}[a_n : n = 1, \ldots]$$

where $\deg a_n = -2n$.

Theorem. (D. Quillen, 1969)
The ring homomorphism

$$\mathbb{Z}[a_n] \simeq \mathcal{A} \to \Omega_U \simeq \mathbb{Z}[a_n]$$

classifying the formal group $\mathcal{F}_U(u,v)$ is an isomorphism.

Thus, the formal group of complex cobordisms was identified with the universal formal group $\mathcal{F}(u,v)$.

Let A be a ring without torsion.

Consider a formal group $F(u,v)$ over A with the exponential $f(t) \in A \otimes \mathbb{Q}[[t]]$. Then the homomorphism $r \colon \mathcal{A} \to A$ classifying $F(u,v)$ gives a Hirzebruch genus $L_f \colon \Omega_U \to A$ such that $F(u,v) = u + v + \sum L_f(a_{i,j}) u^i v^j$.

We obtain one-to-one correspondence

$$(A, F) \Leftrightarrow (L_f \colon \Omega_U \to A).$$

Corollary. $g(u) = u + \sum L_f(\mathbb{C}P^n) \frac{u^{n+1}}{n+1}$.

Thus if we know a formal group law $F(u,v)$ over A we have the values $L_f(\mathbb{C}P^n) \in A$ using the formula

$$\left.\frac{\partial F(u,v)}{\partial v}\right|_{v=0} = \frac{1}{g'(u)}$$

Universal equivariant genera

Consider a $\mathbb{T}^k$-*equivariant normally complex manifold* (M^{2n}, c_ν, θ) where M^{2n} is a smooth manifold with a smooth action θ of the torus $\mathbb{T}^k$ and c_ν is an equivariant complex structure for the normal bundle $\nu(i)$ of an equivariant embedding $i\colon M^{2n} \to \mathbb{C}^l$.

Using the standard action of S^1 on the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$ we obtain the diagonal free action of $\mathbb{T}^k$ on $(S^{2m+1})^k$. Consider the $2(mk+n)$-dim smooth manifold $W_m = (S^{2m+1})^k \times_{\mathbb{T}^k} M^{2n}$ and the l-dim complex vector bundle $q_m\colon E_m \to (\mathbb{C}P^m)^k$, where $E_m = (S^{2m+1})^k \times_{\mathbb{T}^k} \mathbb{C}^l$.

Then $i\colon M^{2n} \to \mathbb{C}^l$ extends to an embedding $i'\colon W_m \to E_m$, and c_ν extends to a complex structure c' on the normal bundle $\nu(i')$.

The composition $p_m\colon W_m \xrightarrow{i'} E_m \xrightarrow{q_m} (\mathbb{C}P^m)^k$ is complex oriented mapping. It determines a complex cobordism class

$$\Phi_m(M^{2m}, c_\nu, \theta) \in U^{-2n}\Big((\mathbb{C}P^m)^k\Big).$$

The standard embedding $\iota_m\colon \mathbb{C}P^m \to \mathbb{C}P^{m+1}$ acts by $\iota_m^*\Phi_{m+1} = \Phi_m$, hence the inverse sequence $\Big(\Phi_m(M,\theta,c_\nu) : m \geqslant 0\Big)$ defines an element of $\varprojlim U^{-2n}\Big((\mathbb{C}P^m)^k\Big) \simeq U^{-2n}\Big((\mathbb{C}P^\infty)^k\Big)$.

Definition. The *universal* $\mathbb{T}^k$*-equivariant genus* of (M^{2n}, c_ν, θ) is

$$\Phi(M^{2n}, c_\nu, \theta)(u) = \varprojlim \Phi_m(M^{2n}, c_\nu, \theta)(u).$$

The rigidity problem.

We have

$$\Phi(M^{2n}, c_\nu, \theta)(u) \in \Omega_U[[u_1, ..., u_k]],$$

where $\Phi(M^{2n}, c_\nu, \theta)(0) = [M^{2n}]$.

Thus for any Hirzebruch genus $L_f : \Omega_U \to A$ we obtain $\mathbb{T}^k$-equivariant genus

$$\Phi_f(M^{2n}, c_\nu, \theta)(u) \in A[[u_1, ..., u_k]].$$

Definition.
The $\mathbb{T}^k$-equivariant genus $\Phi_f(M^{2n}, c_\nu, \theta)$ is rigid if

$$\Phi_f(M^{2n}, c_\nu, \theta)(u) = \Phi_f(M^{2n}, c_\nu, \theta)(0).$$

A compact smooth manifold M^{2n} with an action θ of $\mathbb{T}^k$ and a stably complex θ-invariant structure c_τ on tangent bundle $\tau(M^{2n})$ will be called a *stably complex* $\mathbb{T}^k$*-manifold* (M^{2n}, c_τ, θ).

Any such structure gives a $\mathbb{T}^k$-equivariant normally complex manifold (M^{2n}, c_ν, θ) and we set by definition $\Phi(M^{2n}, c_\tau, \theta)(u) = \Phi(M^{2n}, c_\nu, \theta)(u)$.

Note that on the other hand the structure (M^{2n}, c_ν, θ) does not define uniquely the structure (M^{2n}, c_τ, θ).

Suppose the set of fixed points of the action θ is finite.
The action of $\mathbb{T}^k$ on the fibre $\xi_k \cong \mathbb{C}^{n+l}$ over an isolated point $p \in M^{2n}$ defines an equivariant decomposition $\xi_p = \mathbb{C}^n \oplus \mathbb{C}^l$, where the action of $\mathbb{T}^k$ on $\mathbb{C}^l$ is trivial and the action on $\mathbb{C}^n$ has $0 \in \mathbb{C}^n$ as fixed point.

The signs and weights.

By definition the sign of $p \in M^{2n}$ is $+1$ if the linear map

$$\tau_p(M^{2n}) \xrightarrow{I\oplus 0} \tau_p(M^{2n}) \oplus \mathbb{R}^{2l} \xrightarrow{c_{\tau,p}} \xi_p \cong \mathbb{C}^n \oplus \mathbb{C}^l \xrightarrow{\pi} \mathbb{C}^n$$

conserves the orientation and else the sign is -1.

The action θ of $\mathbb{T}^k$ on $\tau_p(M^{2n})$ defines the representation $\mathbb{T}^k \to U(n)$ that is completely defined by the set of vectors $\{\Lambda_1(p), ..., \Lambda_n(p)\}$, $\Lambda_j(p) \in \mathbb{Z}^k$ of weights of the representation.

Multidimentional power system.

Now let $\{[w](u) \in \Omega^*[[u]] : w \in \mathbb{Z}\}$ denote the power system of the formal group $\mathcal{F}(u,v)$ for complex cobordism.

Denote by $\mathcal{F}_{q=1}^{k} a_q$ the sum of the elements of $a_1, ..., a_k$ in the formal group $\mathcal{F}(u,v)$, that is $\mathcal{F}_{q=1}^{k} a_q = \mathcal{F}(\mathcal{F}_{q=1}^{k-1}, a_k)$.

The w-series $[w](u) = wu$ mod(u^2) is defined uniquely by $[0](u) = 0$ and $[w](u) = \mathcal{F}(u, [w-1](u))$ for $w \in \mathbb{Z}$.

For $\mathbf{w} = (w_1, ..., w_k) \in \mathbb{Z}^k$ and $u = (u_1, ..., u_k)$ one defines

$$[\mathbf{w}](u) = [w](u) \quad \text{for} \quad k = 1 \qquad \text{and}$$

$$[\mathbf{w}](u) = \mathcal{F}_{q=1}^{k}[w_q](u_q), \quad \text{for} \quad k \geq 2.$$

The localization formula.

Theorem.
Let (M^{2n}, c_τ, θ) be a stably-complex $\mathbb{T}^k$-manifold , where θ is an action with isolated fixed points. The localization formula holds:

$$\Phi(M^{2n}, c_\tau, \theta)(u) = \sum_{p \in \mathsf{Fix}\,\theta} \mathsf{sign}(p) \prod_{j=1}^{n} \frac{1}{[\Lambda_j(p)](u)},$$

where $\{\Lambda_j(p), 1 \leqslant j \leqslant n\}$ are the weight vectors of the fixed point $p \in M^{2n}$.

Let $L_f : \Omega_U \to A$ be the Hirzebrugh genus corresponding to the formal group law $F(u,v)$ over A with the exponential $f(t)$. Denote by $[\mathbf{w}]_F$ the power system corresponding to $F(u,v)$. Then

$$\Phi_f(M^{2n}, c_\tau, \theta)(u) = \sum_{p \in \mathsf{Fix}\,\theta} \mathsf{sign}(p) \prod_{j=1}^{n} \frac{1}{[\Lambda_j(p)]_F(u)}.$$

From the localization theorem we obtain

Corollary.

$$\sum_{p \in \mathsf{Fix}\,\theta} \mathsf{sign}(p) \prod_{j=1}^{n} \frac{1}{[\Lambda_j(p)](u)} = [M^{2n}] + \mathcal{L}(u),$$

where $\mathcal{L}(u) \in \Omega_U[[u_1, ..., u_k]]$.

Hurwitz series.
Let $u = (u_1, ..., u_k)$ and $j = (j_1, ...j_k) \in \mathbb{Z}^k_{\geqslant}$.
Set $u^j = u_1^{j_1} \cdot ... \cdot u_k^{j_k}$ and $j! = j_1!...j_k!$.

A *k-dimensional Hurwitz series* over A is a formal power series in the form

$$\varphi(u) = \sum_j \varphi_j \frac{u^j}{j!} \ \in \ A \otimes \mathbb{Q}[[u_1, ...u_k]]$$

with $\varphi_j \in A$ for all j.

Hurwitz series over A form a commutative associative ring $HA[[u]]$ with respect to the usual addition and multiplication of series.

This ring is closed under differentiation and integration with respect to $u_1, ..., u_k$.
If $\varphi(u)$ and $\psi(u) \in HA[[u]]$, $\varphi(0) = 0$, then $\psi\Big(\varphi(u)\Big) \in HA[[u]]$.

For a formal group law $F(u,v)$ over A its exponential $f(t)$ and logarithm $g(u)$ are 1-dimensional Hurwitz series over A.

Set $t=(t_1,...,t_k)$ and $\langle \mathbf{w}, t\rangle = \sum_i w_i t_i$.

Corollary.

$$\sum_{p\in \mathrm{Fix}\,\theta} \mathrm{sign}(p) \prod_{j=1}^{n} \frac{1}{f\Big(\langle \Lambda_j(p), t\rangle\Big)} = L_f(M^{2n}) + \mathcal{L}(t_f),$$

where $\mathcal{L}(t_f) \in HA[[t_1,...,t_k]]$ and $t_f = \Big(f(t_1),...,f(t_k)\Big)$.

Example of $\mathbb{C}P^1$.

Consider the projective line $\mathbb{C}P^1$ with the action of $S^1 = \{z \in \mathbb{C}, |z| = 1\} : g(z_1 : z_2) = (gz_1 : z_2)$.

This action has two fixed points $p_1 = (1 : 0)$ and $p_2 = (0 : 1)$ such that $\mathrm{sign}(p_k) = +1$, $k = 1, 2$ and $\Lambda(p_1) = 1$, $\Lambda(p_2) = -1$.

Thus for any genus $L_f : \Omega_U \to A$ we have

$$\Phi_f(\mathbb{C}P^1)(u) = \frac{1}{u} + \frac{1}{\bar{u}} \in A[[u]],$$

$$\Phi_f(\mathbb{C}P^1)(f(t)) = \frac{1}{f(t)} + \frac{1}{f(-t)} \in HA[[t]].$$

The elliptic formal group law.

The elliptic formal group law $\mathcal{F}_{El}(u,v)$ is given by the addition on the elliptic curve in the Tate form

$$s = u^3 + \mu_1 us + \mu_2 u^2 s + \mu_3 s^2 + \mu_4 us^2 + \mu_6 s^3.$$

Theorem.

$$\mathcal{F}_{El}(u,v) = \left(u + v - uv\frac{(\mu_1 + \mu_3 m) + (\mu_4 + 2\mu_6 m)k}{(1 - \mu_3 k - \mu_6 k^2)}\right) \times$$
$$\times \frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 + \mu_2 n + \mu_4 n^2 + \mu_6 n^3)(1 - \mu_3 k - \mu_6 k^2)},$$

where $\Big(u, s(u)\Big) \in V_\mu$ and $m = \dfrac{s(u) - s(v)}{u - v}$,

$$k = \frac{us(v) - vs(u)}{u - v}, \quad n = m + uv\frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 - \mu_3 k - \mu_6 k^2)}.$$

Theorem. The exponential of an elliptic formal group is

$$f_{El}(t) = -2\frac{\wp(t) - \frac{1}{12}(\mu_1^2 + 4\mu_2)}{\wp'(t) - \mu_1\left(\wp(t) - \frac{1}{12}\mu_1(\mu_1^2 + 4\mu_2)\right) - \mu_3}$$

where $\wp(t) = \wp\Big(t; g_2(\mu), g_3(\mu)\Big)$.

In the general case $f_{El}(t) \in H\mathbb{Z}[\mu][[t]]$ where $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_6)$.

Let the elliptic curve be non-degenerate. Then $f_{El}(t)$ is an elliptic function of order 3 for $\mu_6 \neq 0$ and of order 2 for $\mu_6 = 0$.

Definition.
The Hirzebruch genus with exponentional $f(t) = f_{El}(t)$ is called *the elliptic genus*.

For an elliptic genus we have

$$\Phi_f(\mathbb{C}P^1)(u) = \mu_1 + \frac{\mu_3}{\wp\Big(g(u)\Big) - \frac{1}{12}(4\mu_2 + \mu_1^2)} \in \mathbb{Z}[\mu][[u]].$$

The beginning of this series is

$$\mu_1 + \mu_3 \left(u^2 + \mu_1 u^3 + (\mu_1^2 + \mu_2)u^4 + (\mu_1^3 + 2\mu_1\mu_2 + \mu_3)u^5 + ...\right).$$

Corollary.
The elliptic genus is rigid on $\mathbb{C}P^1$ with the given structure if and only if $\mu_3 = 0$.

The quaternionic stably complex S^1-structure on S^2.

Consider the maximum torus $\mathbb{T}^1$ in Lie group $SP(1)$.
The homogeneous space $SP(1)/\mathbb{T}^1$ is the sphere S^2.
Thus we obtain the stably complex $\mathbb{T}^1$-manifold (S^2, c_τ, θ).

The action θ has two fixed points p_1 and p_2 such that $\mathrm{sign}(p_1) = +1$, $\mathrm{sign}(p_2) = -1$, and $\Lambda(p_1) = \Lambda(p_2) = \big(1\big)$.

Thus for any genus $L_f : \Omega_U \to A$ we have

$$\Phi_f(S^2)(u) = \frac{1}{u} - \frac{1}{u} = 0.$$

Note that $[S^2] = 0$ for the given structure.

Almost compex structure on S^6.

Consider the smallest simple-connected exceptional simple Lie groups G_2. It has the rank 2 and dimension 14 and contains as subgroup $SU(3)$.

The homogeneous space $G_2/SU(3)$ is the sphere S^6 and admits G_2-invariant almost complex structure.

Thus we obtain the $\mathbb{T}^2$-equivariant almost complex SU-structure on S^6. This action has two fixed points p_1 and p_2
such that sign$(p_k) = +1$, $k = 1, 2$ and
$\Lambda(p_1) = \Big((1, 0), (0, 1), (-1, -1)\Big)$, $\Lambda(p_2) = \Big((-1, 0), (0, -1), (1, 1)\Big)$.

Toric genera of S^6.

For any genus $L_f : \Omega_U \to A$ we have

$$\Phi_f(S^6)(u,v) = \frac{1}{uvF(\bar{u},\bar{v})} + \frac{1}{\bar{u}\bar{v}F(u,v)} \in A[[u,v]],$$

$$\Phi_f(S^6)\Big(f(t),f(q)\Big) = \frac{1}{f(t)f(q)f(-t-q)} + \frac{1}{f(-t)f(-q)f(t+q)} \in HA[[t,q]]$$

Corollary. For any odd series $f(t)$ the genus L_f is rigid on S^6 with given structure.

For the Krichever genus L_{Kr} we have

$$f(t) = f_{Kr}(t) = \frac{e^{\frac{\lambda}{2}t}}{\Phi(t;\tau)},$$

where $\Phi(u;\tau)$ is the Baker-Akhiezer function.
By the addition theorem for the Weierstrass σ-function

$$\frac{\sigma(t+\tau)\sigma(\tau-t)}{\sigma(t)^2\sigma(\tau)^2} = \wp(t) - \wp(\tau)$$

we obtain

$$\Phi_f(S^6)(f(t), f(q)) = -\wp'(\tau).$$

It gives the direct proof that the Krichever genus is rigid on the almost complex SU-manifold S^6.

Theorem.
If an elliptic genus is rigid on S^6, then $\Phi_f(S^6) = 3\mu_3 - \mu_1\mu_2$.

An elliptic genus is rigid on S^6 if and only if

$$\begin{aligned} \mu_1\mu_4 &= \mu_3\mu_2, \\ 3\mu_1\mu_6 &= \mu_3\mu_4, \\ 9\mu_3\mu_6 &= \mu_3(\mu_1\mu_2\mu_3 - 3\mu_3^2 + \mu_2\mu_4). \end{aligned}$$

Corollary. The conditions of the theorem are equivalent to

$$\begin{aligned} \mu_1 = 0,\ \mu_3 = 0, &\quad \text{or} \\ \mu_3 = 0,\ \mu_4 = 0,\ \mu_6 = 0, &\quad \text{or} \\ \mu_2 = -\mu_1^2,\ \mu_4 = -\mu_1\mu_3,\ 3\mu_6 = -\mu_3^2, &\quad \text{or} \\ 3\mu_3 = \mu_1\mu_2,\ 3\mu_4 = \mu_2^2,\ 27\mu_6 = \mu_2^3. & \end{aligned}$$

Examples of elliptic genera, not rigid on S^6.

Let $\mu = (0, 0, \mu_3, 0, \mu_6)$ we get $g_2 = 0$, $g_3 = -\mu_3^2 - 4\mu_6$, then

$$\Phi_f(S^6)(f(t), f(q)) = 3\mu_3 + \frac{\mu_3(\mu_3^2 - 3g_3)}{4}\frac{1}{\wp(t)\wp(q)\wp(t+q)} =$$
$$= 3\mu_3 + \frac{1}{4}\mu_3(\mu_3^2 - 3g_3)t^2q^2(t+q)^2 + ...$$

The following example gives us a polynomial answer:

Let $\mu = (\mu_1, -\frac{1}{4}\mu_1^2, -\frac{1}{12}\mu_1^3, \frac{1}{24}\mu_1^4, -\frac{1}{576}\mu_1^6)$, then

$$\Phi_f(S^6)\Big(f(t), f(q)\Big) =$$
$$= -\frac{1}{6912}\mu_1^5\Big(\mu_1^4t^2q^2(t+q)^2 - 12\mu_1^2(t^2+tq+q^2)^2 + 288(q^2+tq+t^2)\Big).$$