Long-time behavior of solutions of the NLS equation with a δ-potential and even initial data

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The Gross-Pitaevskii (GP) equation

$$iu_t + \frac{1}{2}\Delta u + V(x)u + |u|^2 u = 0$$

arises as a model for a wide variety of phenomena in physics. In particular the ground state for GP with trapping potential $V(x)$,

$$u(x,t) = e^{iEt}U(x)$$

provides a model for the Bose-Einstein condensate. Exact solutions of GP are hard to come by, even in 1-d. Physicists have been led to consider 1-d GP with a δ-potential

$$iu_t + \frac{1}{2}u_{xx} + q\delta_0 u + |u|^2 u = 0 \tag{2.1}$$

and, following Zworski and Holmer, we

focus on the following question :

Is the BE condensate for (2.1) stable

under pert's of the initial data ?

The ground state stationary solu. of (2.1)

has the form

(3.1) $$u(x,t) = e^{i\lambda^2 t/2}\, v_\lambda(x)$$

$$v_\lambda(x) = \lambda \operatorname{sech}\left(\lambda|x| + \tanh^{-1}\left(\frac{q}{\lambda}\right)\right)$$

for any $\lambda > |q|$

Solutions of (2.1) corresp. to soln's of

NLS

(3.2) $$i u_t + \frac{1}{2} u_{xx} + |u|^2 u = 0 \quad , \quad x \neq 0$$

which are contin. in x and satisfy

(3.3) $$\frac{1}{2}\left[u_x(0+,t) - u_x(0-,t)\right] + q\, u(0,t) = 0$$

and if $u(x,t)$ is even in x, (3.3) reduces to

$$(4.1) \qquad u_x(0+,t) + q\, u(0+,t) = 0$$

In other words (3.1) with even init. data reduces to an IBVP

$$(4.2) \qquad \begin{aligned} &\cdot\ iu_t + \tfrac{1}{2}u_{xx} + |u|^2 u = 0, && x>0 \\ &\cdot\ u(x,t=0) = u_0(x), && x>0 \\ &\cdot\ u_x(0+,t) + q\, u(0+,t) = 0, && t \geq 0 \end{aligned}$$

This is the problem that we consider: is the BE condensate asymp. stable under even pert's of the init. data?

In order to get some perspective on this prob., we first consider the linear IBVP

$$(4.3) \qquad \begin{aligned} &\cdot\ iu_t + u_{xx} = 0, && x>0 \\ &\cdot\ u(x,0) = u_0(x) && x>0 \\ &\cdot\ u_x(0+,t) + q\, u(0+,t), && t \geq 0 \end{aligned}$$

There are two distinct approaches:

(i) <u>intrinsic approach</u>

Let $H_q^+ = -\dfrac{d^2}{dx^2}$ with domain of s.adj'ness

$$D(H_q^+) = \{u \in L^2(0,\infty) : u, u' \text{ abs. cont}$$
$$u'' \in L^2, \quad u'(0) + q u(0) = 0\}$$

For $q<0$: $\sigma(H_q^+) = (0,\infty) \equiv$ abs. cont. spec

$q>0$: $\sigma(H_q^+) = \underset{\text{abs. cont.}}{\underset{\uparrow}{(0,\infty)}} \cup \underset{\text{pt. spec}}{\underset{\uparrow}{\{-q^2\}}}$

Have solution formulae

$$q<0: \quad u(x,t) = \int_0^\infty e^{-itz^2/2}\, r(z)\, f_q(x,z)\, dz$$

$$q>0: \quad u(x,t) = \int_0^\infty e^{-itz^2/2}\, r(z)\, f_q(x,z)\, dz$$
$$+ 2q e^{itq^2/2} e^{-qx} \int_0^\infty e^{-qy} u_0(y)\, dy$$

where

$$f_q(x,z) = \frac{e^{ixz}}{iz+q} - \frac{e^{-ixz}}{-iz+q}, \quad z \in \mathbb{R}_+$$

(ii) <u>extrinsic approach</u> ("method of images")

Here we seek an extension $u_e(x,t)$ st

- $u_e(x,t)$ solves Schrod. eqtn on $\mathbb{R}\times\mathbb{R}_+$
- $u_e(x,t=0) = u_0(x) \quad , x>0$
- $w(x,t) \equiv (u_e)_x(x,t) + q\, u_e(x,t)$

$$= -w(-x,t) \qquad , x \in \mathbb{R}$$

Such a solu. <u>automatically</u> satisfies the Robin bc at $x=0$. By linearity & homogeneity, enough to require that $w(x,t)$ is odd for $t=0$; i.e. for $x>0$

$$u_e'(-x,0) + q\, u_e(-x,0) = -\left((u_e)'(x,0) + q\, u_e(x,0)\right)$$

Integrating $\Rightarrow$

(6.1) $$u_e(-x,0) = u_0(x) + 2q \int_0^x e^{q(x-s)} u_0(s)\, ds$$

for $x \overset{?}{\geq} 0$.

Set

$$u_e(x,0) = u_0(x) \quad , \quad x>0$$
$$= u_0(-x) + 2q\int_0^{-x} e^{-q(x+s)} u_0(s)\,ds\ ,$$
$$x<0$$

Easily see that $u_e(x,0)$ is C^1 and indeed gives rise, in principle, to a soln. of the IBVP for the Schröd. Eqtn. But there is a problem! If $q<0$ one sees from (6.1) that $u_e(x,0)$ decays as $x\to-\infty$ and so gives rise to a stand. prob. on $\mathbb{R}$ for which $\exists^{ce}$ and ! is easily established. Eventually one is led back to the soln. formula in terms of H_q^+ above. But for $q>0$, $u_e(x,t) \underset{x\to-\infty}{\sim} e^{q|x|}\int_{\mathbb{R}} e^{-qy}u_0(y)\,dy$, so that $u_e(x,0) \not\to 0$ generically as $x\to-\infty$. Thus method of images breaks down. Said differently, the soln. formula in terms of H_q^+ cannot be "unfolded" on the line with $u_e(x)$ suff. smooth and decaying.

It is useful to view (i) vs (ii) from a geometric pt. of view: method (i) involves only the "intrinsic geometry". Method (ii) attempts to imbed the problem into an "extrinsic geometry".

Now what about the non-linear problem? We know from the classic work of Zakharov and Shabat that NLS on the line (or more generally AKNS) is $\equiv$ to an iso-spec. deform. of the assoc. Lax oper.

$$L = iz\sigma + Q, \quad Q = \begin{pmatrix} 0 & u(x,t) \\ -\bar{u}(x,t) & 0 \end{pmatrix}, \quad \sigma = \frac{\sigma_3}{2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

(8.1) $$\text{NLS} \quad \equiv \quad (\partial_x - L)(\partial_t - E) = (\partial_t - E)(\partial_x - L)$$

where

$$E = -iz^2\sigma + \frac{1}{2}\begin{pmatrix} i|u|^2 & iu_x - zu \\ i\bar{u}_x + z\bar{u} & -i|u|^2 \end{pmatrix}$$

Questions (i) Is the IBVP (4.2) for NLS integrable?

(ii) can the prob. also be viewed geometrically? i.e. is there an "intrinsic" and also an "extrinsic" way of solving the problem?

Here I am taking "integrable" in the sense of the 17th century: can I express the soln., and in particular its long-time behavior, in terms of an explicit func. th' that I understand?

The answer to (i) is not at all clear a priori. As we see from the GP eqtn, the origin of the bdry cond. $u_x + qu|_{x=0+}$ is as a potential (corres. classly to a force) in the eqtn. Although the forces corres. to standard NLS are "integrable", it is by no means clear that the extended dyn. syst. including the force assoc. with $V = q\,\delta_0$ is integrable.

It turns out that, indeed, the IBVP (4.2) is integ. in the above class. sense and that, moreover, the prob. can be viewed geometrically. The intrinsic approach, in which one never utilizes, or mentions, the full line $\mathbb{R}$, is due to Fokas, and also, in terms of the current prob., Alexander Its. The extrinsic approach, a non-linear method of images, is due to Bikbaev and Tarasov, who in turn utilize ideas of Sklyanin. We follow the method of Bikbaev and Tarasov (it turns out that the specific IBVP at hand is non-generic & singular for the Fokas-Its approach).

How does the method work? Although the

method of images works in a str. forward way for the nonlin. Dirichlet and Neumann prob's, in the present Robin - bound. cond. situation there is no obvious analog of $w \equiv u_x + q u$. We do, however, have one powerful tool of integ. th^y at our disposal viz. the concept of a Backlund transformation, which enables us to construct new solutions of NLS out of old solutions in an algebraically contolled manner. The idea of Bik. Tarasov is:

(i) to construct a new soln. $\tilde{u}(x,t)$ out of $u(x,t)$, $x>0$, and

(ii) to transport this soln. to a soln of NLS on $x<0$

$$\tilde{u}(x,t) = -\tilde{u}(-x,t) \quad , \quad x<0$$

(iii) in such a way that

$$u_e(x,t) \equiv u(x,t)\ , \ x>0$$
$$\equiv -\tilde{u}(-x,t)\ , \ x<0$$

gives rise to a classical, smooth solution of NLS on the whole line with the property that the condition

$$(u_e)_x(0,t) + q\, u_e(0,t) = 0$$

is automatically satisfied. We call $u_e(x,t)$ the Backlund extension of $u(x,t)$ to $\mathbb{R}$.

As the essent. algebra in the construction is the Backlund trans., for which all the relev. quant's can be computed explicitly, we end up with a scatt./inverse scatt. prob. on $\mathbb{R}$ where the scatt. data is known explicitly in terms of the init. data. By the steepest-descent method for Riemann-Hilbert prob's, one can then compute the long-time Zakharov - Manakov - type behavior of the soln. to any desired accuracy ....

How does the Backlund trans. work?

<u>First</u> step is to transform, for each fixed t,

$$L = iz\sigma + Q, \qquad Q = \begin{pmatrix} 0 & u(x,t) \\ -\bar{u}(x,t) & 0 \end{pmatrix}$$

to

$$\tilde{L} = iz\sigma + \tilde{Q}, \qquad \tilde{Q} = \begin{pmatrix} 0 & \tilde{u}(x,t) \\ -\bar{\tilde{u}}(x,t) & 0 \end{pmatrix}$$

in such a way that $\partial_x - \tilde{L}$ evolves iso-spectrally. This is clearly achieved if L and $\tilde{L}$ are related by

$$(z+P)(\partial_x - L) = (\partial_x - \tilde{L})(z+P) \tag{13.1}$$

for some $P = P(x,t)$, for then

$$(\partial_x - \tilde{L}) = (z+P)(\partial_x - L)(z+P)^{-1}$$

and so

$$\begin{gathered} \partial_x - L(t) \quad \text{iso-spec.} \\ \Longleftrightarrow \\ \partial_x - \tilde{L}(t) \quad \text{iso-spec.} \end{gathered}$$

A str. forw. calc. shows that (13.7) $\Rightarrow$ $P = P(x,t)$ solves, for each fixed t, a dift. eqtn., itself an iso-spec. deformation

(14.1) $$\frac{\partial P}{\partial x} = [Q(x) - i\sigma P, P]$$

which has a genl. soln.

$$P(x,t) = \varphi(x,t)\, P_0(t)\, \varphi(x,t)^{-1}$$

for $P_0(t) = P(0,t)$, for some invert. matrix func. $\varphi(x,t)$, $\varphi(0,t) = I$. Diff. choices of $P_0(t)$ achieve diff. goals, giving rise to diff. iso-spec. deformations. For exple., if $\xi \in \mathbb{C}^+$, the choice $P_0(t) = -\begin{pmatrix} \xi & 0 \\ 0 & \bar{\xi} \end{pmatrix}$ gives rise to a soln of NLS with a bound state added in at $z = \xi$ and at $z = \bar{\xi}$. For our IBVP, it turns out that the relevant choice is

$$(15.1) \qquad P_0(t) = -iq\sigma_3 = \begin{pmatrix} -iq & 0 \\ 0 & iq \end{pmatrix}$$

It turns out, and this is the miracle of the matter (in integrable th^y, there is always a miracle, inexplicable and, at best, only intuited) is that with this choice, if $u(x,t)$ solves NLS, then $\tilde{L} = \partial_x - \tilde{Q}$ will undergo the iso-spec deform. corresp. to NLS i.e. $\tilde{u}(x,t)$ will solve NLS. Moreover it will produce the desired solu'n of our IBVP via the prescription

$$u_e(x,t) \equiv u(x,t) \quad , \; x>0$$
$$\equiv -\tilde{u}(-x,t) \quad , \; x<0$$

i.e. $u_e(x,t)$ is a smooth solution of NLS on $\mathbb{R}$, $u_e(x,0) = u(x,0)$ for $x>0$ and automatically

$$(u_e)_x(0,t) + q\, u_e(0,t) = 0$$

One more remark on the Back. trans.:

just a hint of why it works and how it fits together. If $\psi_{ZS} = (\psi_1^+, \psi_2^+)(x,t,z)$ are the Zak.-Shab. soln's for L i.e.

$$(\partial_x - L)\,\psi_{ZS} = 0$$

then (8.1) $\Rightarrow$

$$\frac{\partial}{\partial t}\begin{pmatrix} (\psi_1^+)_1 \\ (\psi_1^+)_2 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} |u|^2 & u_x + izu \\ \bar{u}_x - iz\bar{u} & z^2 + |u|^2 \end{pmatrix}\begin{pmatrix} (\psi_1^+)_1 \\ (\psi_1^+)_2 \end{pmatrix}$$

and we see that if $u_x(0,t) + q\, u(0,t) = 0$, or $\bar{u}_x(0,t) + q\,\bar{u}(0,t) = 0$, then $\bar{u}_x - iz\bar{u} = 0$ at $x = 0$ for $\boxed{z = iq}$ and $\forall t$. Hence if $(\psi_1^+)_2(0,t,iq) = 0$ for $t = 0$, it <u>remains true $\forall t$</u>. This fixed "boundary [redacted] condition" guarantees that $p_0(t) = -iq\,\sigma_3$ is the right choice!

<u>Comment</u>: It turns out that not only is the Backlund method a <u>non-lin analog</u> of the method of images, it is a <u>non-linearization</u> of <u>the</u> method.

So what are the <u>results</u>? It turns out that the BE condensate is asymptotically stable if $q>0$ and unstable (under even pert's) if $q<0$ i.e. stable if $q\,\delta_0 u$ and $|u|^2 u$ have the same sign, and unstable otherwise. More precisely, let

$$0 < |q| < \mu_0$$

and consider (2.1) with init. data

$$u(x,0) = v_{\mu_0}(x) + \epsilon\, w(x)$$

where $w(x)$ is even with bded norm

$\|w\|_{H^{1,1}} \le \epsilon$, $H^{1,1} = \{ w \in L^2(\mathbb{R}) :$ w abs. cont, $w' \in L^2$, $x\,w(x) \in L^2 \}$

Let $u_e(x,0)$ be the Back. ext. of

$u(x,0)\big|_{\mathbb{R}_+}$ to $\mathbb{R}$ and let $a(z)$, $z \in \overline{\mathbb{C}}_+$,

be the assoc. scatt. func. (trans. coeff.) Then for

some explicit $\chi_0 = \chi_0(\mu_0)$ and any

$$0 \le \varepsilon \le \chi_0 |q|^{1/2}$$

(i) $a(z)$ has one simple zero $i\mu_1$, $\mu_1 > 0$, if

$\boxed{q > 0}$, and

(ii) $a(z)$ has (generically) two simple zeros,

$i\mu_1$, $i\mu_2$, if $\boxed{q < 0}$, $\mu_1 > \mu_2 > 0$.

In case (i), $u(x,t) \sim$ 1 soliton soln. of

NLS at $t \to \infty$, the cont. spec. radiating

away,

$$u(x,t) \underset{t\to\infty}{\sim} e^{i(\mu_1^2 t/2 + p_1 + \ell_1)} v_{\mu_1}(x)$$

&

$\mu_1 = \mu_0 + O(\varepsilon)$, ℓ_1 and p_1 explic. consts of $O(\varepsilon)$

Thus we have stability of the condensate in the above sense if $\boxed{q>0}$. In case (ii)

$$u(x,t) \underset{t\to\infty}{\sim} \text{2 soliton soln. of NLS}$$

with 2 charac. freq's $e^{i\mu_1^2 t/2}$ and $e^{i\mu_2^2 t/2}$

where $\mu_1 = \mu_0 + O(\varepsilon)$, $\mu_2 = -q + O(\varepsilon^2)$.

This shows the instability of the condensate if $\boxed{q<0}$.

A technical remark: The analytical origin of the root $a(i\mu_2) = 0$ for $q<0$ is very unusual: this root comes out of "nowhere" as there is no root $a(-iq) = 0$ for $\varepsilon = 0$. It turns out that $a(z)$ is of the form $a(z) = \dfrac{G(z)}{z - i\beta}$,

where $\beta = \pm q$. Here $G(z)$ is anal. in $\mathbb{C}^+$: for $\varepsilon = 0$, one has $\beta = -q > 0$ and $G(i\beta) = 0$, $G'(i\beta) \neq 0$. Hence $a(i\beta) = a(-iq) \neq 0$. But when $\varepsilon > 0$, β flips and we have $\beta = q < 0$: by Rouché, $G(z) = G(z;\varepsilon)$ has a root $i\mu_2$ near $-iq$, $\mu_2 = -q + O(\varepsilon^2)$, but now $z - i\beta \neq 0$ near

$-q$ and so the root $a(i\mu_2)$ emerges, as it were, from "behind a cloak".

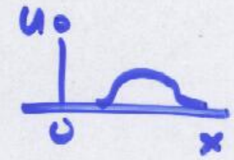
As mentioned earlier, we were led to our problem by earlier work of Holmer and Zworski. They considered (2.1) with even data in the case $\varepsilon = q$ $(\leq \gamma_0 |q|^{\frac{1}{2}})$. They used general dyn. system methods and did not utilize the integ. of the prob. at all. Their results were then limited in t, $t \leq c\,|q|^{-2/7}$. For such times they conclude that the BE condensate is stable, in the above sense, whether $q>0$ or $q<0$. It turns out from our calculations, however, that for $\varepsilon = q$, there is a critical time scale, $t \simeq |q|^{-2}$. For $t \ll |q|^{-2}$, the 2^{nd} soliton, corres. to $i\mu_2$, has a negligible effect. It only begins to contribute significantly to the solution for times $t \gg |q|^{-2}$. As $|q|^{-2} \gg |q|^{-2/7}$, this explains the calc's in Zworski-Holmer.

Finally, another technical comment (a puzzle!)

We only consider solutions in $H^{1,1}$: with more work we could consider solutions in $H^{k,k}$ for any $k \geq 1$, but there are no new phenomena.

Here is a natural question: suppose $u(x,t)$ solves the IBVP

- $iu_t + \frac{1}{2} u_{xx} + |u|^2 u = 0 \quad , x > 0, t > 0$
- $u(x,0) = u_0(x) \in C_0^{\infty}(0,\infty)$



- $u_x(0+,t) + \alpha\, u(0+,t) = 0 \quad , t > 0$
  $\alpha \in \mathbb{R} \setminus 0$.

Is $u(x,t)$ a classical solution? i.e. C^2 in x and C^1 in t? The only way I know how to prove this is to use the Backlund extension method. I have asked the "usual suspects", but no-one seems to know how to prove this fact using standard PDE techniques. What if $|u|^2 u$ is replaced by, say, $|u|^4 u$? This is a basic question in PDE!