Elliptic Calogero-Moser systems for crystallographic complex reflection groups.

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joint work with G. Felder, X. Ma, and A. Veselov,

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(J. of algebra, C. De Concini 60-th birthday volume)

1. Complex reflection groups.

Let W be a finite group acting faithfully on a f.d. complex vector space $\mathfrak{h}$. One says that W is a complex reflection group if it is generated by elements s such that $\mathrm{rk}(1-s)=1$.

Chevalley's theorem: W is a complex reflection group iff $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial algebra (or, equivalently, $S\mathfrak{h}^W$ is a polynomial algebra).

Irreducible complex reflection groups were classified by Shepard and Todd in 1954.

Examples: 1) Weyl groups (and more generally, Coxeter groups)

2) $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, $\mathfrak{h}=\mathbb{C}^n$.

2. Crystallographic complex reflection groups.

Definition. A complex reflection group W is <u>crystallographic</u> if it preserves a lattice L in $\mathfrak{h}$ (of rank $2\dim_{\mathbb{C}} \mathfrak{h} = \dim_{\mathbb{R}} \mathfrak{h}$). Classification of such groups is known (due to Popov).

Examples. 1) Weyl groups

2) $S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ with $r=1,2,3,4,6$ (we use triangular lattice for $r=3,6$ and square lattice for $r=4$).

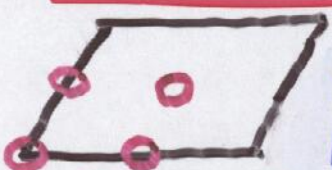
3. Elliptic Calogero-Moser systems.

Let W be a crystallographic complex reflection group preserving a lattice $L \subset \mathfrak{h}$. Then we can consider the complex torus $X = \mathfrak{h}/L$, which carries an action of W.

If W acts irreducibly on $\mathfrak{h}$, this torus is automatically an abelian variety (so let us assume this).

Let $s \in W$ be a complex reflection. Let X^s be the fixed set of s on X. This is a union of hypertori in X which are called reflection hypertori.

Example. 1) $W = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathfrak{h} = \mathbb{C}$.

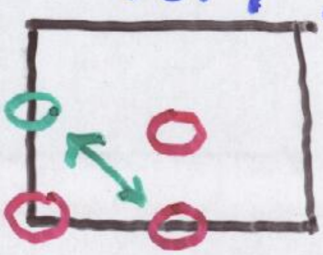


the 4 pts of order 2 are the reflection hypertori in $E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ (elliptic curve)

2) $W = \mathbb{Z}/3\mathbb{Z}$, $\mathfrak{h} = \mathbb{C}$

3 points are reflection hypertori.

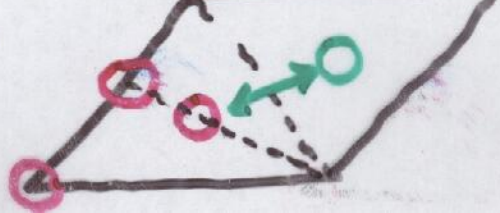
3) $W = \mathbb{Z}/4\mathbb{Z}$, $\mathfrak{h} = \mathbb{C}$



3 points are reflection hypertori.

4) $W = \mathbb{Z}/6\mathbb{Z}$, $\mathfrak{h} = \mathbb{C}$

3 points are reflection hypertori



If $T \subset X^5$ is a reflection hypertorus, then we can define the subgroup $W_T \subset W$ of elements which preserve all points of T. This subgroup is isomorphic to $\mathbb{Z}/m_T\mathbb{Z}$ for some $m_T \geq 2$. (order of T_T)

Let $\mathfrak{A}$ be the set of pairs (T, j), where T is a reflection hypertorus, and $1 \leq j \leq m_T - 1$ is an integer.

There is a natural action of W on $\mathfrak{A}$. Let $C = \mathbb{C}[\mathfrak{A}]^W$.

Example. 1) $W = \mathbb{Z}_2$ The orders of all the 4 points are 2, so $\dim C = 4$ (D4)

2) $W = \mathbb{Z}_3$. The orders of the 3 points are 3 so $\dim C = 6$ (E6)

3) $W = \mathbb{Z}_4$. The orders of the 3 pts are 2, 4, 4, so $\dim C = 7$ (E7)

4) $W = \mathbb{Z}_6$; The orders of the 3 pts are 2, 3, 6, so $\dim C = 8$ (E8)

Main result. (with Felder, Ma, Veselov) 2010

There exists a family of classical and quantum integrable systems, parametrized by $c \in \mathbb{C}$, defined on the abelian variety X, whose symbols are generators of the polynomial algebra $(S\mathfrak{h})^W$, $P_1, \ldots, P_n$ (where $n = \dim \mathfrak{h}$). These systems have meromorphic coefficients on X with poles on reflection hypertori. The quantum systems have regular singularities, and their monodromy gives rise to representations of generalized Cherednik (= double affine Hecke) algebras.

Examples. 1) W is a Weyl group, $\mathfrak{h}$ its standard representation.

In this case the Hamiltonian of the quantum system is

$$H = \Delta - \sum_{\alpha>0} c_\alpha (c_\alpha + 1)(\alpha,\alpha)\wp((\alpha,x),\tau)$$

on $X = \mathfrak{h}/L = E_\tau \otimes P^\vee$ ← dual weight lattice

This is the elliptic Calogero-Moser system.

2) $W = S_n \ltimes \mathbb{Z}_2^n$, $\mathfrak{h} = \mathbb{C}^n$, $L = \mathbb{Z}^n \oplus \tau\mathbb{Z}^n$, $X = E_\tau^n$. The Hamiltonian of the quantum system depends on 5 parameters and is given by

$$H = \Delta - \sum_{i \neq j} k(k+1)\left(\wp(x_i - x_j;\tau) + \wp(x_i + x_j;\tau)\right)$$

$$- \sum_{e=1}^{4} \sum_{j=1}^{n} c_e(c_e+1)\wp(x_j - \xi_e, \tau),$$

where ξ_e are the points of order 2.

This is the Inozemtsev system, which is a deformation of example 1 (where we had 2 parameters).

In the $n=1$ case there are 4 parameters, and we get the Darboux operator, which is a 4-parameter generalization of the Lamé operator

$$L = D^2 - m(m+1)\wp(x,\tau)$$

3) **New example.** Let $W = S_n \ltimes \mathbb{Z}_r^n$ where $r=3,4,6$. Then $X = E_\tau^n$, where E_τ is an elliptic curve with special symmetry. Then the basic invariants of W are $\Sigma p_i^r, \Sigma p_i^{2r}, \dots, \Sigma p_i^{nr}$. So the lowest hamiltonian has symbol Σp_i^r. Let us give the explicit expression for $r=3$.

In this case we have seven parameters $k, a_0, b_0, a_1, b_1, a_2, b_2$, and the Hamiltonian looks like

$$H=\sum_{i=1}^{n}\partial_i^3+\sum_{i=1}^{n}\left(a_0\wp(x_i)+a_1\wp(x_i-\eta_1)\right.$$
$$\left.+a_2\wp(x_i-\eta_2)\right)\partial_i$$
$$-3k(k+1)\sum_{i<j}\sum_{p=0}^{2}\wp(x_i-\varepsilon^p x_j)(\partial_i+\varepsilon^{-p}\partial_j)$$
$$+\sum_{i=1}^{n}\left(b_0\wp'(x_i)+b_1\wp'(x_i-\eta_1)+b_2\wp'(x_i-\eta_2)\right)$$

where $\tau=\varepsilon=e^{2\pi i/3}$, $\wp(x)=\wp(x,\tau)$, $\eta_1=i\sqrt{3}/3$, $\eta_2=-i\sqrt{3}/3$.

4. Construction of crystallographic elliptic Calogero-Moser systems.

We will use the method of Buchstaber - Felder Veselov, closely related to the method of Cherednik. This method is based on the notion of Elliptic Dunkl Operators.

BFV (E-Ma)

4a. Elliptic Dunkl Operators.

Fix a generic line bundle $\mathcal{L} \in \mathrm{Pic}_0(\mathfrak{X})$ (i.e., such that $\mathcal{L} \not\cong {}^g\mathcal{L}$ $\forall g \in W$).

For every $(T, j) \in \mathfrak{S}$, there is a global meromorphic section $f^{\mathcal{L}}_{T,j}$ of the vector bundle

$$\left(s_T^j \mathcal{L}\right)^* \otimes \mathcal{L} \otimes \mathfrak{h}_T^*$$

(where s_T is the generator of W_T which acts on the normal bundle of T with eigenvalue $e^{2\pi i/m_T}$, and $\mathfrak{h}_T$ is the tangent space to T).

which has a simple pole at T with residue 1 and no other singularities (residue is well defined since we can interpret sections of this bundle as 1-forms, as $\mathfrak{h}_T^* \subseteq \mathfrak{h}^*$).

$(\mathfrak{h} = \mathfrak{h}_T \oplus N_T)$

Let c be a W-invariant function on $\mathfrak{S}$. Let $v \in \mathfrak{h}$. let ∇ be the flat unitary connection on $\mathcal{L}$.

Definition. The elliptic Dunkl operator is

$$D^{\mathcal{L}}_{v,c} \overset{\mathrm{def}}{=} \nabla_v + \sum_{(T,j) \in \mathfrak{S}} c(T,j)\left(f^{\mathcal{L}}_{T,j}, v\right) s_T^j.$$

Example. Let W be a Weyl group. Then the elliptic Dunkl operators are the operators defined by BFV:

$$D^{\lambda}_{v,c} = \partial_v + \sum_{\alpha>0} c_\alpha (\alpha, v)\, \sigma_{(\alpha^\vee,\lambda)}(\alpha)\, s_\alpha, \qquad \lambda \in \mathfrak{g}^*.$$

where
$$\sigma_\mu(z) = \frac{\theta(z-\mu)\,\theta'(0)}{\theta(z)\,\theta(-\mu)}, \qquad \theta = \theta_1(z,\tau).$$

The elliptic Dunkl operators act on sections of $\mathcal{L}$.

Theorem. (BFV, E.–Ma)

1) $[D^{\alpha}_{v,c}, D^{\alpha}_{v',c}] = 0$

2) $g \circ D^{\alpha}_{v,c} \circ g^{-1} = D^{g\alpha}_{gv,c}$.

This gives rise to the idea of proof of the main result which goes back to BFV. The idea is (by analogy with Heckman, Dunkl, Cherednik...) to construct the integrals of the elliptic crystallographic CM system by applying the symmetric polynomials P_i to the ELLIPTIC DUNKL OPER.

Reminder on classical Dunkl and Calogero-Moser operators.

$S \subset W$ set of reflections

$c: S \to \mathbb{C}$ is a W-invariant function.

$$D^0_{v,c} = \partial_v + \sum_{s \in S} \frac{2c(s)\alpha_s(v)}{(1-\lambda_s)\alpha_s} s,$$

where λ_s is the nontrivial eigenvalue of s on $\mathfrak{h}^*$, and α_s are the roots corresponding to s.

$$P_i^c(p,\lambda) = m\left(P_i(D^0_{\cdot,c})\right),$$

where $m\left(\sum \gamma_g g\right) = \sum \gamma_g$.

<u>Remark</u>. In fact don't need m (as $P_i^c(p,\lambda)$ is already a function). But this is not obvious and not true in the quantum case.

However, there is a problem with this approach: the elliptic Dunkl operators depend on the line bundle $\mathcal{L}$ and are not W-invariant: under the action of w, $\mathcal{L}$ goes to $g\mathcal{L}$. They have a chance of becoming W-invariant only when $\mathcal{L}$ tends to the trivial bundle, but in this limit they develop a singularity (recall that $\mathcal{L}$ had to be "generic"). So the idea has to be modified, so that we are able to "subtract" these singularities and successfully pass to the limit $\mathcal{L} \to$ trivial bundle. To this end, it turns out that classical rational Calogero-Moser system comes handy.

5. To explain this method, let us parametrize line bundles $\mathcal{L}$ on X by points $\lambda \in \mathfrak{h}$, identifying

identifying $\mathfrak{h}$ with its Hermitian dual $\mathfrak{h}^\vee$ by means of a W-invariant positive Hermitian inner product B on $\mathfrak{h}$ (if W is irreducible, this inner product is unique up to scaling). We will write this parametrization as $\lambda \mapsto \delta_\lambda$, and denote $D^{\alpha}_{v,c}$ by $D^{\lambda}_{v,c}$. Also let $P_i^c(p,\lambda)$ be the rational classical Calogero-Moser Hamiltonians attached to $(W,\mathfrak{h})$ and parameter c (which is a function on the set of reflections in W invariant under conjugation).

Theorem 1) For an appropriate linear function $c = c(C)$, the operators

$$L_i^{c,\lambda} = P_i^{c(C)}(D^{\lambda}_{\cdot,C}, \lambda)$$

are holomorphic near $\lambda = 0$.

2) The operators $\overline{L}_i^c = L_i^{c,0}$ a W-invariant pairwise commuting elements of $\mathbb{C}W \ltimes D(X_{reg})$, where X_{reg} is the complement of reflection hypertori in X.

3) The restrictions L_i^c of $\overline{L}_i^c$ to W-invariant méromorphic functions on X are commuting differential operators on X_{reg}, whose symbols are the polynomials P_i.

The operators L_i^c are the Hamiltonians of the crystallographic elliptic Calogero-Moser system.

The classical version of this system is constructed by the classical version of this construction.

6. A geometric construction of the same system (in the style of Beilinson & Drinfeld).

Let X be a complex alg. variety, and W a finite group acting faithfully on X. For $g \in W$, consider the fixed set X^g. If $Y \subset X^g$ is a connected component, it is called a reflection hypersurface. Given Y, we have a cyclic group $W_Y \subset W$ of elements fixing Y pointwise, whose order we will call n_Y. Let $\mathcal{S}$ be the set of pairs (Y, j), where Y is a reflection hypersurface, and $j = 1, .., n_Y - 1$. Let c be a W-invariant function on $\mathcal{S}$, and let ψ be a twisting for differential operators on X (as in Beilinson-Bernstein

In 2004, to this data I attached a quasicoherent sheaf of algebras on X/W, denoted $H_{c,\psi,W,X}$.
On affine open sets U, the algebras of sections $H_{c,\psi,W,X}(U)$ are generated by $\mathcal{O}_X$, W, and "generalized dunkl operators". Also, defining $e = \frac{1}{|W|} \sum_{g \in W} g$, we define a sheaf $eH_{c,\psi,W,X}e$. These sheaves are called the sheaf of Cherednik algebras and the sheaf of spherical Cherednik algebras.

Remark. 1) $H_{0,\psi,W,X} = \mathbb{C}W \ltimes \mathcal{D}_\psi(X)$;
2) $H_{c,\psi,W,X} = \mathbb{C}W \ltimes \mathcal{D}_\psi(X)$ after localization on the reflection hypersurfaces;

3) If $X = \mathfrak{h}$ is a linear representation of W then $H_{c,\psi,W,X}(X) = H_c(W,\mathfrak{h})$ is the usual rational Cherednik algebra.

4) If U is the formal neighborhood of a point $\bar{x} \in X/W$ then $H_{c,\psi,W,X}(U)$ is Morita equivalent to the rational Cherednik algebra of the stabilizer W_x of a preimage $x \in X$ of $\bar{x}$, acting on T_xX.

Now let us come back to the situation when X is an abelian variety, and W an irreducible crystallographic reflection group. In this case twistings ψ lie in $H^2(X)^W = \mathbb{C}\omega$, where ω is the Kähler form.

Theorem. There is a linear function $\ell : \mathbb{C}[\mathfrak{A}]^W \to \mathbb{C}\cdot\omega$ such that:
$\Gamma(eH_{\psi,c,w,x}e) = \mathbb{C}$ unless
$\psi = \ell(c)$ ("critical level condition")
In this case,
$\Gamma(eH_{\psi,c,W,x}e)$ is a polynomial algebra in n generators.

Evaluating this algebra in $\mathbb{C}W \ltimes \mathcal{D}_\psi(X_{reg})$ after localization, we get the elliptic crystallographic CM system.

This is analogous to the Beilinson & Drinfeld construction of the quantum Hitchin system in the Geometric Langlands theory:

quantum Hitchin $= \Gamma(\mathcal{D}_{-h^\vee}(Bun_G(X)))$

7. Applications to finite-gap operators (with E. Rains).

From the work of Airault, McKean, and Moser it's known that elliptic finite-gap potentials are parametrized by critical points of elliptic Calogero-Moser Hamiltonians. More generally, Krichever showed that elliptic CM flow provides elliptic solutions of the KP hierarchy. This application is toward extension of these results to operators with symmetries.

Let E be a special elliptic curve with $\mathbb{Z}_r$ symmetry, $r=2,3,4,6$. Let L be a differential operator on E of the form

$$\partial^r + a_{r-2}(z)\partial^{r-2} + \dots + a_0(z)$$

which is $\mathbb{Z}_r$-invariant, has poles at the fixed points η_i of $\mathbb{Z}_r$ and also some points $z_1,..,z_N$ (taken from distinct $\mathbb{Z}_r$-orbits) to-gether with their images. Let r_j be the order of stabilizer of η_j, and let L behave near η_j as L_j^{r/r_j}, where L_j is a homogeneous operator with any integer indices. Also let us fix the indices at the poles $z_1,..,z_N$ to be the "smallest interesting case": $-1, 1, .., e-2, e$.

<u>Conjecture</u>. Such operators correspond to critical points of the elliptic crystallographic CM lowest Hamiltonian.

For $r=2$, this is classical (Inozemtsev system). For $r=3$, this is checked in my paper with Rains. Open for $r=4,6$.

Generalization to non-integer indices at η_i: require trivial monodromy only around z_j, $j=1,..,N$ for $L\psi=\lambda\psi$.

Generalization to higher Hamiltonians:

Consider operators of order nr with special conditions at poles.

Also have rational degenerations of these conjectures which make sense for any $r\in\mathbb{N}$.

Happy birthday,
Igor Moiseevich!