Quantum Entanglement and the Finite-Gap Integration

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• Entanglement in XY spin chain

1.1. The model.

$$H_{XY} = - \sum_{n=-N}^{N} (1+\gamma)\,\sigma_n^x \sigma_{n+1}^x + (1-\gamma)\,\sigma_n^y \sigma_{n+1}^y + h\,\sigma_n^z$$

$\gamma > 0 \quad , \quad h \geq 0$

$\mathcal{H} = (\mathbb{C}^2)^{\otimes N} \qquad \sigma_n = I \otimes \cdots \otimes \sigma \otimes \cdots \otimes I$

$\sigma^x \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^y \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\sigma^z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{N = \infty}$$

The model was solved in 60-70 by:

E. Lieb, T. Schultz, D. Mattis

E. Barouch, B.M. McCoy, M. Dresden

The correlation functions has been studied since 1976 work of

B. McCoy, C. Tracy, T. Wu

1.2. The density matrix.

$T = 0 \Rightarrow$ the system is at its ground state, Ψ_g

$$\Downarrow$$

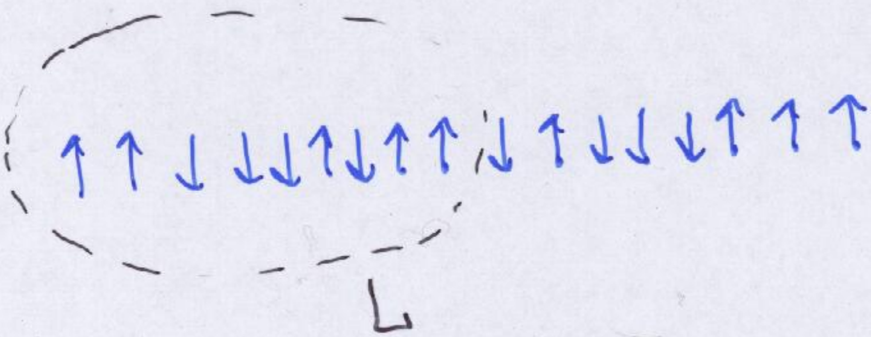
The density matrix $\rho = |\Psi_g\rangle\langle\Psi_g|$:

$\forall A \in \mathcal{L}(\mathcal{H})$ (the algebra of observables)

$$\langle A \rangle_{T=0} = \operatorname{trace} \rho A = \langle \Psi_g | A | \Psi_g \rangle$$

1.3. Reduced density matrix of a subsystem.

Subsystem of L neighboring spins:



The reduced density matrix of the block of the L spins is defined as

$$\rho_L := \mathrm{Trace}_{N-L}\, \rho$$

Note ! :

if $\Psi_g = \Psi_g^L \otimes \Psi_g^{N-L}$, i.e. the block is

unentangled from the rest of the chain,

$$\rho_L = |\Psi_g^L\rangle\langle\Psi_g^L|,$$

$$\text{Spectr}\,\rho_L = \{0, 1\}$$

in general,

$$\text{Spect}\,\rho_L = \{\lambda_i\}_{i=1}^{M}, \quad \lambda_i \geq 0$$

$$\lambda_1 + \lambda_2 + \ldots + \lambda_M = 1.$$

1.4. The von Neumann entropy as a measure of entanglement.

$$S_L = -\mathrm{Trace}\,(\rho_L \ln \rho_L)$$

$$(N = \infty)$$

Properties:

- If the block is unentangled, $S_L = 0$
- If $S_L > 0 \Rightarrow \rho_L$ is not a projection

$\Rightarrow$ the block is entangled to the rest of the chain

$$\Downarrow$$

S_L - a measure of entanglement

C. Bennett, H. Bernstein, S. Popescu, B. Schumacher (1996)

1.5. The Renyi Entropy.

$$\boxed{S_L(\alpha) = \frac{1}{1-\alpha} \ln \mathrm{Tr}\,(\rho_L^{\alpha}) \qquad \alpha > 0\,,\ \alpha \neq 1}$$

Note: $\lim_{\alpha \to 1} S_L(\alpha) = S_L$ - von Neumann

A. Rényi (1970), S. Lloyd (1993, Science 261)

S. Abe, A.K. Rajagopal (1999)

C. Bennett, D. DiVincenzo (2000, Nature 404)

H. Brandt (2006, Quantum Information and Computation IV, Proc. SPIE, v.6244)

(know the trace of ρ_L^{α}, $\forall \alpha$ $\Rightarrow$

know the state of the mixed quantum system)

Main question:

$$S_L, \; S_L(\alpha) \quad \text{for } L = \infty$$

- $\gamma = 0$ (XX0): B.-Q. Jin, V.E. Korepin (2004)
- $\gamma > 0$ $\quad S_\infty$: B.-Q. Jin, V.E. Korepin, A.R.I; I. Peschel (2004–2005)
- $\gamma > 0$ $\quad S_\infty(\alpha)$: F. Franchini, B.-Q. Jin, V.E. Korepin, A.R. Its

(G. Vidal, J. Latorre, E. Rico, A. Kitaev, 2003, conjecture: $S_\infty > 0$)

- Generalization: The Keating - Mezzadri chains.

$$H_{KM} = -\frac{\delta}{2} \sum_{0 \le j \le k \le N}' \Big[(A_{j-k} + \gamma B_{j-k}) \sigma_j^x \sigma_k^x \Big(\prod_{l=j+1}^{k-1} \sigma_l^z \Big)$$

$$+ (A_{j-k} - \gamma B_{j-k}) \sigma_j^y \sigma_k^y \Big(\prod_{l=j+1}^{k-1} \sigma_l^z \Big) \Big]$$

$$- \sum_{j=0}^{N}' \sigma_j^z$$

(quadratic in σ, translation invariant, solvable)

S_∞ : F. Mezzadri, M. Mo, A. I. (2008)

- Evaluation of the Renyi entropy. Block Toeplitz determinants and their asymptotics.

2.1 Starting formula.

$$S_L(\alpha) = \frac{1}{1-\alpha} \sum_{k=1}^{L} \ln \left[\left(\frac{1+\nu_k}{2}\right)^{\alpha} + \left(\frac{1-\nu_k}{2}\right)^{\alpha} \right]$$

$\pm i\nu_k$ - zeros of $D_L(\lambda) = \det T_L[\Phi]$

$$T_L = \left\{ \Phi_{j-k} \right\}_{j,k=0,\ldots,L-1}$$

$$\Phi_l = \frac{1}{2\pi i} \oint_{|z|=1} \Phi(z) \frac{dz}{z^{l+1}}$$

$$\Phi(z) \equiv \Phi(z;\lambda) = i\lambda I_2 + \Phi(z;0) :$$

$$\Psi(z) = \begin{pmatrix} i\lambda & \varphi(z) \\ -\varphi^{-1}(z) & i\lambda \end{pmatrix}, \quad \varphi(z) = \left(\frac{\bar{z}_1}{\bar{z}_2} \, \frac{(z - \bar{z}_1^{-1})(z - z_2)}{(z - \bar{z}_2^{-1})(z - z_1)} \right)^{1/2}$$

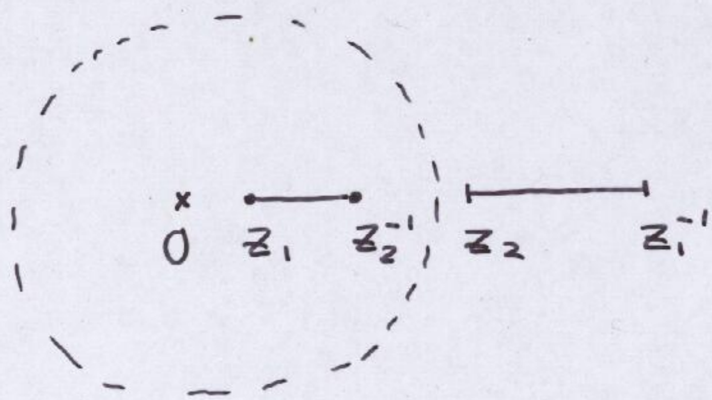
Cas 1a : $4(1-\gamma^2) < h^2 < 4$

Case 1b : $h^2 < 4(1-\gamma^2)$

Case 2 : $h > 2$

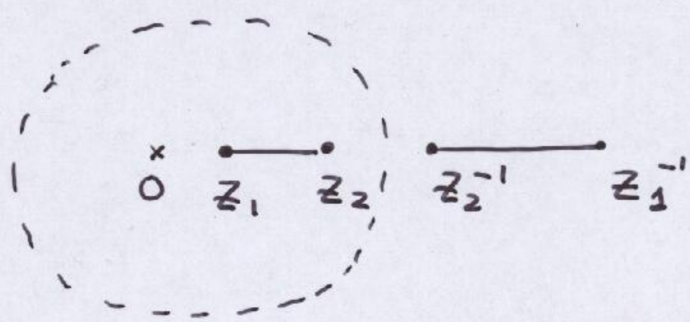
$$\left.\begin{aligned} z_1 &= \frac{h - \sqrt{h^2 - 4(1-\gamma^2)}}{2(1+\gamma)} \\ z_2 &= \frac{1+\gamma}{1-\gamma} \, z_1 \end{aligned}\right\} \quad \text{case 1a, 2}$$

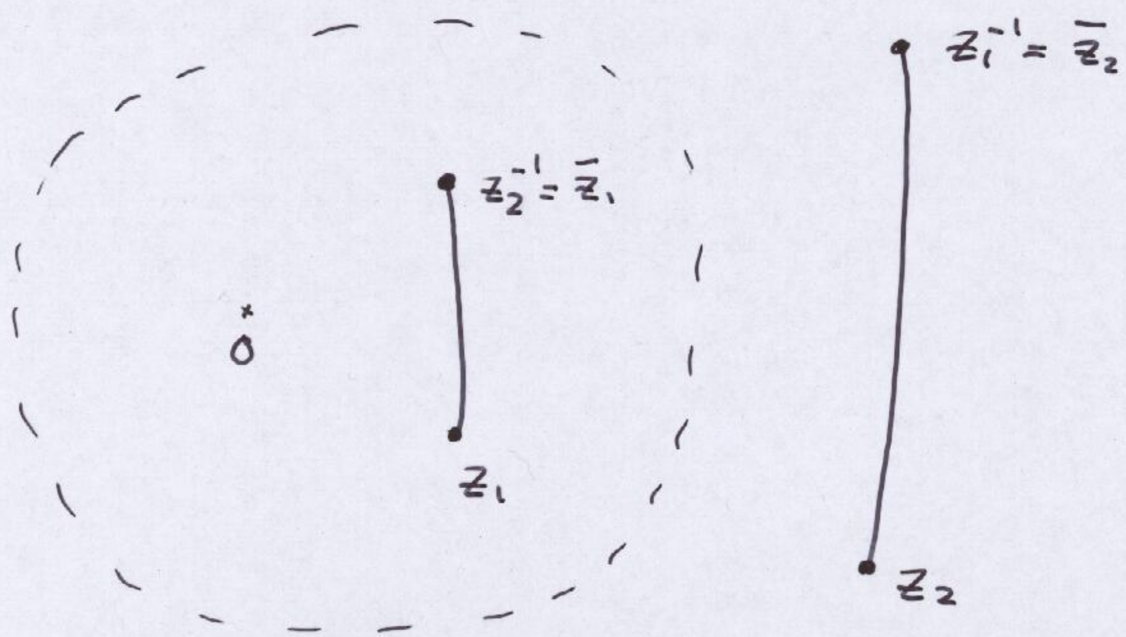
$$\left.\begin{aligned} z_1 &= \frac{h - i\sqrt{4(1-\gamma^2) - h^2}}{2(1+\gamma)} \\ z_2 &= 1/\bar{z}_1 \end{aligned}\right\} \quad \text{Case 1b.}$$



Case 1a

Case 2





Case 1b

B.-G. Jin & V. Korepin – based on

J. Latorre, E. Rico, J. Vidal (2003)

2.2 Widom's Theorem. (1974)

$$\boxed{D_L[\Phi] \sim G^L E} \quad L \to \infty$$

$$G[\Phi] = \exp\left\{ \frac{1}{2\pi i} \oint_{|z|=1} \ln \det \Phi(z) \, \frac{dz}{z} \right\}$$

$$E[\Phi] = \det\left(T_\infty[\Phi] \, T_\infty[\Phi^{-1}] \right)$$

$$\left(\begin{array}{c} \sum |\Phi_k| + |k| \, |\Phi_k|^2 < \infty \\ \det \Phi \neq 0 \qquad \Delta \arg \det \Phi = 0 \end{array} \right)$$

"Intermediate" Widom's theorem:

$$\Phi(z) = i\lambda I + \Phi_0(z)$$

$$\frac{d}{d\lambda}\ln D_L(\lambda) \sim L\frac{d}{d\lambda}\oint_{|z|=1} \ln\det\Phi(z)\frac{dz}{2\pi i z}$$

$$+\frac{1}{2\pi}\oint_{|z|=1} \operatorname{trace}\left(\frac{dU_+}{dz}U_+^{-1}(z) + V_+^{-1}(z)\frac{dV_+}{dz}\right)\Phi^{-1}(z)\,dz$$

$$\Phi(z) = U_+(z)U_-(z) = V_-(z)V_+(z) \; -$$

- canonical Wiener-Hopf factorization

(re-discovered by B.-Q. Jin, V. E. Korepin and A. R. Its in 2004 ....)

via the Integrable Kernels & Riemann-Hilbert techniques

- A "Finite-Gap" approach to the Wiener-Hopf factorization.

$$\Phi(z) = \begin{pmatrix} i\lambda & \varphi(z) \\ -\varphi^{-1}(z) & i\lambda \end{pmatrix}$$

$$\varphi^2(z) = \prod_{j=1}^{2n} \frac{z - z_j}{1 - z_j z} \qquad n \geq 1$$

$$\{ z_1, \dots, z_{2n}, z_1^{-1}, \dots, z_{2n}^{-1} \}$$

$$\equiv \{ \lambda_1, \lambda_2, \dots, \lambda_{4n} \}$$

$$|\lambda_i| < 1 \qquad i = 1 \dots 2n$$

$$|\lambda_i| > 1 \qquad i = 2n+1, \dots, 4n$$

$n = 1$ - XY chain

$n > 1$ - Keating-Mezzadri chain

The question is to find

$U_+(z)$ - analytic , $|z| < 1$

$U_-(z)$ - analytic , $|z| > 1$

$$U_-(\infty) = I$$

such that

$$\boxed{\Phi(z) = U_+(z)U_-(z) , \quad |z| = 1.}$$

Step 1.

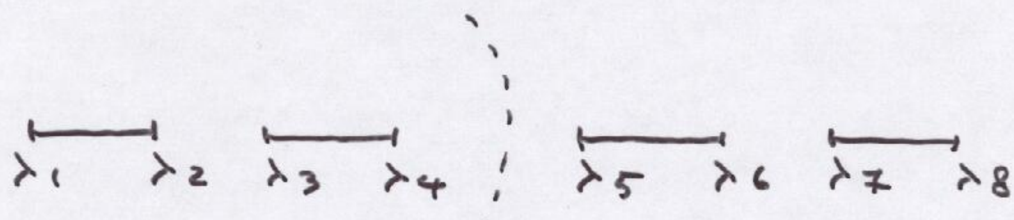
$$\Psi(z) = Q(z)\,\Lambda\,Q^{-1}(z)$$

$$Q(z) = \begin{pmatrix} \varphi(z) & -\varphi(z) \\ i & i \end{pmatrix} \qquad \Lambda = i\begin{pmatrix} \lambda+1 & 0 \\ 0 & \lambda-1 \end{pmatrix}$$

Note:

- $Q(z) \in H(\mathbb{C} \setminus \Sigma)$

$$\Sigma = \bigcup_{j=1}^{2n} [\lambda_{2j-1}, \lambda_{2j}] \equiv \bigcup_{j=1}^{2n} \Sigma_j$$



- $Q_+(z) = Q_-(z)\,\sigma_1 \qquad z \in \Sigma$

- $Q(z) \longmapsto Q(\infty) \quad , \; z \to \infty$

Remark:

$$Q(z) = Q_k(z) \begin{pmatrix} (z-\lambda_k)^{\pm 1/2} & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$Q_k(z)$ - holomorphic at $z = \lambda_k$

"+" if $\lambda_k = z$;

"−" if $\lambda_k = z^{-1}$;

Step 2.

$$Q \Lambda Q^{-1} = U_+ U_- \Rightarrow U_+^{-1} Q = U_- Q \Lambda^{-1}$$

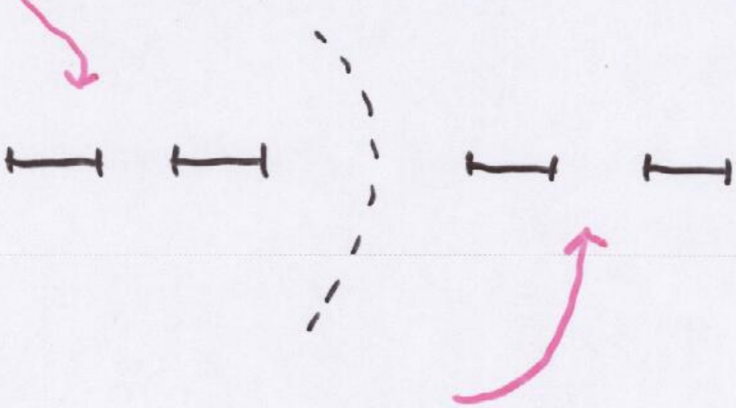
$$\Downarrow$$

$$S(z) := \begin{cases} U_-(z) Q(z) \Lambda^{-1} & |z| > 1 \\ U_+^{-1}(z) Q(z) & |z| < 1 \end{cases}$$

- $S(z) \in H(\mathbb{C} \setminus \Sigma')$

- $S_+(z) = S_-(z)\, \mathcal{J}_1$,

  $z \in \Sigma'_j$, $j = 1, \dots, n$



- $S_+(z) = S_-(z)\, \Lambda\, \mathcal{J}_1\, \Lambda^{-1}$

$$= S_-(z) \begin{pmatrix} 0 & \frac{\lambda+1}{\lambda-1} \\ \frac{\lambda-1}{\lambda+1} & 0 \end{pmatrix}$$

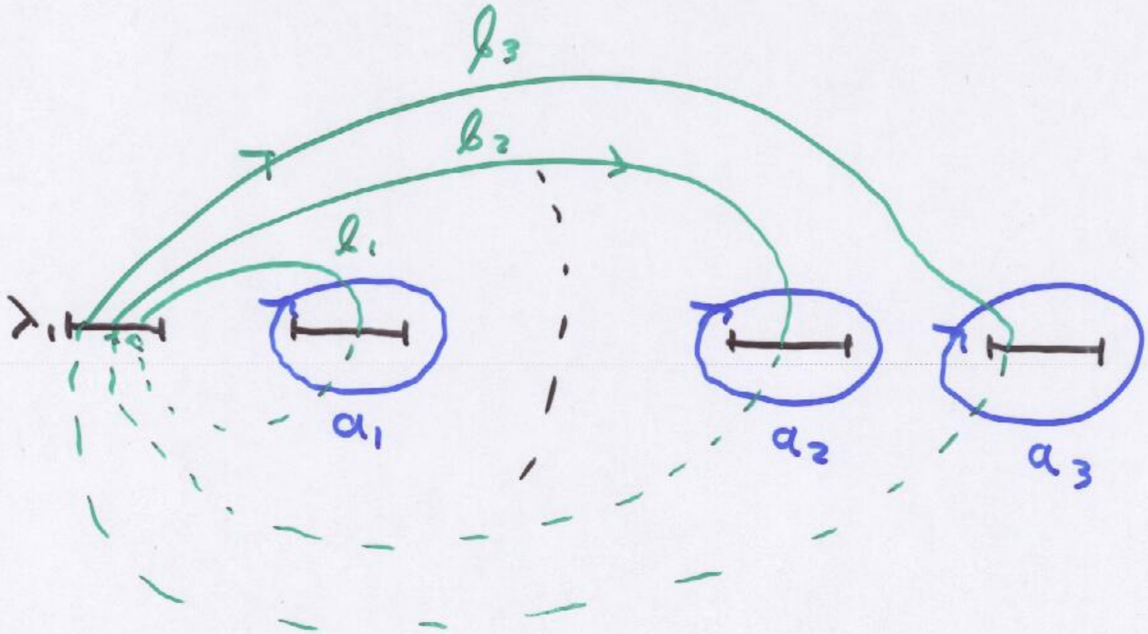
  $z \in \Sigma'_j$, $j = n+1, \dots, 2n$

- $S(\infty) = Q(\infty)\, \Lambda^{-1}$

Step 3.

$$X = \Big\{ P = (z, w) : \qquad w^2 = \prod_{j=1}^{4n} (z - \lambda_j) \Big\}$$

$g = 2n-1$



$$\omega_j = \int_{\lambda_1}^{P} d\omega_j \quad , \quad \int_{a_j} d\omega_k = \delta_{jk}$$

$$B_{jk} = \int_{b_j} d\omega_k$$

"Krichever's ghost":

$$\Omega(P) = \int_{\lambda_1}^{P} d\Omega \quad \text{- second kind}$$

$$\Omega(P) = \pm\,\Omega_\infty(z) + o(1)$$

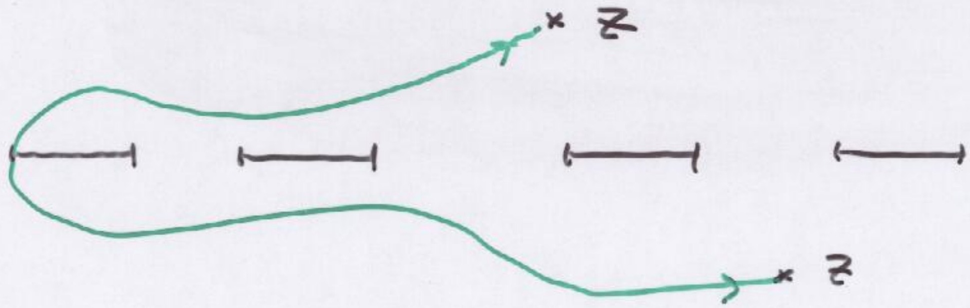
$$P \to \infty^{\pm}, \quad \Omega_\infty(z) = \sum_{k=1}^{m}{}' \, t_k z^k$$

$$\int_{a_j} d\Omega = 0 \quad , \quad \int_{b_j} d\Omega = 0$$

$$j = 1, \dots, 2n-1 \qquad j = 1, \dots, n-1$$

$$\int_{b_j} d\Omega = \ln \frac{\lambda - 1}{\lambda + 1}, \quad j = n, \dots, 2n-1.$$

$$S(z) \longmapsto \Psi(z) := S(z)\, e^{\Omega(z)\sigma_3}$$



- $\Psi(z) \in H(\mathbb{C} \setminus \Sigma')$

- $\Psi_+(z) = \Psi_-(z)\,\sigma_1 \qquad z \in \Sigma'$

- $\Psi(z) \underset{z\to\infty}{=} Q(\infty)\Lambda^{-1}\left(I + O(1/z)\right) \times e^{\Omega_\infty(z)\sigma_3}$

$$\Downarrow$$

$$\Psi(z) = \left(\vec{\Psi}(P), \vec{\Psi}(P^*) \right)$$

$$\pi(P) = \pi(P^*) = z$$

- $\vec{\Psi}(P)$ - meromorphic on $X \setminus \{\infty^\pm\}$

- $\left(\vec{\Psi}(P)\right)_{poles} = \sum_{j=1}^{2n} z_j^{-1}$

- $\vec{\Psi}(P) \longrightarrow \vec{C}_\pm \, e^{\pm \Omega_\infty(z)}$

$$P \to \infty^\pm$$

$\vec{\Psi}(P) \equiv$ a Baker-Akhiezer function

can be written down

in terms of Riemann theta-functions

$$\Theta(\vec{z}) \equiv \Theta(\vec{z} \mid B)$$

$$\Downarrow$$

$$D_L(\lambda) \simeq (1-\lambda^2)^L$$

$$\times \frac{\Theta\left(\frac{1}{2\pi i}\ln\frac{\lambda+1}{\lambda-1}\vec{e} + \tau/2\right)\Theta\left(\frac{1}{2\pi i}\ln\frac{\lambda+1}{\lambda-1}\vec{e} - \tau/2\right)}{\Theta^2(\tau/2)}$$

$$\vec{e} = (\underbrace{0,0,\dots,0}_{n-1},1,\dots,1)$$

$$\tau/2 = -\sum_{j=1}^{2n} \omega(z_j^{-1}) - K$$

$n = 1$: B.-Q. Jin, V. Korepin, A.I. (2004)

$n > 1$: F. Mezzadri, M. Mo, A.I. (2008)

A Story.

- Late 70-s - "Finite-gap" ≡ "algebrogeometric" method in Soliton Theory.

  Krichever's scheme.

- 1997 - Deift, Zhou, I - use of the f-g techique in the random matrix theory

- 1999 - Deift, Kapaev, Zhou, I —"—"— in the inverse monodromy problem for Fuchsian Systems

- Late 90-s - DKMVZ - use of the ∂-g techniques in the theory of orthogonal polynomials (Aptekarev - 1984)

- 2007 - M. Cafasso - more general block Toeplitz determinants. A direct use of Krichever's KP-theory

(2007 - Basor, Ehrhardt - Block Toeplitz from dimer model. Reduction to the half-truncated symbol)

Big Expectation:

Use of the finite-gap approach to the Wiener-Hopf factorization in diffraction.

<u>Main Challenge</u>: in the case of higher matrix case, the problem of analytical description of the underline Riemann surface.

(D. Korotkin: 2001 - inverse monodromy problem for higher matrix size, quasi-permutation monodromy matrices)

• Back to the XY model.

~~2.3. Our main result~~

~~2004~~

$$\varkappa = \frac{\sqrt{h^2 - 4(1-\gamma^2)}}{2\gamma} \qquad \text{Case 1a}$$

$$\varkappa = \frac{\sqrt{4(1-\gamma^2) - h^2}}{\sqrt{4-h^2}} \qquad \text{Case 1b}$$

$$\varkappa = \frac{2\gamma}{\sqrt{h^2 - 4(1-\gamma^2)}} \qquad \text{Case 2}$$

$$\varkappa' = \sqrt{1-\varkappa^2} \quad , \quad I(\varkappa) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\varkappa^2 x^2)}}$$

$$\tau = i \, \frac{I(\varkappa')}{I(\varkappa)}$$

$$\Theta_3(s) \equiv \Theta_3(s;\tau) = \sum_{m=-\infty}^{\infty} e^{\pi i \tau m^2 + 2\pi i m s}$$

$$D_L(\lambda) = (1-\lambda^2)^L \equiv (G[\varphi])^L$$

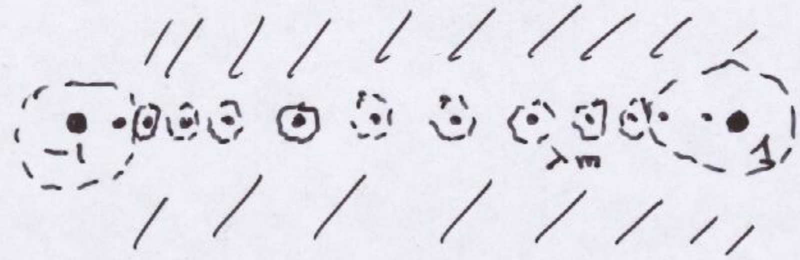
$$\times \frac{\theta_3\left(\frac{1}{2\pi i}\ln\frac{\lambda+1}{\lambda-1} + \frac{\delta\tau}{2}\right)\theta_3\left(\frac{1}{2\pi i}\ln\frac{\lambda+1}{\lambda-1} - \frac{\delta\tau}{2}\right)}{\theta_3^2\left(\frac{\delta\tau}{2}\right)} \equiv E[\varphi]$$

$\delta = \alpha'$

$$\times \left(1 + O(\varrho^{-L})\right), \quad L \to \infty$$

$\varrho = |z_2|$, case 1 , $\varrho = |z_2^{-1}|$, case 2

$\lambda \in$



$\pm\lambda_m$, $\lambda_m = \tanh\left(m + \frac{1-\delta}{2}\right)\pi\tau_0$

$\tau_0 = -i\tau = I(x')/I(x)$

$\delta = 1$ Case 1 $\qquad \delta = 0$ Case 2

Conclusion:

$$\boxed{\nu_{2m}, \nu_{2m+1} \longmapsto \lambda_m}$$

$$\Downarrow$$

$$\boxed{\begin{gathered} S_\infty(\alpha) = \frac{1}{1-\alpha} \sum_{m=-\infty}^{\infty} \ln\left[\left(\frac{1+\lambda_m}{2}\right)^\alpha + \left(\frac{1-\lambda_m}{2}\right)^\alpha\right] \\ \lambda_m = \tanh\left(m + \frac{1-\sigma}{2}\right)\pi\tau_0 \,, \quad \tau_0 = \frac{I(x')}{I(x)} \end{gathered}}$$

Remark. $$S_\infty(1) = \sum_{m=-\infty}^{\infty} (1+\lambda_m) \ln \frac{2}{1+\lambda_m}$$

$$= \frac{1}{2} \int_1^\infty \ln \frac{\theta_3\left(\frac{1}{2\pi i}\ln\frac{\lambda+1}{\lambda-1} + \sigma\tau/2\right)\theta_3\left(\frac{1}{2\pi i}\ln\frac{\lambda+1}{\lambda-1} - \sigma\tau/2\right)}{\theta_3^2(\sigma\tau/2)} d\lambda$$

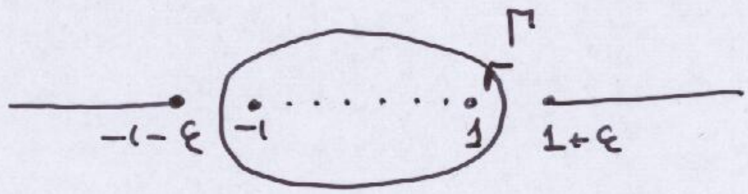
B.-Q. Jin, V. Korepin, A. I.
I. Peschel (2004)

Remark. what is actually proven is this:

$$S_L = \lim_{\varepsilon \to 0} \frac{1}{4\pi i} \int_\Gamma d\lambda \, e(1+\varepsilon, \lambda) \frac{d}{d\lambda} \ln \left(D_L(\lambda) (\lambda^2 - 1)^{-L} \right)$$

$$e(x, \nu) = -\frac{x+\nu}{2} \ln \frac{x+\nu}{2} - \frac{x-\nu}{2} \ln \frac{x-\nu}{2}$$

Γ:



$$S_\infty := \lim_{\varepsilon \to 0} \left[\lim_{L \to \infty} \frac{1}{4\pi i} \int_\Gamma d\lambda \, e(1+\varepsilon, \lambda) \frac{d}{d\lambda} \ln \left(D_L(\lambda) (\lambda^2 - 1)^{-L} \right) \right]$$

$$\overset{?}{=} \lim_{L \to \infty} \left[\lim_{\varepsilon \to 0} \int_\Gamma \cdots\cdots \right]$$

• Summing up. The Renyi Entropy as a modular function.

The principal observation:

$$S_\infty(\alpha) = \frac{2}{1-\alpha} \sum_{m=0}^{\infty} \ln\left(1 + q^{\alpha(2m+1)}\right)$$

$$- \frac{2\alpha}{1-\alpha} \sum_{m=0}^{\infty} \ln\left(1 + q^{2m+1}\right) \qquad h > 2$$

$$S_\infty(\alpha) = \frac{2}{1-\alpha} \sum_{m=1}^{\infty} \ln\left(1 + q^{2m\alpha}\right)$$

$$- \frac{2\alpha}{1-\alpha} \sum_{m=1}^{\infty} \ln\left(1 + q^{2m}\right) + \ln 2 \qquad h < 2$$

$$q = e^{-\pi \tau_0} = e^{-\pi I(x')/I(x)}$$

The classics (Jacobi identities)

$$\prod_{m=0}^{\infty} (1 + q^{2m+1}) = \left(\frac{16\, q}{\lambda (1-\lambda)} \right)^{1/24}$$

$$\prod_{m=1}^{\infty} (1 + q^{2m}) = \left(\frac{\lambda^2}{32 q^2 (1-\lambda)} \right)^{1/24}$$

$$\lambda \equiv \lambda(\tau) = \frac{\theta_2^4 (0;\tau)}{\theta_3^4 (0;\tau)} \equiv \varkappa^2(q)$$

$$q = e^{\pi i \tau}$$

$$1 - \lambda(\tau) = \frac{\theta_4^4 (0;\tau)}{\theta_3^4 (0;\tau)} \equiv \varkappa'^2(q)$$

$$\boxed{\lambda(\tau) \text{ - Klein's elliptic } \lambda\text{-function}}$$

Using the Jacobi identities we arrive at the following representations for the Renyi Entropy.

$$S_\infty(\alpha) = \frac{1}{6}\frac{\alpha}{1-\alpha}\ln(\varkappa\varkappa')$$

$$- \frac{1}{12}\frac{1}{1-\alpha}\ln\Big(\lambda(\alpha\tau)\big(1-\lambda(\alpha\tau)\big)\Big)$$

$$+ \frac{1}{3}\ln 2, \qquad h > 2$$

$$S_\infty(\alpha) = \frac{1}{6}\frac{\alpha}{1-\alpha}\ln\frac{\varkappa'}{\varkappa^2}$$

$$+ \frac{1}{12}\frac{1}{1-\alpha}\ln\frac{\lambda^2(\alpha\tau)}{1-\lambda(\alpha\tau)} + \frac{1}{3}\ln 2$$

$$h < 2$$

F. Franchini, B.-Q. Jin, V. E. Korepin, A.R.I.
(2007)

Remark 1.

$$\alpha \to 1 \Rightarrow \lambda(\alpha\tau) \mapsto \lambda(\tau) = \varkappa^2$$

$$\Downarrow$$

$$S_\infty(1) = \frac{1}{6}\left[\ln\frac{4}{\varkappa\varkappa'} + (\varkappa^2 - \varkappa'^2)\frac{2I(\varkappa)I(\varkappa')}{\pi}\right]$$

$h > 2$

$$S_\infty(1) = \frac{1}{6}\left[\ln\frac{4\varkappa^2}{\varkappa'} + (2 - \varkappa^2)\frac{2I(\varkappa)I(\varkappa')}{\pi}\right]$$

$h < 2$

Can be obtained directly by summing up the series for $S_\infty(1)$ - I. Peschel (2004)

- Work in progress.

Multi-Interval Case

$$\underbrace{\uparrow\downarrow\uparrow\downarrow\uparrow}_{L}\ \downarrow\downarrow\downarrow\ \underbrace{\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow}_{M}$$

"two-interval" case

$$T_L[\varphi] \longmapsto$$

$$\hat{T} \equiv \begin{pmatrix} T_{LL}[\varphi^{(11)}] & T_{LM}[\varphi^{(12)}] \\ T_{ML}[\varphi^{(21)}] & T_{MM}[\varphi^{(2,2)}] \end{pmatrix}$$

B.-Q. Jin, V. Korepin

A. Kuijlaars: the RH representation

for

$\det \hat{T}$

<u>current efforts</u>: the asymptotic analysis of those RH problems.

L. Brightmore, V. Korepin,

A. Kuijlaars, F. Mezzadri,

M. Mo, A. I.

- Entanglement Spectrum

$$\text{spectrum } \rho_L \underset{L\to\infty}{\equiv} \{ (\lambda_n, a_n) \}_{n=0}^{\infty} \; ?$$

Observe:

$$\sum_{n=0}^{\infty} a_n \lambda_n^{\alpha} = \mathrm{Tr}\, \rho_\infty^{\alpha}$$

$$\equiv \zeta_{\rho_\infty}(\alpha)$$

$$= \exp\left\{ (1-\alpha)\, S_\infty(\alpha) \right\}$$

- From q-series:

$$\zeta_{\varrho_\infty}(\alpha) = e^{\alpha\left(\pi\tilde{\tau}_0/12 + 1/6 \ln \frac{\varkappa\delta'}{4}\right)}$$

$$\times \prod_{m=0}^{\infty} \left(1 + q^{(2m+1)\alpha}\right)^2$$

$$= e^{\alpha\left(\pi\tilde{\tau}_0/12 + 1/6 \ln \frac{\varkappa\delta'}{4}\right)} \sum_{n=0}^{\infty} P_{\Theta}^{(2)}(n)\, q^{\alpha n}$$

$P_{\Theta}^{(2)}(n) = \#\{$ partition of n into positive odd numbers, where each integer can contribute at most twice $\}$

$$\Downarrow$$

$$\lambda_n = e^{\frac{1}{6}\ln\frac{\varkappa\varkappa'}{4} - \pi\frac{I(\varkappa')}{I(\varkappa)}\left(n - 1/12\right)}.$$

$$a_n = P_{\mathcal{O}}^{(2)}(n)$$

$$= \sum_{l=0}^{n}{}' P_{\mathcal{O}}^{(1)}(n) P_{\mathcal{O}}^{(1)}(n-l)$$

$$P_{\mathcal{O}}^{(1)}(n) = \#\left\{ (m_1, \dots, m_k) : \begin{array}{l} m_j \text{ - odd} \\ m_1 > m_2 > \dots > m_k, \quad n = m_1 + m_2 + \dots + m_k \end{array} \right\}$$

- Asymptotics of a_n.

$$f(z) := \sum_{n=0}^{\infty}{}' a_n z^n = e^{(1-\alpha) S_\infty(\alpha) - \alpha\left(\frac{1}{6}\ln\frac{\varkappa\varkappa'}{4} + \frac{\pi\tau_0}{12}\right)}$$

$$\alpha = -\frac{1}{\pi\tau_0}\ln z$$

$$z \approx 1 \iff \alpha \approx 0$$

$\Downarrow$ from the modular representation of $S_\infty(\alpha)$

$$S_\infty(\alpha) = \frac{1}{\alpha(1-\alpha)}\frac{\pi}{12}\frac{1}{\tau_0} + \frac{\alpha}{1-\alpha}\frac{1}{6}\ln\frac{\varkappa\varkappa'}{4} + O\left(e^{-\pi/\alpha\tau_0}\right)$$

$\alpha \to 0$

$$-\frac{\pi}{2} < \arg\alpha < \frac{\pi}{2}.$$

$$\Downarrow$$

$$a_n = \frac{1}{2\pi i} \int\limits_{|z| = 1-\varepsilon} f(z)/z^{n+1} \, dz$$

$$\sim 2^{-3/2} \, 3^{-1/4} \, n^{-3/4} \, e^{\pi \sqrt{n/3}}$$

$$n \to \infty$$

Asymptotics of the

Hardy- Ramanujan- Rademacher type.

F. Franchini, V. Korepin, L. Takhtajan, A.I.

Remark.

$$P(n) = \# \{ (m_1, \dots, m_k),\ m_j > 0,\ n = m_1 + \dots + m_k \}$$

Rademacher's series:

$$P(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\,\frac{d}{dn}\left(\frac{\sinh\left\{ \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right\}}{\sqrt{n - \frac{1}{24}}} \right)$$

Similar formula for $a_n = P_{\mathbb{G}}^{(2)}(n)$?

Physical meaning ?