Lie groups, Toda chains and cluster variables

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The Versatility of Integrability

Columbia University, May 6, 2011

joint work with V. Fock (Strasbourg and ITEP),
"Quantum groups and relativistic Toda chains",
Nucl.Phys. **56B** (Proc. Suppl.) (1997) 208-214;
and hopefully smth to appear . . .

Lie groups and integrable systems: the Poisson structure on group manifold

$$\left\{ g \underset{,}{\otimes} g \right\} = -\frac{1}{2} \; [r, g \otimes g]$$
$$r = \sum_{\alpha \in \Delta_+} e_\alpha \wedge f_\alpha = \sum_{\alpha \in \Delta_+} (e_\alpha \otimes f_\alpha - f_\alpha \otimes e_\alpha) \tag{1}$$

For $G = SL(N)$ (or $GL(N)$) this is explicitly

$$\left\{ g^i_j, g^k_l \right\} = -\frac{1}{2} \; g^i_l g^k_j \left(\epsilon(i,k) + \epsilon(j,l) \right)$$
$$\epsilon(i,k) = \begin{cases} 1 & i < k \\ 0 & i = k \\ -1 & i > k \end{cases} \tag{2}$$

Obvious rank $G = N-1$ functionally independent Poisson commuting functions, since

$$\left\{ g^n \underset{,}{\otimes} g^k \right\} = [r, g^n \otimes g^k] \tag{3}$$

then

$$\begin{aligned} \left\{ \mathrm{Tr} g^n, \mathrm{Tr} g^k \right\} = \mathrm{Tr}_{12} \left\{ g^n \underset{,}{\otimes} g^k \right\} = \\ = \mathrm{Tr}_{12}[r, g^n \otimes g^k] = 0 \end{aligned} \tag{4}$$

One can also take $\mathrm{Tr}_\mu g$ for different representations μ etc.

An *integrable system* on symplectic leaf of dimension $2 \cdot \mathrm{rank}\ G$! Quantization of this integrable system (in conventional quantum-mechanical sense) - quantum group!

What is its name? (middle of 90-s ...)

To answer: an $SL(2)$ **example**

For the functions

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2) \tag{5}$$

the Poisson brackets are

$$\begin{array}{ccc} \{a,b\} = -\frac{1}{2}\ ab & \{a,c\} = -\frac{1}{2}\ ac & \{b,c\} = 0 \\ \{d,b\} = \frac{1}{2}\ db & \{d,c\} = \frac{1}{2}\ dc & \{a,d\} = bc \end{array} \tag{6}$$

from

$$r = e \otimes f - f \otimes e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

Symplectic leaf: fixed by two Casimirs

$$\begin{aligned} C_1 &= ad - bc \\ C_2 &= b/c \end{aligned} \tag{8}$$

and the constant value levels of these functions are

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} Pf & Q \\ Q & P^{-1}f \end{pmatrix} \tag{9}$$

with

$$\begin{aligned} Q = e^{q/2}, \qquad P = e^{p} \\ f = \sqrt{1 + e^{q}} \end{aligned} \tag{10}$$

so that

$$H \sim \mathsf{Tr}\ g = \left(e^{p} + e^{-p}\right) \sqrt{1 + e^{q}} \tag{11}$$

Relativistic Toda chain: in the non-relativistic limit

$$g = 1 + L + ... = 1 + \begin{pmatrix} p & e^{q/2} \\ e^{q/2} & -p \end{pmatrix} + ... \qquad (12)$$

with

$$L = \begin{pmatrix} p & e^{q/2} \\ e^{q/2} & -p \end{pmatrix} = p \cdot h + e^{q/2} \cdot (e + f) \in sl(2) \qquad (13)$$

being the Lax operator for ordinary non-periodic Toda chain.

Can be constructed with the linear bracket

$$\left\{ L \underset{,}{\otimes} L \right\} = [r, (L \otimes 1 + 1 \otimes L)] \qquad (14)$$

with the Hamiltonian $H \sim \mathrm{Tr}\ g^2$.

How to construct the symplectic leave of dim $= 2 \cdot \text{rank } G$? For $G = SL(N)$ fixing rank $G = N - 1$ Casimirs gives the symplectic leaf of dimension

$$\dim\ G - \text{rank } G = N(N-1)$$

instead of

$$2 \text{ rank } G = 2(N-1)$$

(which coincide only for $N = 2$).

Naively: a "collection" of $SL(2)$ subgroups ...
Using cluster variables for constructing symplectic leaves.

Graph (or *quiver*) Γ with $|\Gamma|$ vertices, connected by (any number of) oriented edges. Variables $\{z_i | i \in \Gamma\}$ - a chart in some manifold $(\mathbb{C}^*)^{|\Gamma|}$.

The Poisson bracket

$$\{z_i, z_j\} = \epsilon_{ij} z_i z_j, \qquad i, j = 1, \ldots, |\Gamma| \tag{15}$$

with the incidence matrix

$$\epsilon_{ij} = \#\text{arrows } (i \to j) = -\epsilon_{ji} \tag{16}$$

(in the most generic setup not even integer-valued!)

Properties:

- For any $\Gamma' \subset \Gamma$ just put

$$\epsilon_{ij'} = 0, \quad \forall\ i \in \Gamma, \ \ \forall\ j' \in \Gamma' \tag{17}$$

  "forgetting" all vertices from Γ'.

- Gluing: Γ_1 and Γ_2 by *identifying* subsets $\Gamma'_1 = \Gamma'_2 = \Gamma'$, with the variables $\{z_{i_1} = z^{(1)}_{i_1} | i_1 \in \Gamma_1 \diagdown \Gamma'_1\}$, $\{z_{i_2} = z^{(2)}_{i_2} | i_2 \in \Gamma_2 \diagdown \Gamma'_2\}$, and $\{z_{i'} = z^{(1)}_{i'} z^{(2)}_{i'} | i' \in \Gamma'\}$, so that

$$\epsilon_{i'j'} = \epsilon^{(1)}_{i'j'} + \epsilon^{(2)}_{i'j'}, \qquad i', j' \in \ \Gamma' \tag{18}$$

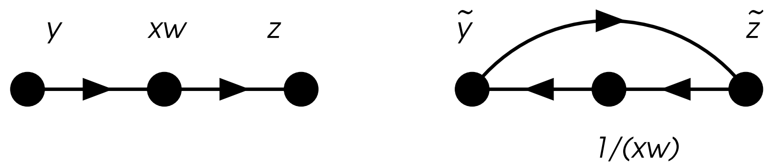
- *Mutations* of the graph and the z-variables

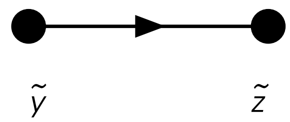
$$\mu_j: \quad z_j \to \frac{1}{z_j}, \qquad z_i \to z_i \left(1 + z_j^{\mathrm{sgn}(\epsilon_{ij})}\right)^{\epsilon_{ij}}, i \neq j \qquad (19)$$

(requires integer ϵ_{ij}).

Constructing the Poisson manifolds – two basic examples: Lie groups and Teichmüller spaces.

Group structure: *gluing* of $(y \to x) \cdot (w \to z) = y \to xw \to z$ together with *mutation* and *forgetting*





gives rise to multiplication law of the upper-triangular subgroup of $SL(2)$ (or $PGL(2)$)

$$y \to y(1 + xw) = \tilde{y}, \quad z \to z\left(1 + \frac{1}{xw}\right)^{-1} = \tilde{z} \tag{20}$$
$$\{\tilde{y}, \tilde{z}\} = \tilde{y}\tilde{z}$$

i.e.

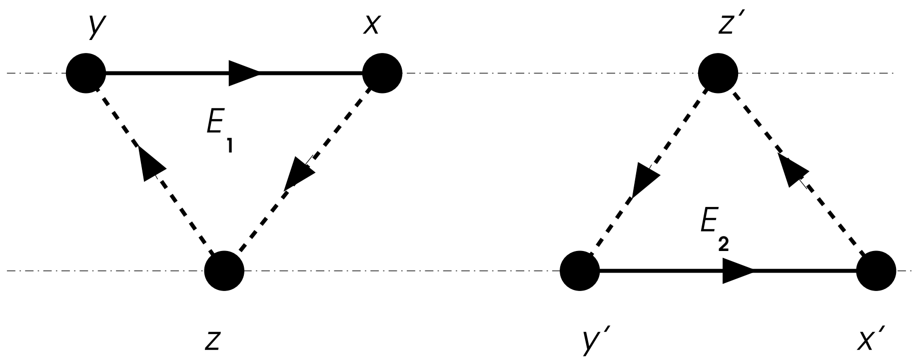
$$y \underset{E}{\longrightarrow} x = YEX = \frac{1}{\sqrt{xy}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{yx} & \sqrt{\frac{y}{x}} \\ 0 & \frac{1}{\sqrt{yx}} \end{pmatrix} \in SL(2) \qquad (21)$$

The Poisson structure $\{y, x\} = yx$ coincides with the r-matrix one $\left\{g \underset{,}{\otimes} g\right\} = -\frac{1}{2}\ [r, g \otimes g]$ for $SL(2)$, restricted to (21).

Obvious notations for $SL(N)$: $E_i = \exp e_i$, $F_i = \exp f_i$, $H_i(x) = \exp(xh_i) = X_i$, $i = 1, \ldots,$ rank G with

$$[h_i, e_j] = \delta_{ij} e_j, \quad [h_i, f_j] = -\delta_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} C_{ik} h_k \qquad (22)$$

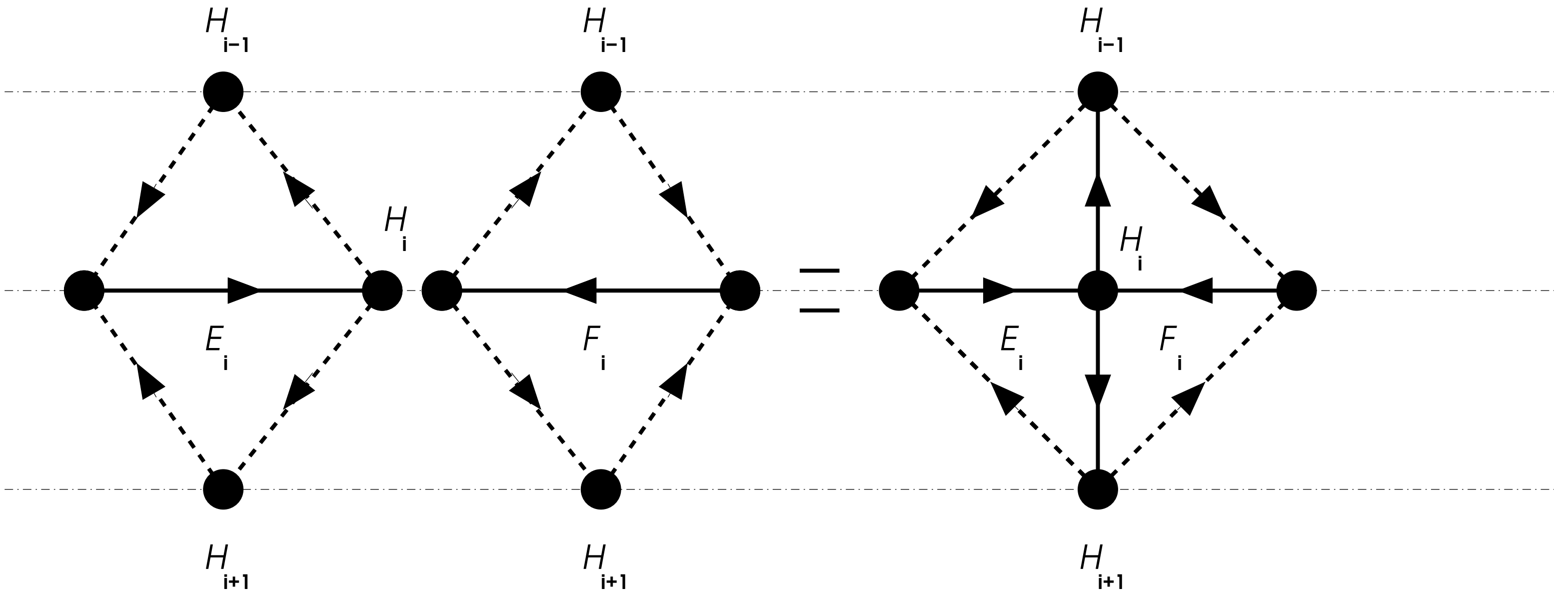
Similarly, the subgroups of $SL(3)$ (generated by simple E_1 and E_2) with $\{y, x\} = yx$, and $\{x, z\} = \frac{1}{2}xz$, $\{z, y\} = \frac{1}{2}zy$



e.g. for the left triangle: $H_1(y)E_1H_1(x)H_2(z) \sim$

$$\sim \begin{pmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23)$$

Generally, for $SL(N)$ the building block is rhombus, e.g.



for the simple E_i and F_i, $1 < i < N-1$, with the Cartan generators H_i, and $H_{i\pm1}$ at each level correspondingly.

All halves come from comparison with $\{g \underset{,}{\otimes} g\} = -\frac{1}{2}\ [r, g \otimes g]$ and finally disappear.

Statement: *The symplectic leaves of relativistic Toda correspond to the periodic sub-configurations of the square lattice, with the incidence matrix being the Cartan matrix of a Lie group.*

SL(2)

$$g(x,y) \sim Y^{1/2} \cdot E \cdot X \cdot F \cdot Y^{1/2} \sim E \cdot X \cdot F \cdot Y =$$
$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} =$$
$$= \frac{1}{\sqrt{xy}} \begin{pmatrix} xy + y & 1 \\ y & 1 \end{pmatrix} \tag{24}$$

Graph $y^{1/2} \overset{E}{\rightarrow} x \overset{F}{\leftarrow} y^{1/2}$ or $y \Rightarrow x$ induces the Poisson structure

$$\{y, x\} = 2yx \tag{25}$$

The Darboux variables

$$x = e^{-q}, \quad y = \frac{e^{2p}}{1+x} \tag{26}$$

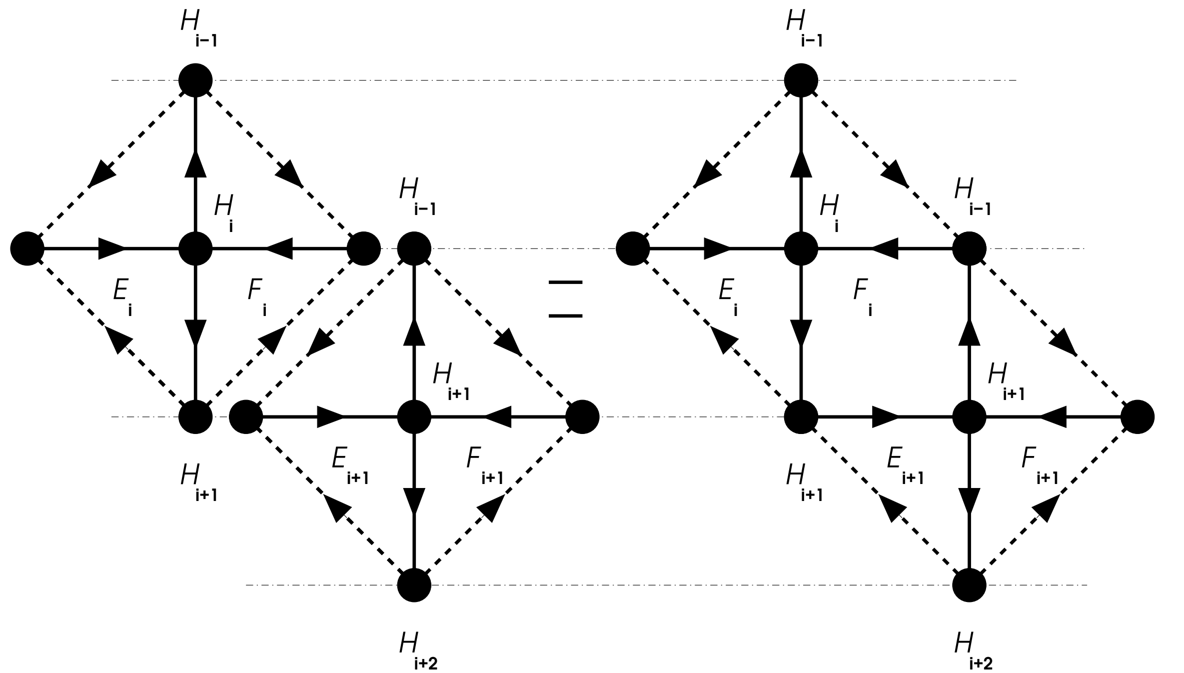
so that the Hamiltonian Tr $g(x, y)$ is

$$H = \sqrt{xy} + \sqrt{\frac{y}{x}} + \frac{1}{\sqrt{xy}} = \left(e^{p} + e^{-p}\right)\sqrt{1+e^{q}} \tag{27}$$

Remark: in fact $y = \exp(2P)$ with

$$2P = 2p + \frac{\partial}{\partial q}\mathrm{Li}_2(-e^{q}) \tag{28}$$

Generally:



gluing the subgroups of $SL(N)$ one gets an element of the two-dimensional square lattice, turning to the graph with $\epsilon_{ij} = C_{ij}$ after imposing periodicity in horizontal direction.

Open SL_N case: the product over the simple roots

$$\underbrace{E_1X_1F_1Y_1}_{g_1} \cdot \underbrace{E_2X_2F_2Y_2}_{g_2} \cdot \ldots \cdot \underbrace{E_{N-1}X_{N-1}F_{N-1}Y_{N-1}}_{g_{N-1}} \sim \qquad (29)$$
$$\sim g_N(\mathbf{x}, \mathbf{y}) \in SL(N)$$

and the graph $\begin{pmatrix} y_1 \Downarrow \overset{\leftarrow}{\rightarrow} \Uparrow x_2 \overset{\rightarrow}{\leftarrow} \Downarrow y_3 \overset{\leftarrow}{\rightarrow} \ldots \overset{\rightarrow}{\leftarrow} \Downarrow y_{N-1} \\ x_1 \qquad\quad y_2 \qquad\quad x_3 \qquad\qquad\qquad x_{N-1} \end{pmatrix}^T$ induces the Poisson bracket

$$\{y_i, x_j\} = C_{ij} y_i x_j, \quad i,j = 1,\ldots,N-1 \qquad (30)$$

with the Cartan matrix

$$C_{ij} \underset{sl_N}{=} 2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}, \quad i,j = 1,\ldots,N-1 \qquad (31)$$

The Hamiltonians are coefficients of

$$\det\left(g_N(\mathbf{x},\mathbf{y}) - \lambda\cdot\mathbf{1}\right) = \sum_{j=0}^{N} R_j(\mathbf{x},\mathbf{y})(-\lambda)^j \tag{32}$$

(with $R_0 = R_N = 1$) do Poisson-commute

$$\begin{gathered} \{R_i(\mathbf{x},\mathbf{y}), R_j(\mathbf{x},\mathbf{y})\} = 0, \qquad i,j = 1,\dots,N-1 \\ R_1 = \mathrm{Tr}\ g^{-1}, \quad R_{N-1} = \mathrm{Tr}\ g \end{gathered} \tag{33}$$

Moreover, explicitly $R_j(\mathbf{x},\mathbf{y}) = \prod_k (x_k y_k)^{-C^{-1}_{jk}} \cdot Z_j(\mathbf{x},\mathbf{y})$ and the *polynomials* $Z_j(\mathbf{x},\mathbf{y})$ are “almost dimer partition functions”

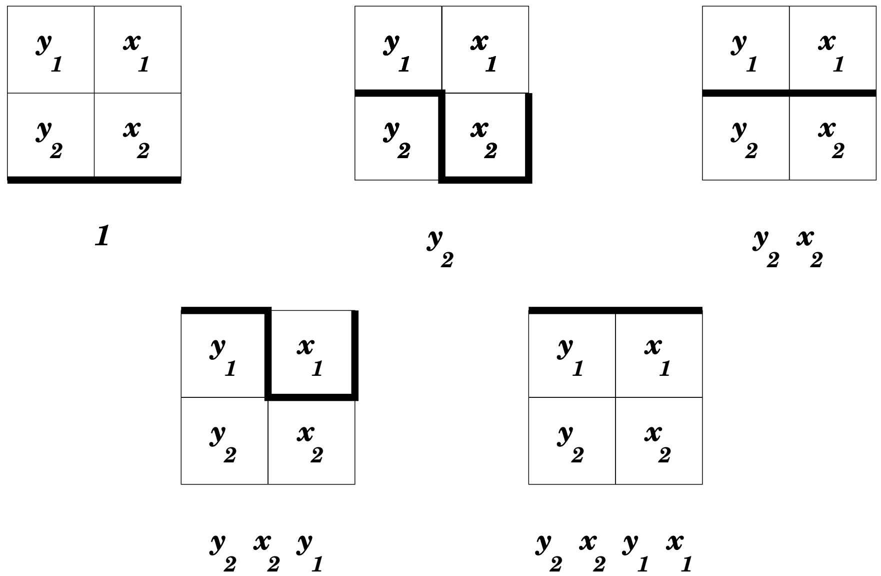
$$Z_j = \sum_{0\le m_i\le\epsilon_i}^{m_j\ge m_{j\pm1}\ge m_{j\pm2}\ge\dots} \sum_{m_i-1\le n_i\le m_i} \prod_i y_i^{m_i} x_i^{n_i} \tag{34}$$

with $\epsilon_i = 1,2$ (number of edges entering the i-th vertex of the Dynkin diagram).

Say, for $SL(3)$:

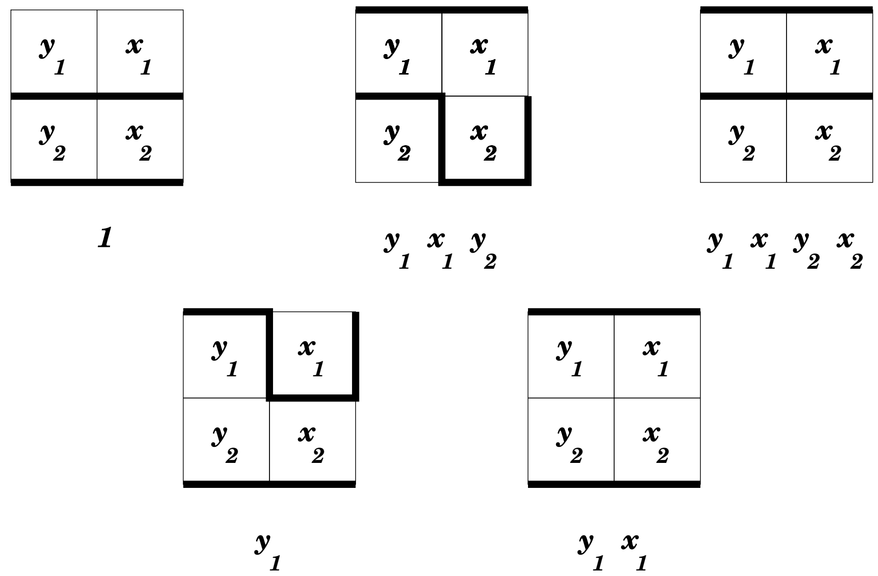
$$Z_1 = 1 + y_1 + y_1 x_1 + y_1 x_1 y_2 + y_1 x_1 y_2 x_2$$
$$Z_2 = 1 + y_2 + y_2 x_2 + y_2 x_2 y_1 + y_2 x_2 y_1 x_1 \qquad (35)$$

or

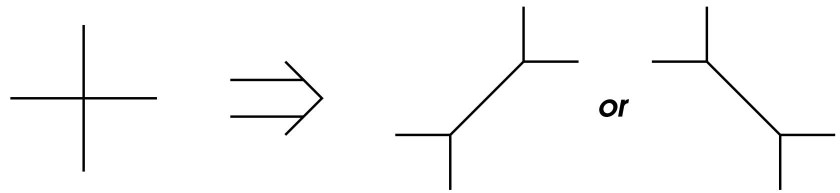


diagramatically for $Z_2(\mathbf{x}, \mathbf{y})$, with each diagram corresponding to the particular term in the matrix product $(YEXF)_{ii}$.

Similarly, for $Z_{N-k} \sim \mathrm{Tr}\ \overbrace{g_N \wedge g_N \wedge \ldots \wedge g_N}^{k}$ one gets the diagrams with k broken lines, e.g.



for the $Z_1(\mathbf{x},\mathbf{y}) \sim \mathrm{Tr} g^{-1} \sim \mathrm{Tr}\,(g \wedge g)$ of $SL(3)$. Almost dimers: after "resolving" the vertices:



The Darboux co-ordinates are again

$$x_i = \exp(-\boldsymbol{\alpha}_i \cdot \mathbf{q}), \qquad y_i = \exp(\boldsymbol{\alpha}_i \cdot (\mathbf{P} + \mathbf{q})) \tag{36}$$

where

$$\mathbf{P} = \mathbf{p} + \frac{\partial}{\partial \mathbf{q}} \left(\frac{1}{2} \sum_{k=1}^{N-1} \mathrm{Li}_2 \left(-\exp(\boldsymbol{\alpha}_k \cdot \mathbf{q}) \right) \right) \tag{37}$$

and the Hamiltonian

$$\begin{aligned} H &= \mathrm{Tr}\left(g + g^{-1}\right) = R_1 + R_{N-1} = \\ &= \sum_{i=1}^{N} \left(\exp(\boldsymbol{\nu}_i \cdot \mathbf{p}) + \exp(-\boldsymbol{\nu}_i \cdot \mathbf{p})\right) \cdot \\ &\cdot \sqrt{1 + \exp(\boldsymbol{\alpha}_i \cdot \mathbf{q})} \sqrt{1 + \exp(\boldsymbol{\alpha}_{i-1} \cdot \mathbf{q})} \end{aligned} \tag{38}$$

only to replace $\sqrt{1 + \exp(\boldsymbol{\alpha}_{0,N} \cdot \mathbf{q})}$ by unities, otherwise with $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_N$ affine simple root - the *periodic* Toda.

Notations:

$$\boldsymbol{\alpha}_i = \sum_j C_{ij} \boldsymbol{\mu}_j, \quad \boldsymbol{\mu}_i = \sum_j C^{-1}_{ij} \boldsymbol{\alpha}_j, \quad \boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j = C_{ij} \tag{39}$$

and, for $SL(N)$

$$\boldsymbol{\nu}_i = \boldsymbol{\mu}_i - \boldsymbol{\mu}_{i-1}, \qquad i = 1, \ldots, N \tag{40}$$

where it is implied that $\boldsymbol{\mu}_0 = \boldsymbol{\mu}_N = \mathbf{0}$. Their scalar products are

$$\boldsymbol{\nu}_i \cdot \boldsymbol{\nu}_j = \delta_{ij} - \frac{1}{N}, \qquad i, j = 1, \ldots, N \tag{41}$$

also

$$\boldsymbol{\alpha}_i = \boldsymbol{\nu}_i - \boldsymbol{\nu}_{i+1} = \mathbf{e}_i - \mathbf{e}_{i+1}, \qquad i = 1, \ldots, N-1 \tag{42}$$

Periodic relativistic Toda chain: degenerate Cartan matrix, embedding of (the collection of) $SL(2)$-like symplectic leaves into $\widehat{SL(N)}$ ...

Practically - the co-extended loop group $\widehat{SL(N)}$:
• simple roots E_i and F_i, $i \in \mathbb{Z}_N$, with $E_0 = \mathsf{E}(u)$, $F_0 = \mathsf{F}(u)$; e.g. for $SL(2)$

$$\mathsf{E}(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \qquad \mathsf{F}(u) = \begin{pmatrix} 1 & 1/u \\ 0 & 1 \end{pmatrix} \tag{43}$$

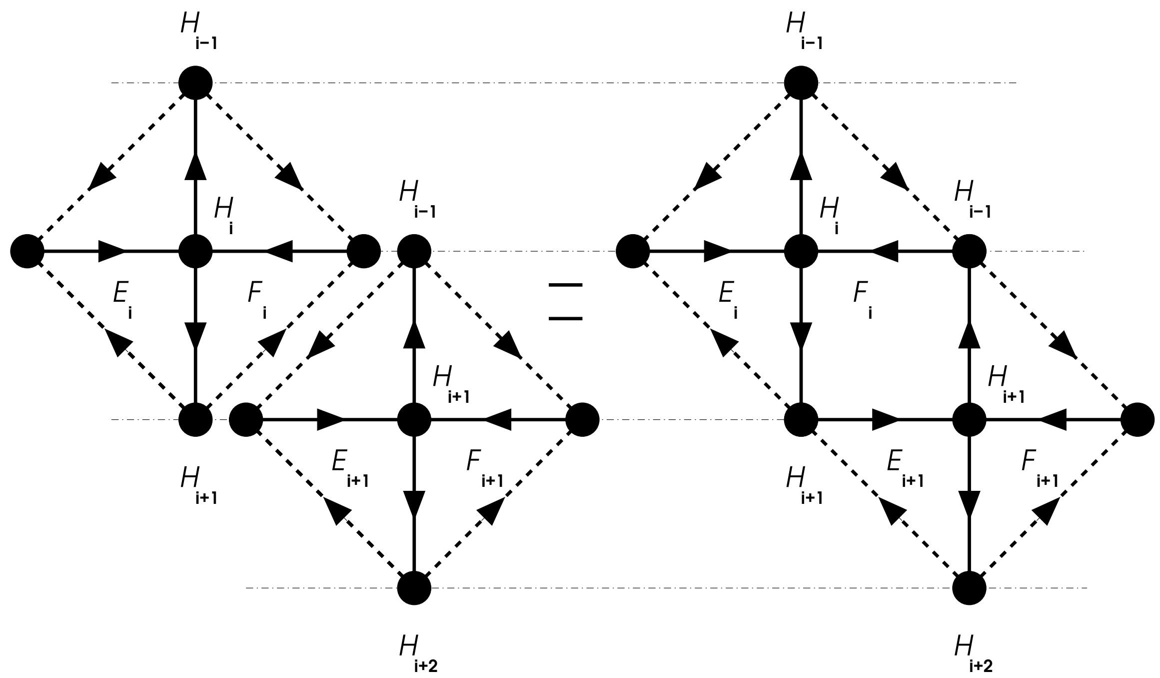
• Cartan $H_0(x) = T_x = x^{d/du}$, $H_i(x) = X_i T_x$ for $i = 1, \ldots, N-1$.

The spectral parameter dependent matrices

$$g(u) \sim \prod_{j\in\mathbb{Z}_N} E_j H_j(x_j) F_j H_j(y_j) \sim \prod_{j\in\mathbb{Z}_N} E_j X_j T_{x_j} F_j Y_j T_{y_j} \tag{44}$$

(trivial shift due to $\prod_{j\in\mathbb{Z}_N} x_j = \prod_{j\in\mathbb{Z}_N} y_j = \text{const}$).

again corresponding for $\widehat{SL(N)}$ to



gluing the oriented plaquettes of the same two-dimensional lattice. However, it turns into the graph with $\epsilon_{ij} = \widehat{C}_{ij}$ after imposing periodicity both in horizontal and vertical directions!

Localizing the shifts by $(x_j = x_{\nu_j}/x_{\nu_{j+1}})$

$$
\begin{gathered}
g(u) \sim \prod_{j\in\mathbb{Z}_N} E_j X_j T_{x_{\nu_j}} T^{-1}_{x_{\nu_{j+1}}} F_j Y_j T_{y_{\nu_j}} T^{-1}_{y_{\nu_{j+1}}} \sim \\
\sim \prod_{j\in\mathbb{Z}_N} T^{-1}_{x_{\nu_j}} T^{-1}_{y_{\nu_j}} E_j X_j T_{x_{\nu_j}} F'_j Y_j T_{y_{\nu_j}} \sim \mathsf{g}_0(u)\cdot g_N(\mathbf{x},\mathbf{y})
\end{gathered}
\tag{45}
$$

with $g_N \in SL(N)$ being the Lax of open Toda, and

$$
\begin{gathered}
\mathsf{g}_0(u) = T^{-1}_{x_{\nu_0}} T^{-1}_{y_{\nu_0}} E_0 T_{x_{\nu_0}} F'_0 T_{y_{\nu_0}} = \\
= \mathsf{E}_0\left(\frac{u}{x_{\nu_0} y_{\nu_0}}\right) \mathsf{F}_0\left(\frac{u}{y_{\nu_0} x_{\nu_1}}\right)
\end{gathered}
\tag{46}
$$

contains the spectral parameter.

The Poisson bracket is “old” $\{y_i, x_j\} = C_{ij} y_i x_j$, $1 \le i,j \le N-1$ after fixing the Casimir functions $\prod_{j\in\mathbb{Z}_N} x_j$ and $\prod_{j\in\mathbb{Z}_N} y_j$.

Spectral curves

The characteristic polynomial

$$\mathcal{P}(\lambda; u) = \det\left(\lambda \cdot \mathbf{1} - g(u)\right) = \\ = \sum_{j=0}^{N} \lambda^j (-)^{N-j} R_j(\mathbf{x}, \mathbf{y}) - \lambda^{N-1} u - \frac{\lambda}{u} \tag{47}$$

gives the relativistic Toda spectral curve

$$w + \frac{1}{w} = P_N(\lambda) = \lambda^{-N/2} \sum_{j=0}^{N} \lambda^j (-)^{N-j} R_j(\mathbf{x}, \mathbf{y}) \\ w = u\lambda^{N/2-1} \tag{48}$$

The Krichever data: $\oint \frac{d\lambda}{\lambda} = \text{const}$, $\oint \frac{dw}{w} = \text{const}$, genus $= N - 1 = \text{rank}\, G$.

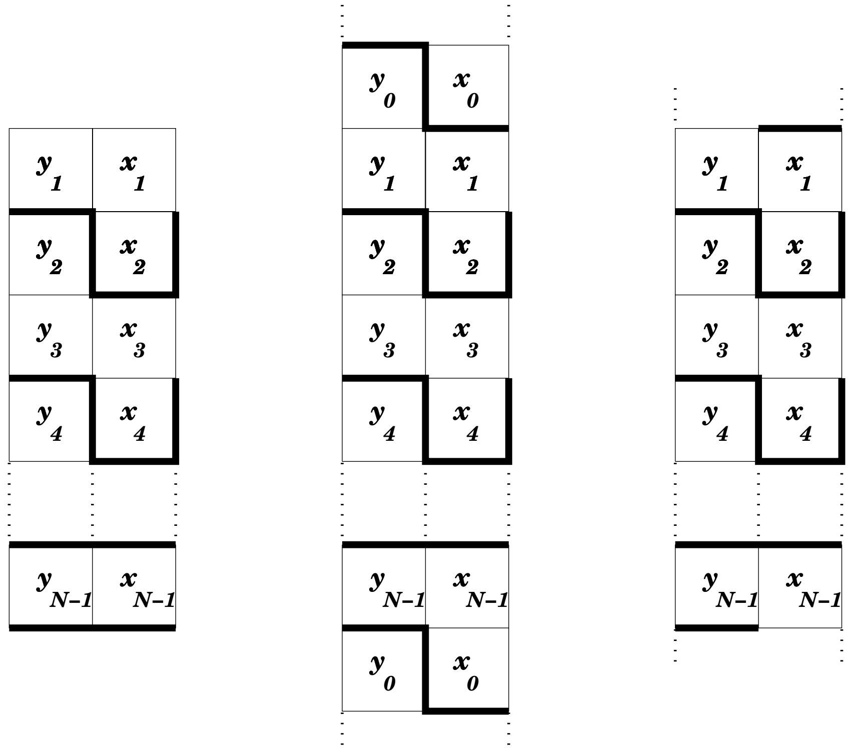
The Poisson commuting Hamiltonians

$$\{R_j, R_k\} = 0, \quad j \neq k \tag{49}$$

(again $R_0 = R_N = 1$) w.r.t. the "open" bracket. E.g. for $\widehat{SL(3)}$ explicitly

$$R_1 = \text{res Tr } g^{-1}(u)\frac{du}{u} = \prod_k (x_k y_k)^{-C^{-1}_{1k}} \cdot$$
$$\cdot \left(1 + y_1 + y_1 x_1 + y_1 x_1 y_2 + y_1 x_1 y_2 x_2 + \underbrace{x_1 x_2}_{\text{new}}\right)$$
$$R_2 = \text{res Tr } g(u)\frac{du}{u} = \prod_k (x_k y_k)^{-C^{-1}_{2k}} \cdot \tag{50}$$
$$\cdot \left(1 + y_2 + y_2 x_2 + y_2 x_2 y_1 + y_2 x_2 y_1 x_1 + \underbrace{x_1 x_2}_{\text{new}}\right)$$

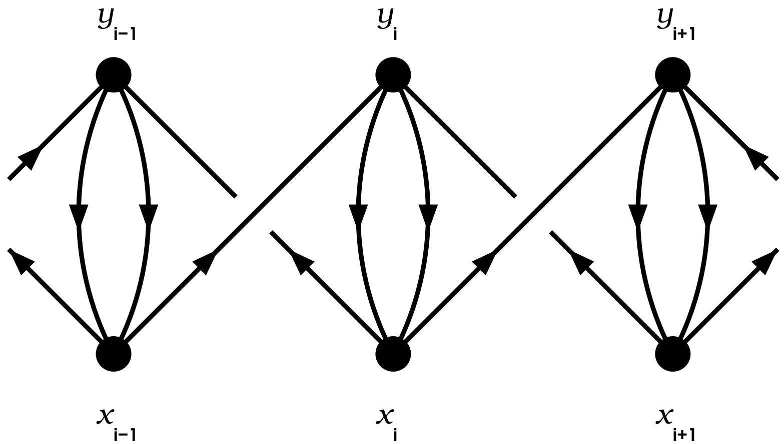
Diagram picture for the periodic Toda



for $\widehat{Z_4(\mathbf{x},\mathbf{y})}$ function: left diagram for $SL(N)$ is "completed" to $\widehat{SL(N)}$ - becoming periodic in the vertical direction, and by substitution $1 \to 1 + 1/x_0 \underset{\prod_{j\in\mathbb{Z}_N} x_j = 1}{=} 1 + \prod_{j=1}^{N-1} x_j$ in $\mathrm{g}_0(u)$.

2×2 **formalism** (monodromy matrices?)

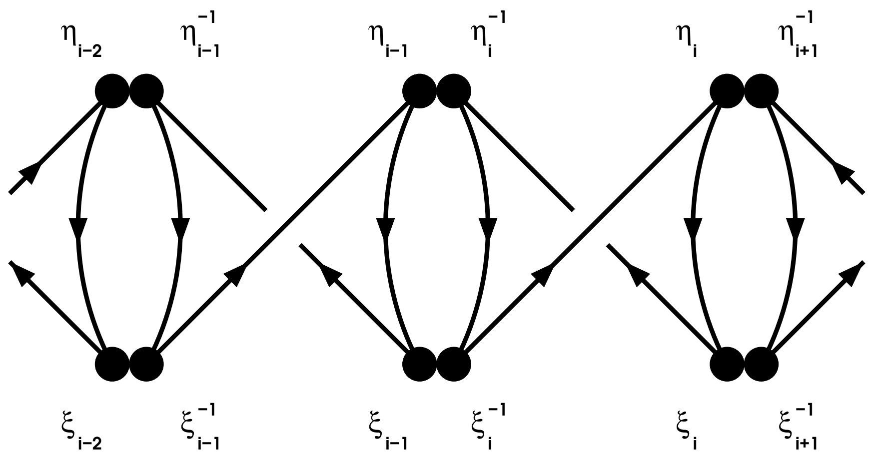
The horizontally depicted graph $\Downarrow^{y_0}_{x_0} \begin{smallmatrix}\leftarrow\\ \rightarrow\end{smallmatrix} {}^{x_1}_{y_1} \Uparrow \begin{smallmatrix}\rightarrow\\ \leftarrow\end{smallmatrix} {}^{y_2}_{x_2} \Downarrow \ldots \Downarrow \begin{smallmatrix}\leftarrow\\ \rightarrow\end{smallmatrix} {}^{x_0}_{y_0}$ or, after twist,



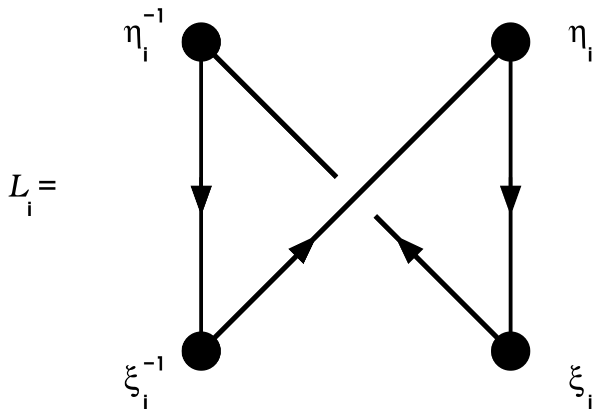
Introducing the $GL(N)$-variables

$$x_i = \frac{\xi_i}{\xi_{i+1}}, \quad y_i = \frac{\eta_i}{\eta_{i+1}}, \quad i = 1, \ldots, N-1$$
$$y_N = y_0, \quad \xi_N = \xi_0, \quad \eta_N = \eta_0 \tag{51}$$

this acquires the form



of the product of $L_j(u) = L(\eta_j, \xi_j; u)$ operators



being (for $j = 1, \ldots, N$)

$$L_j(u) = \Big(\mathsf{H}(\eta_j)\mathsf{H}_0(\xi_j)\Big)^{-1} \Phi(u)\mathsf{H}(\eta_j)\mathsf{H}_0(\xi_j) =$$

$$= \begin{pmatrix} 0 & \sqrt{\dfrac{\xi_j}{u\eta_j}} \\ \sqrt{\dfrac{u\eta_j}{\xi_j}} & \sqrt{\dfrac{u}{\eta_j\xi_j}} + \sqrt{\dfrac{\eta_j\xi_j}{u}} \end{pmatrix} \tag{52}$$

where

$$\Phi(u) = \mathsf{E}(u)\omega(u)\mathsf{F}(u) =$$

$$= \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} 0 & u^{-1/2} \\ u^{1/2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/u \\ 0 & 1 \end{pmatrix} \tag{53}$$

with $\omega E_j \omega = E_{j+1}$, $\omega F_j \omega = F_{j+1}$ and $\omega \mathsf{H}_j \omega = \mathsf{H}_{j+1}$ ($j \in \mathbb{Z}_2$).

In more common terms

$$\xi_i = \exp(-q_i), \quad \eta_i = \exp(P_i + q_i), \quad i = 1, \ldots, N$$
$$P_i = p_i + \frac{\partial}{\partial q_i}\left(\tfrac{1}{2}\sum_{k=1}^{N} \mathrm{Li}_2\left(-\exp(q_k - q_{k+1})\right)\right) \tag{54}$$

these are

$$L_j(u) = \begin{pmatrix} 0 & \dfrac{e^{-P_j/2 - q_j}}{\sqrt{u}} \\ \sqrt{u}e^{P_j/2 + q_j} & \sqrt{u}e^{-P_j/2} + \dfrac{e^{P_j/2}}{\sqrt{u}} \end{pmatrix} \tag{55}$$

the FT operators for (relativistic) Toda, satisfying

$$\left\{L_i(u) \underset{,}{\otimes} L_j(u')\right\} = -\tfrac{1}{2}\delta_{ij}\left[r_{\mathrm{trig}}\left(u, u'\right), L_i(u) \underset{,}{\otimes} L_j(u')\right] \tag{56}$$

with the trigonometric classical r-matrix

$$r_{\mathrm{trig}}(u,u') = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dfrac{u+u'}{u-u'} & -\dfrac{2u'}{u-u'} & 0 \\ 0 & -\dfrac{2u}{u-u'} & \dfrac{u+u'}{u-u'} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{57}$$

being essentially

$$\sum_{\alpha\in\Delta_+} e_\alpha \wedge f_\alpha = \sum_{n\geq 0} e_n \wedge f_{-n} + \sum_{n\geq 1} \Big(f_n \wedge e_{-n} + \tfrac{1}{2} h_n \wedge h_{-n} \Big) \tag{58}$$

antisymmetric r-matrix for $\widehat{sl}_2$ in ev$(u) \otimes$ ev(u').

Advanced issues

- Quantization: quantum group! Easy in cluster variables - to get the Barnes functions etc. Looks as straightforward, but still to be done.

- Invariance under the discrete "flows"

$$\begin{aligned}(y,x) \mapsto \left(y^{-1}(1+x)^{-2}, x^{-1}(1+y(1+x)^2)^2\right) = \\ = \mu_{y(1+x)^2} \circ \mu_x(y,x) = \mu_{y(1+x)^2}\left(y(1+x)^2, x^{-1}\right)\end{aligned} \tag{59}$$

  being a composition of mutations

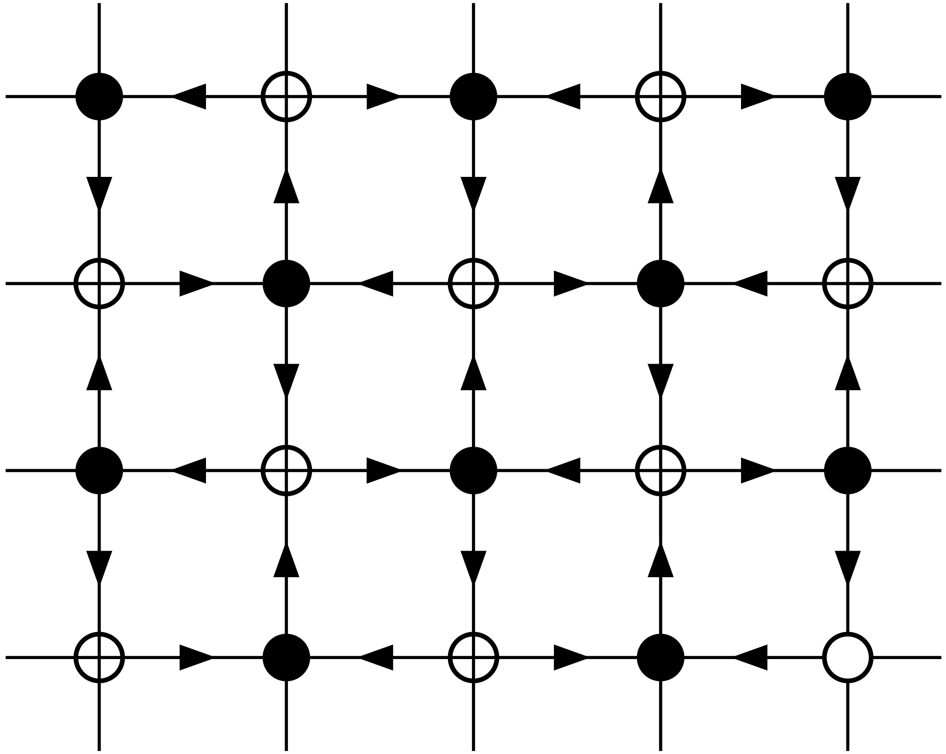
$$\begin{aligned}\mu_x(y,x) = \left(y(1+x)^2, x^{-1}\right) \\ \mu_y(y,x) = \left(y^{-1}, x\left(1+y^{-1}\right)^{-2}\right)\end{aligned} \tag{60}$$

Generally

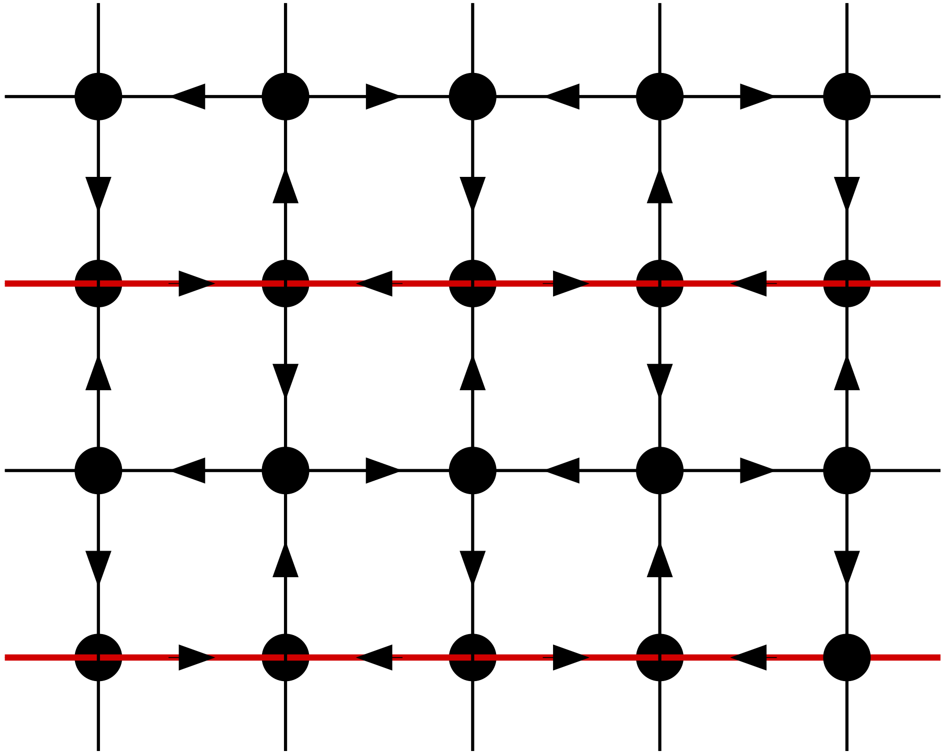
$$\begin{aligned}\tilde{y}_i^{-1} &= y_i \prod_j \left(1 + x_j^{\mathsf{sgn}(C_{ij})}\right)^{C_{ij}} \\ \tilde{x}_i &= x_i^{-1} \prod_j \left(1 + \tilde{y}_j^{-\mathsf{sgn}(C_{ij})}\right)^{C_{ij}}\end{aligned} \tag{61}$$

are again constructed as a sequence of mutations of the graph ${}^{y_1}_{x_1} \Downarrow \begin{smallmatrix}\leftarrow \\ \rightarrow\end{smallmatrix} \Uparrow {}^{x_2 \rightarrow}_{y_2 \leftarrow} \Downarrow {}^{y_3 \leftarrow}_{x_3 \rightarrow} \ldots \begin{smallmatrix}\rightarrow \\ \leftarrow\end{smallmatrix} \Downarrow {}^{y_{N-1}}_{x_{N-1}}$.

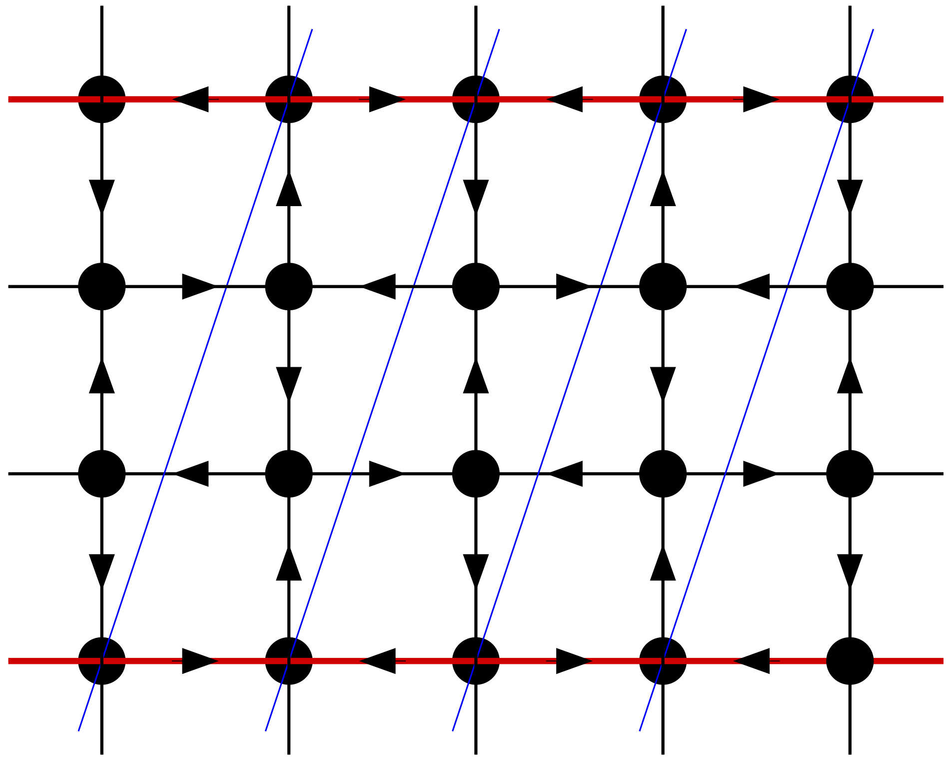
Their properties? For a graph with only single arrows (rectangle $m \times n$) the $(n + m + 2)$-th power of such map is identity (Y-system's conjecture of Al. Zamolodchikov: relation to TBA, quantization of the Hitchin systems, ...)



Rectangular lattice built from oriented square plaquettes. Consequence of black and white mutations returns graph to itself, but transforms the variables in the vertices.



Rectangular lattice with vertical 2-periodicity: 2×2 formulation of N-particle affine Toda after imposing horizontal N-periodicity. New integrable systems?



Vertical 3-periodic lattice with (necessary) horizontal shift: the 3×3-formalism. For horizontal N-periodicity one gets the OT discretized Boussinesq (pentagram map), if $N=3$ this is $\widehat{SL(3)}$-Toda. Spectral curves of $g=2\left(\left[\frac{N+1}{2}\right]-1\right)$, the Krichever data, commuting Hamiltonians etc.

- Other Lie groups (beyond $SL(N)$) - “extra gluing” ...

- Cluster variables - Lie groups and Teichmüller spaces, conformal blocks from quantum Teichmüller spaces etc ...

  Closed relativistic Toda: invariants in the “evaluation representation” of the operators like $q^{L_0}e^{tH_0}e^{J_{\pm 1}}\ldots$.
  Look similar to each other ...

  Towards $\mathcal{N}=2$ supersymmetric gauge theory, SW prepotentials, Krichever tau-functions etc.

Thank you, Igor!

I still have many questions to you...