The Versatility of Integrability
Celebrating Igor Krichever's 60th Birthday

Quantum Integrability and Gauge Theory

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IHES

This is a work on experimental theoretical physics

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In collaboration with

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(ITEP)

and

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Hep-th [arXiv:1103.3919]

Earlier work

G.Moore, NN, S.Shatashvili.,
arXiv:hep-th/9712241;
A.Gerasimov, S.Shatashvili.
arXiv:0711.1472, arXiv:hep-th/0609024
NN, S.Shatashvili,
arXiv:0901.4744, arXiv:0901.4748, arXiv: 0908.4052
NN, E.Witten
arXiv:1002.0888

Earlier work on instanton calculus

A.Losev, NN, S.Shatashvili.,
arXiv:hep-th/9711108, arXiv:hep-th/9911099;
NN arXiv:hep-th/0206161;
The papers of
N.Dorey, T.Hollowood , V.Khoze, M.Mattis,

Earlier work on separated variables and D-branes

A.Gorsky, NN, V.Roubtsov,
arXiv:hep-th/9901089;

In the past few years
a connection
between the following
two seemingly unrelated
subjects
was found

The supersymmetric gauge theories

with as little as 4 supersymmetries

on the one hand

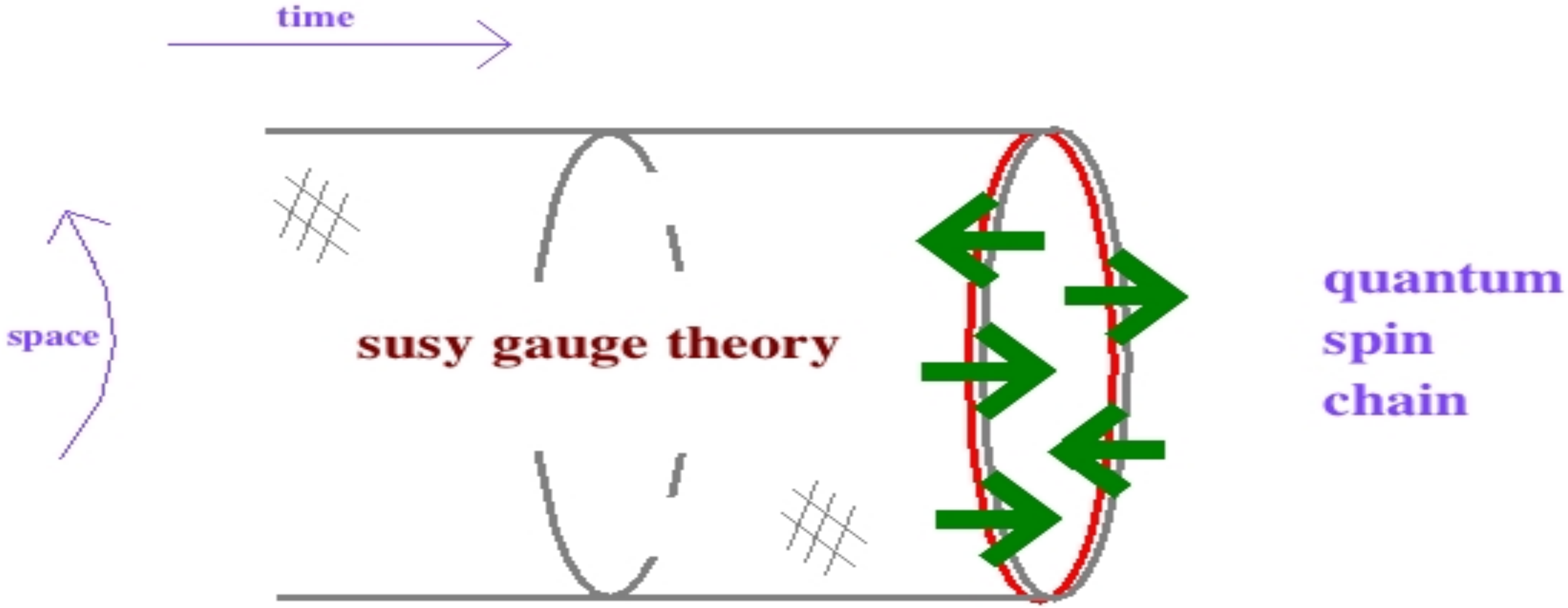
and

Quantum integrable systems

soluble by Bethe Ansatz

on the other

The **supersymmetric vacua** of
the (finite volume)
gauge theory



are **the stationary states** of
a quantum integrable system

Operators

The « **twisted chiral ring** » operators

$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots, \mathcal{O}_n$

map to the **quantum Hamiltonians**

$\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \ldots, \mathbf{H}_n$

Eigenvalues

The vacuum expectation values of
the **twisted chiral ring** operators

$$E_k(\lambda) = \langle\, \lambda \,|\, \mathcal{O}_k \,|\, \lambda \,\rangle$$

Identify with the **energy** and other **eigenvalues** on the integrable side

$$H_k \, \Psi_\lambda = E_k(\lambda) \, \Psi_\lambda$$

The main ingredient of the correspondence:

The effective twisted superpotential of the gauge theory

=

The Yang-Yang function of the quantum integrable system

The effective twisted superpotential of the gauge theory

$$\mathbf{A} = a + \vartheta^+ \psi_+ + \bar{\vartheta}^- \bar{\psi}_- + \vartheta^+ \bar{\vartheta}^- (F_A + iD)$$

$$\mathcal{L}^{\text{eff}} = \text{g}_{ij} da_i \wedge * d\bar{a}_j + \text{g}^{ij} \left(\text{Re} \left(\frac{\partial \widetilde{W}}{\partial a_i} \right) \text{Re} \left(\frac{\partial \widetilde{W}}{\partial a_j} \right) + F_i \wedge * F_j \right) + i \, \text{Im} \left(\frac{\partial \widetilde{W}}{\partial a_i} \right) F_i$$

The effective twisted superpotential leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

The effective twisted superpotential

$$\widetilde{W}^{\text{eff}}(a_1, \ldots, a_N; \varepsilon; \tau, m)$$

Is a multi-valued function on
the Coulomb branch of the theory,
depends on the parameters of the theory

The Yang-Yang function of the quantum integrable system

The YY function was introduced
by C.N.Yang and C.P.Yang in 1969
For the non-linear Schroedinger problem.

The miracle of Bethe ansatz: The spectrum of the quantum system is described by a classical equation

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

EXAMPLE:
Many-body system

Calogero-Moser-Sutherland system

$$H_{\mathrm{eCM}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \mathrm{g}^2 \sum_{i<j} U(x_i - x_j; \mathrm{q})$$

$$p_k = -i\hbar \frac{\partial}{\partial x_k}$$

The elliptic Calogero-Moser system

$\mathcal{N}$ identical particles on

a circle of radius β

subject to the two-body interaction

elliptic potential

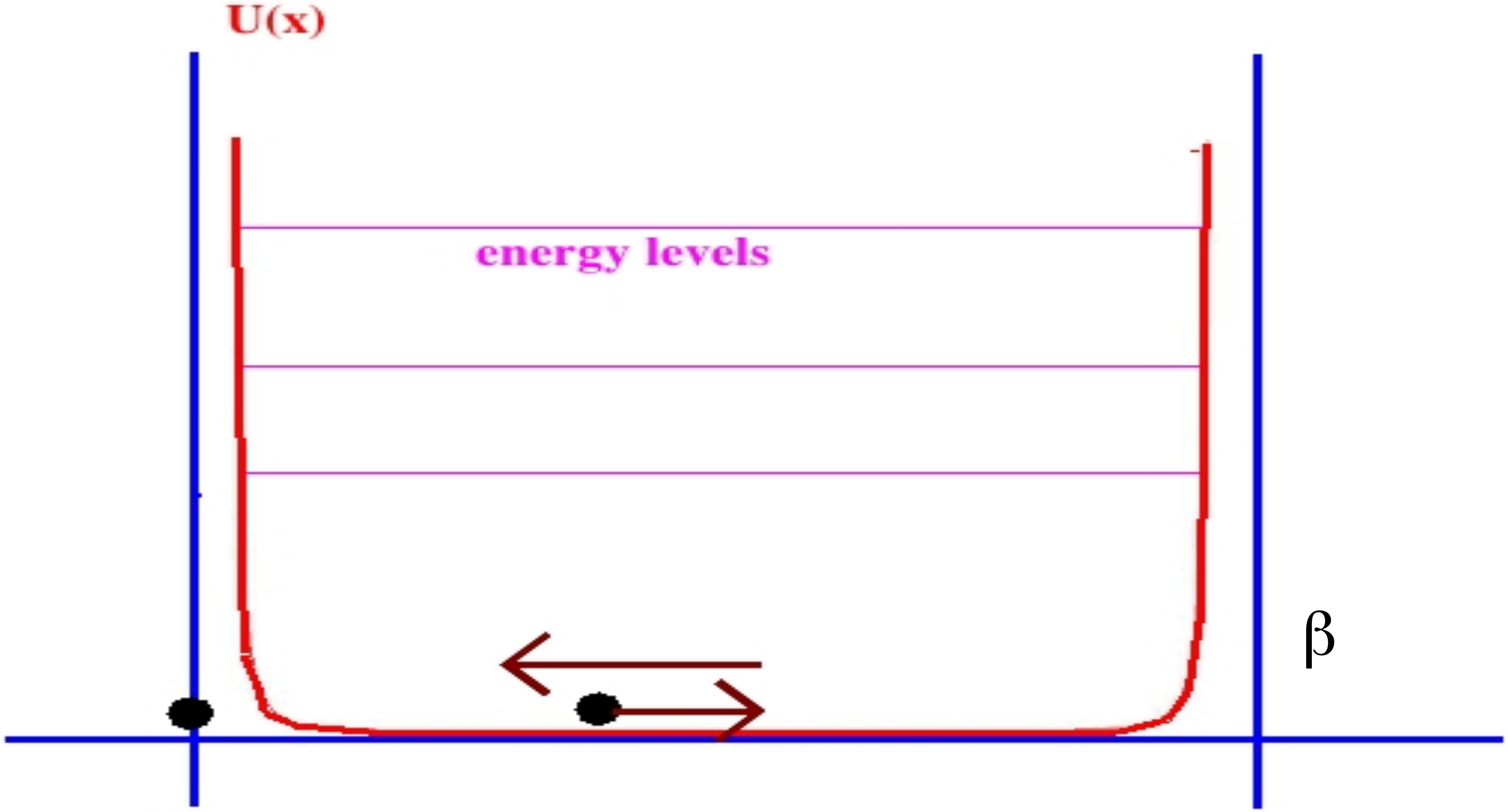
$$U(x;\mathrm{q}) = U(-x;\mathrm{q}) = \sum_{n\in\mathbf{Z}} \frac{1}{\sinh^2(x + 2\pi n\beta)}$$

Quantum many-body systems

One is interested in the β-periodic symmetric, L^2-normalizable wavefunctions

$$\Psi(x_1, \ldots, x_N)$$

It is clear that one should get an infinite discrete energy spectrum



Many-body system
vs
gauge theory

The

infinite discrete spectrum

of

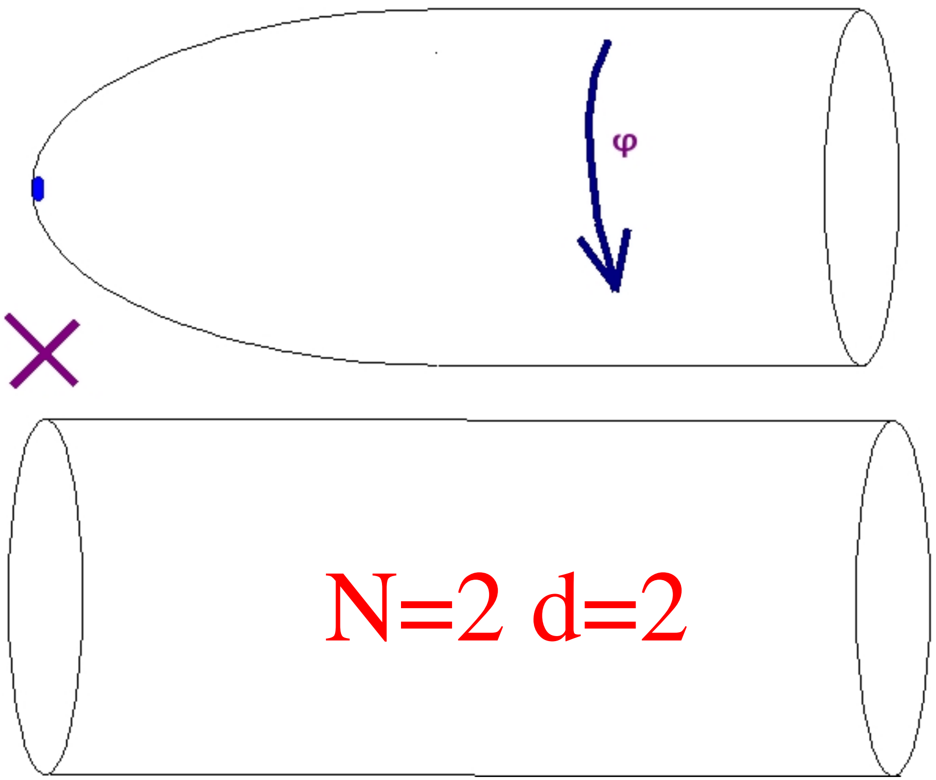
the integrable many-body system

=

The vacua of the $N=2$ $d=2$ theory

The gauge theory

The N=2 d=2 theory, obtained by subjecting the N=2 d=4 theory to an Ω-background in R^2



$$\Phi \rightarrow \Phi - \varepsilon D_{\varphi}$$

The four dimensional gauge theory on $\Sigma \times R^2$, viewed $SO(2)$ equivariantly, can be formally treated as an infinite dimensional version of a two dimensional gauge theory

The two dimensional theory

Has an
effective twisted
Superpotential!

The effective twisted superpotential

$$\widetilde{W}^{\text{eff}}(a_1, \ldots, a_N; \varepsilon; \tau, m)$$

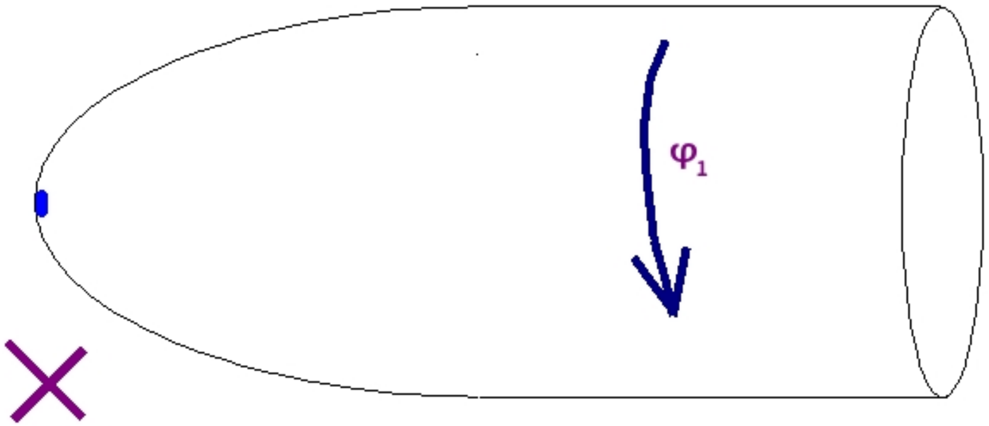
Can be computed from the
N=2 d=4
Instanton partition function

$$Z(a, \varepsilon_1, \varepsilon_2; m, \tau)$$

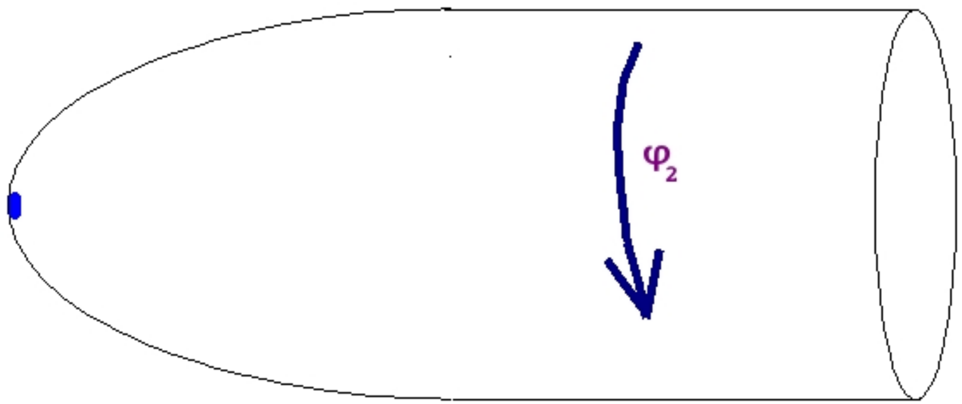
The effective twisted superpotential

$$Z(a,\varepsilon_1,\varepsilon_2;m,\tau) \sim e^{\frac{1}{\varepsilon_2}\widetilde{W}^{\text{eff}}(a_1,\ldots,a_N;\varepsilon_1;\tau,m)+\ldots}$$

as $\varepsilon_2 \to 0$



$\Phi \rightarrow \Phi - \varepsilon_1 D_{\varphi_1} - \varepsilon_2 D_{\varphi_2}$



The effective twisted superpotential has one-loop perturbative and all-order instanton corrections

$$\widetilde{W}^{\text{eff}}(a;\tau) = \widetilde{W}^{\text{pert}}(a) + \sum_{n=1}^{\infty} q^n \widetilde{W}_{n-\text{inst}}(a)$$

In particular, for the $\mathcal{N}=2^*$ theory

(adjoint hypermultiplet with mass m)

$\mathcal{N}$=2* theory

$$\exp \frac{\partial \widetilde{W}^{\text{pert}}(a)}{\partial a_i} =$$

$$e^{\frac{\pi i \tau a_i}{\varepsilon}} \prod_{j \neq i} S(a_i - a_j)$$

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)} \frac{\Gamma\left(1 - \frac{x}{\varepsilon}\right)}{\Gamma\left(1 + \frac{x}{\varepsilon}\right)}$$

Bethe equations
Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)\Gamma\left(1-\frac{x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)\Gamma\left(1+\frac{x}{\varepsilon}\right)}$$

This is the two-body scattering
In hyperbolic Calogero-Sutherland

Bethe equations
Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)} \frac{\Gamma\left(1-\frac{x}{\varepsilon}\right)}{\Gamma\left(1+\frac{x}{\varepsilon}\right)}$$

$$U_0(x) = \frac{1}{\sinh^2(x)}$$

Two-body potential

Bethe equations
Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)} \frac{\Gamma\left(1-\frac{x}{\varepsilon}\right)}{\Gamma\left(1+\frac{x}{\varepsilon}\right)}$$

Harish-Chandra, Gindikin-Karpelevich,
Olshanetsky-Perelomov, Heckmann,
final result: Opdam

The full superpotential of $\mathcal{N}$=2* theory leads to the vacuum equations

Momentum phase shift

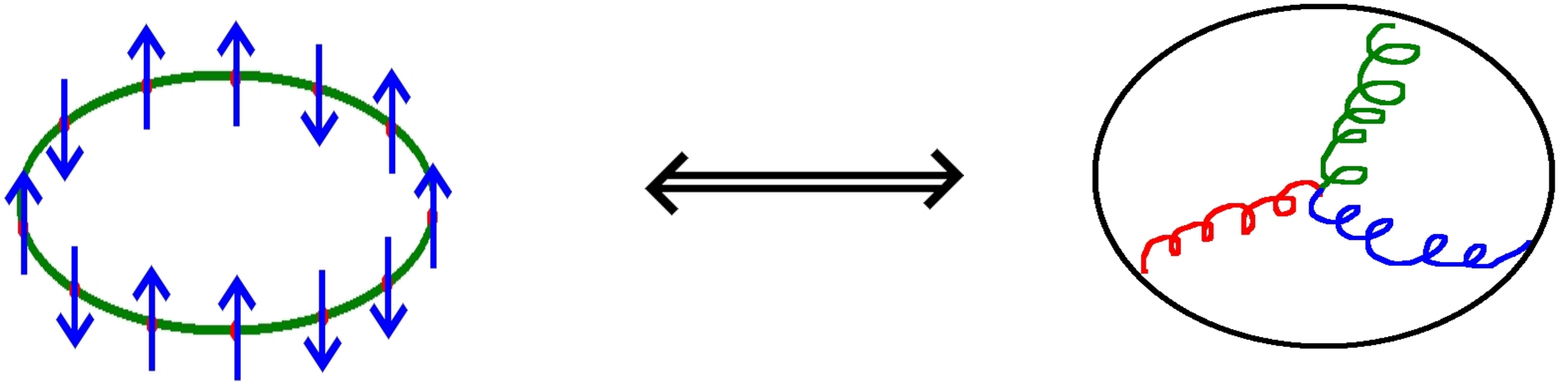
$$e^{\frac{\pi i \tau a_i}{\varepsilon}} \prod_{j\neq i} S(a_i - a_j) \times \left[1 + q \sum_{k\neq i} \prod_{l\neq k} \mathrm{rational}(a_i, a_l, a_k, m(m+\varepsilon), \varepsilon) + \ldots \right]$$

Two-body scattering

The finite size corrections

$$q = \exp(-N\beta)$$

Dictionary



Dictionary

Elliptic CM system $\Longleftrightarrow$ $\mathcal{N}=2^*$ theory

Dictionary

classical Elliptic CM system $\Longleftrightarrow$ 4d $\mathcal{N}=2^*$ theory

Dictionary

quantum Elliptic CM system	$\Longleftrightarrow$	4d $N=2^*$ Theory in 2d Ω-background

Dictionary

The (complexified) system size β $\Longleftrightarrow$ The gauge coupling τ

Dictionary

The Planck constant $\Longleftrightarrow$ The Equivariant parameter ε

Dictionary

The correspondence
Extends to other
integrable sytems:
Toda, relativistic Systems,
Perhaps all 1+1 iQFTs

Two ways of getting two dimensional theory starting with a higher dimensional one

1) Kaluza-Klein reduction, e.g. compactification on a torus with twisted boundary conditions…
2) Boundary theory, localization on a cosmic string….

Two ways of getting two dimensional theory starting with a higher dimensional one

1) Kaluza-Klein reduction: gives the spin chains, e.g. XYZ
2) Boundary theory, localization on a cosmic string: gives the many-body systems, e.g. CM, more generally, a Hitchin system

Plan

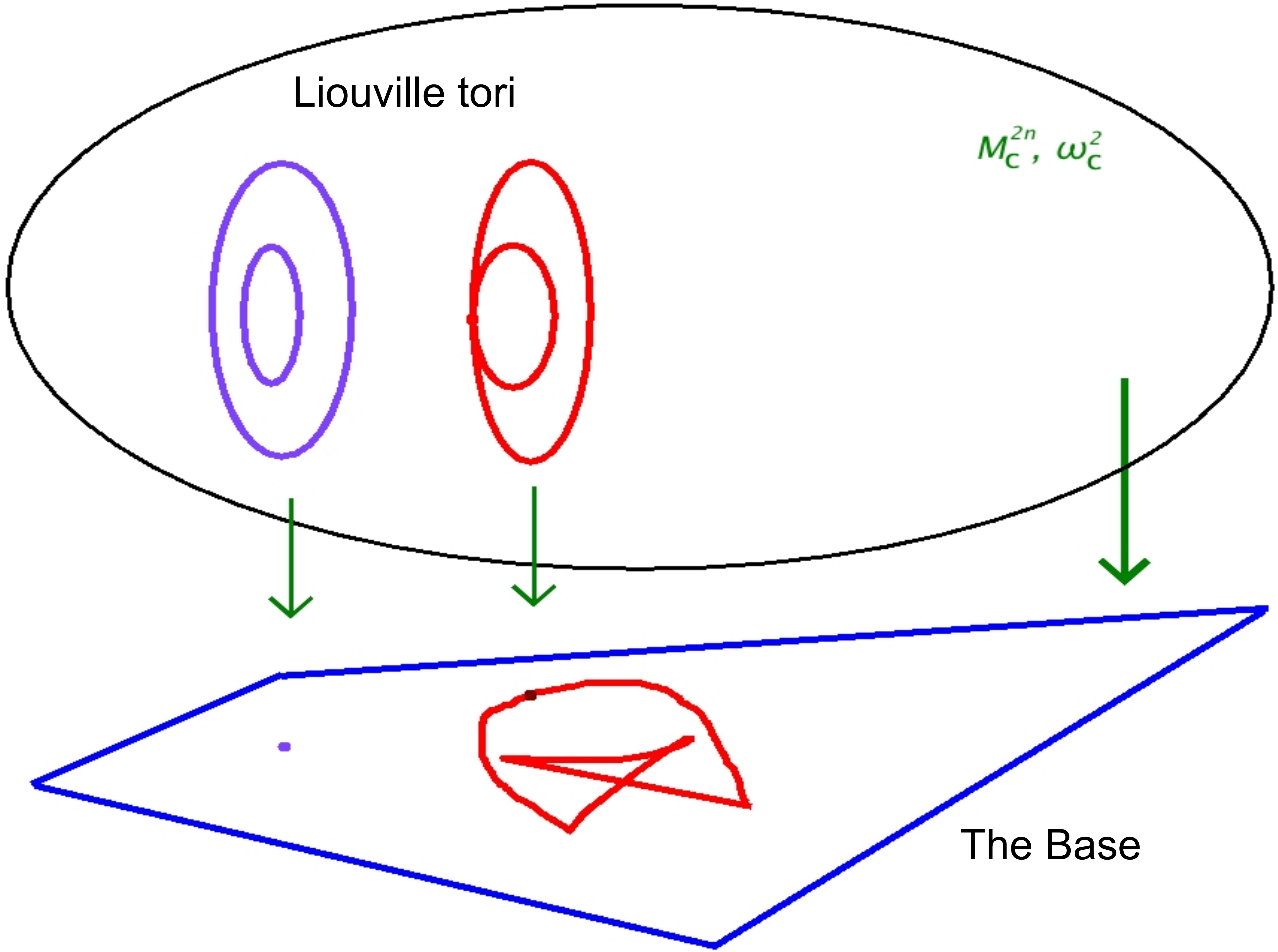
Now let us follow
The quantization procedure
more closely,
Starting on the gauge theory
side

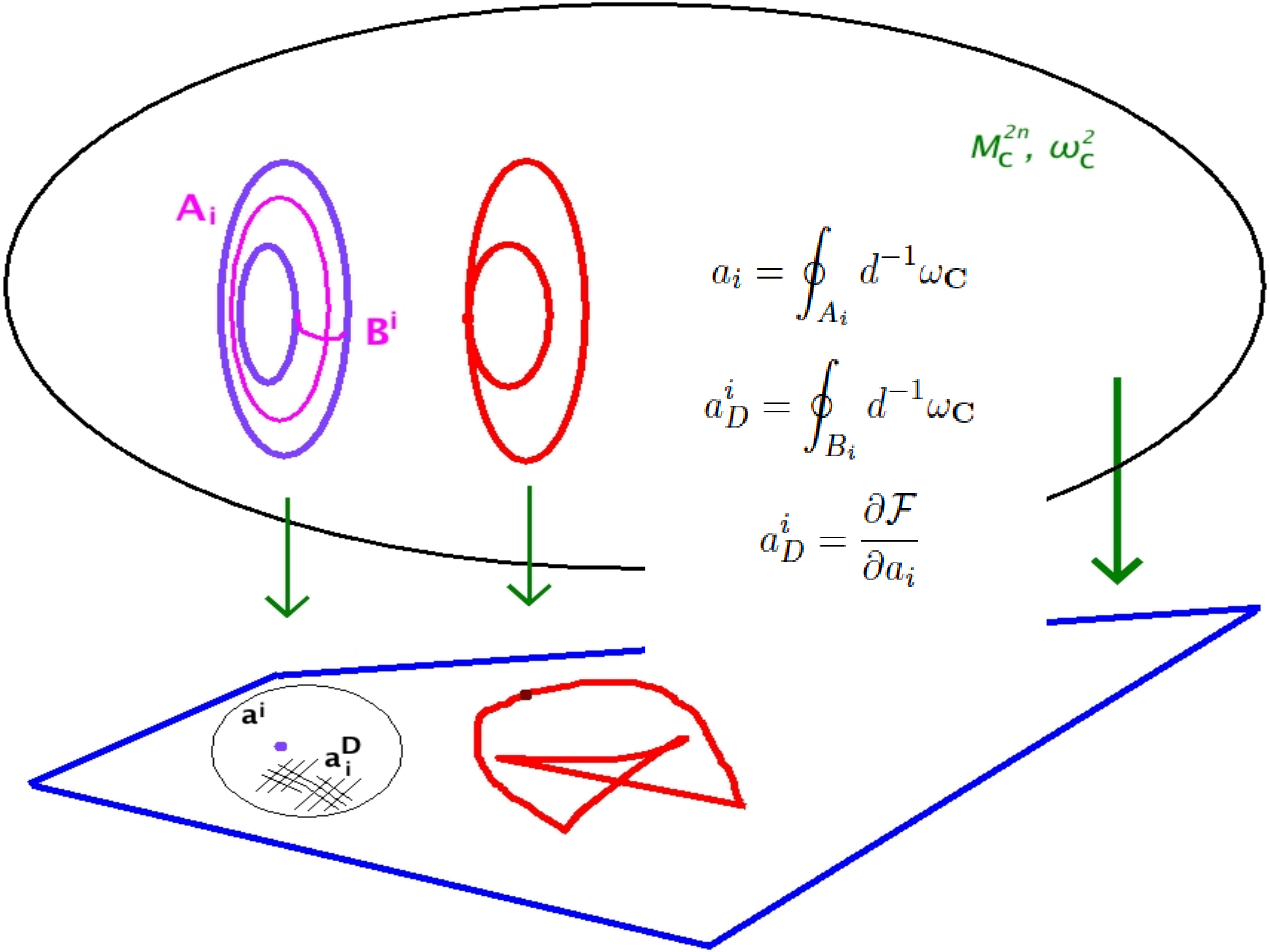
In the mid-nineties of the 20 century it was understood, that

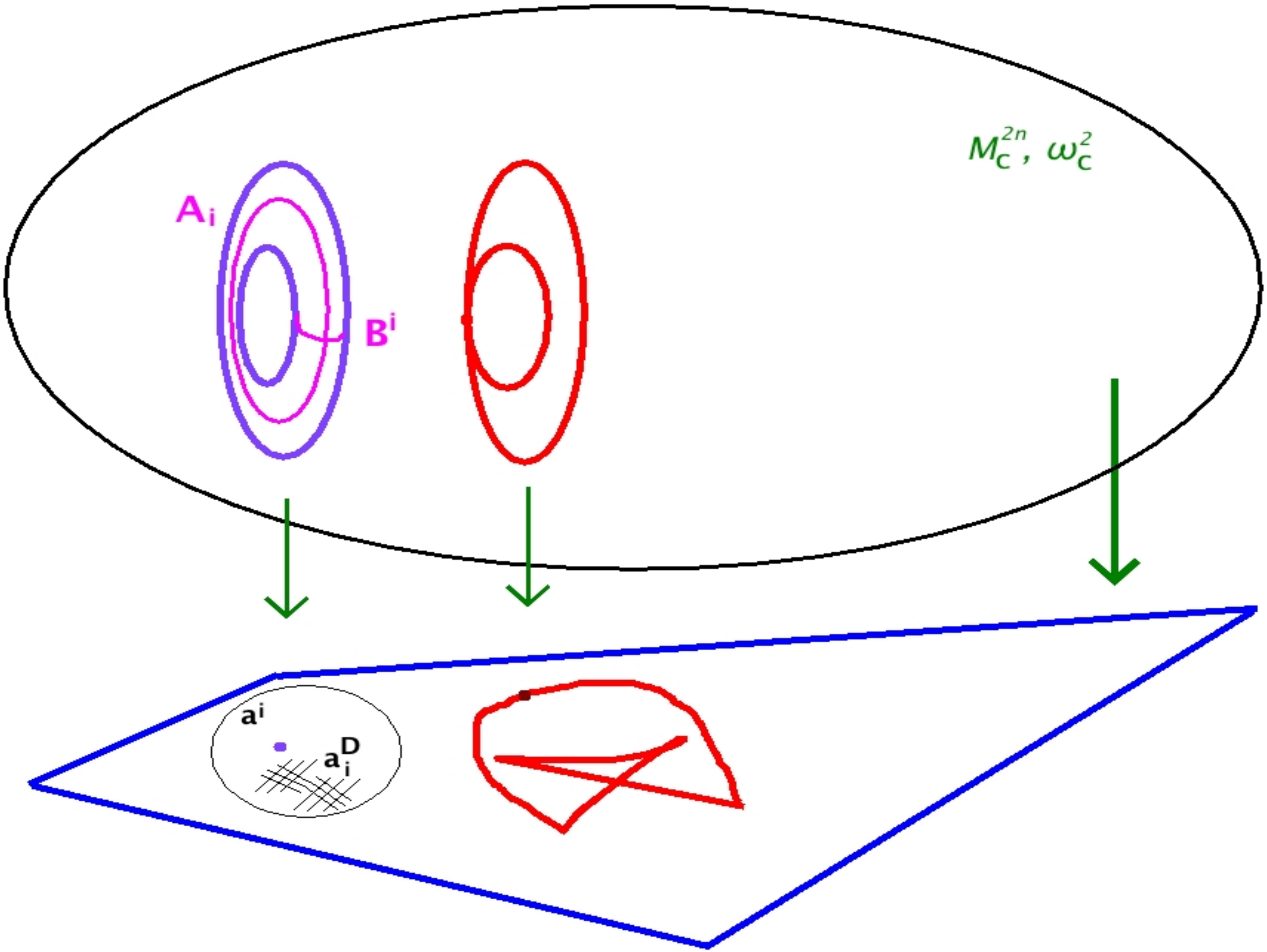
The geometry of the moduli space of vacua of $N=2$ supersymmetric gauge theory is identified with that of a base of an algebraic integrable system

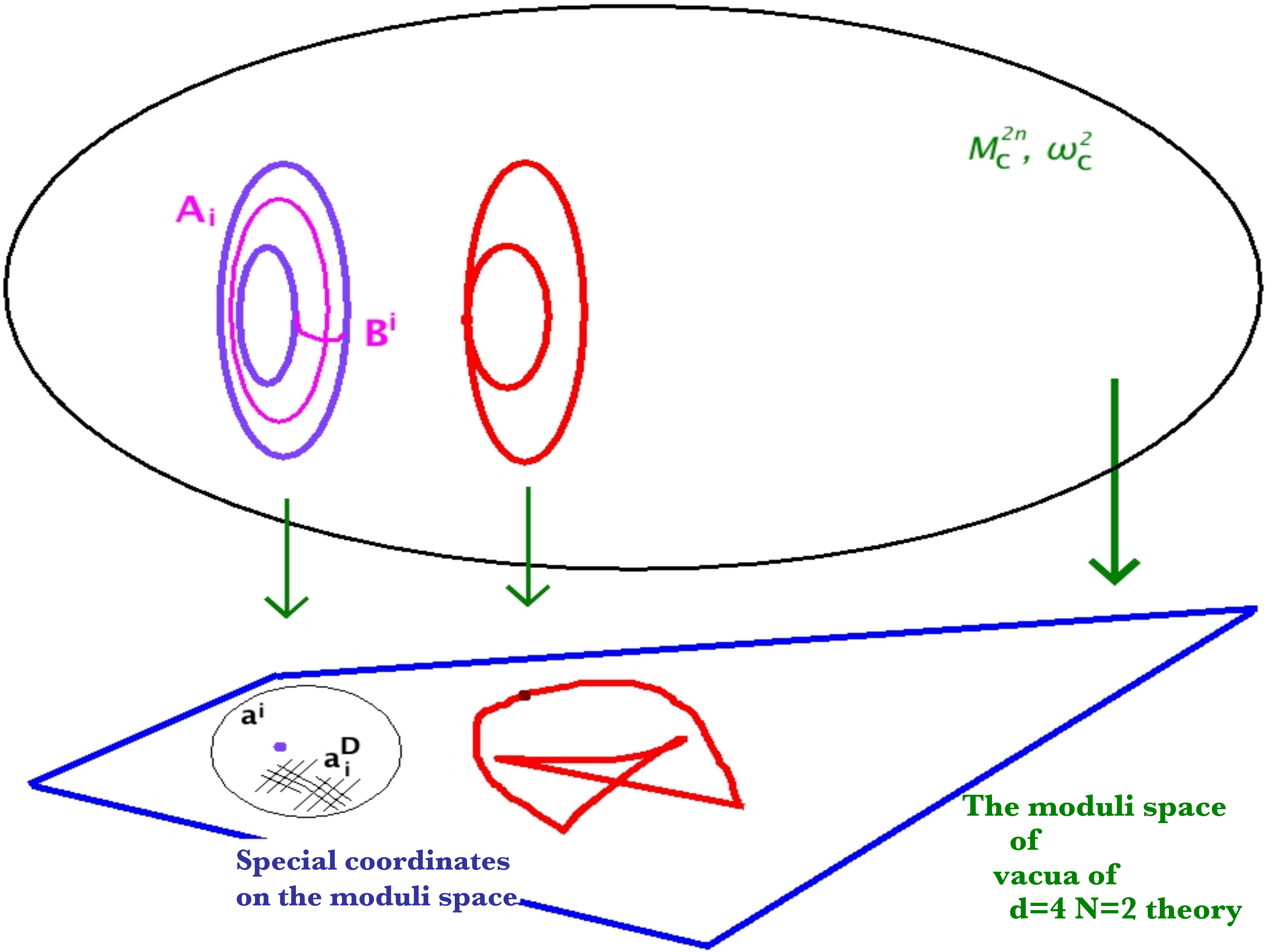
Donagi, Witten

Gorsky, Krichever, Marshakov, Mironov, Morozov


Liouville tori
M_c^{2n}, ω_c^2
The Base


M_C^{2n}, ω_C^2
A_i
B^i
$a_i = \oint_{A_i} d^{-1}\omega_{\mathbf{C}}$
$a_D^i = \oint_{B_i} d^{-1}\omega_{\mathbf{C}}$
$a_D^i = \frac{\partial \mathcal{F}}{\partial a_i}$
a^i
a_i^D

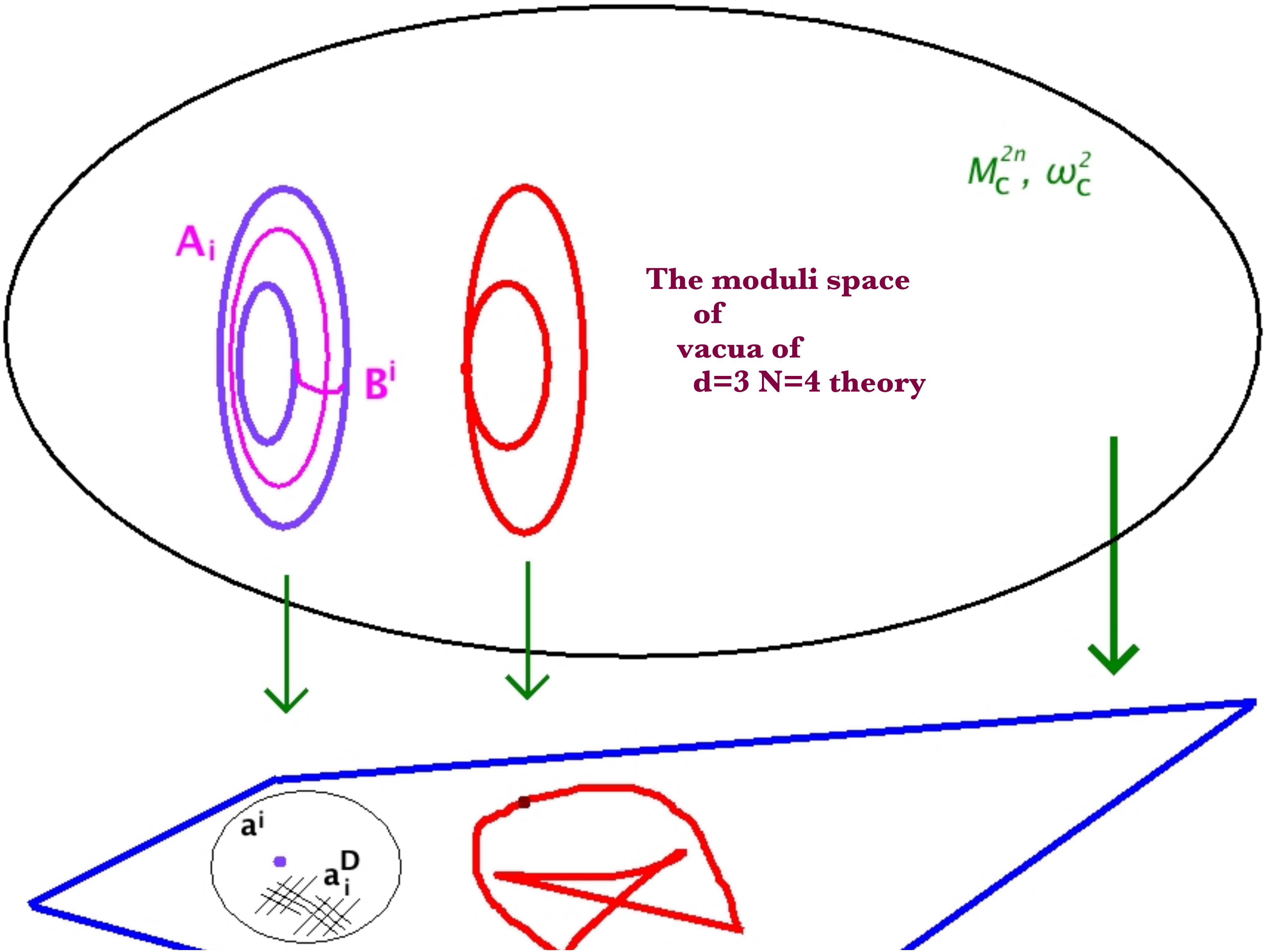

M_c^{2n}, ω_c^2
A_i
B^i
a^i
a_i^D


M_c^{2n}, ω_c^2
A_i
B^i
a^i
a_i^D
Special coordinates
on the moduli space
The moduli space
of
vacua of
d=4 N=2 theory

The Coulomb branch
of the moduli space
of vacua of the
d=4 N=2 supersymmetric
gauge theory
is the base
of a complex (algebraic)
integrable system

The Coulomb branch
of the same theory,
compactified on a circle down
to three dimensions
is the phase
space of
the same integrable system

This moduli space is a
hyperkahler manifold, and
it can arise both as a Coulomb
branch of one susy gauge
theory and as a Higgs branch
of another susy theory.
This is the 3d mirror
symmetry.



M_c^{2n}, ω_c^2
A_i
B^i
The moduli space
of
vacua of
d=3 N=4 theory
a^i
a_i^D

In particular, one can start
with a
six dimensional (0,2)
ADE
superconformal field theory,
and compactify it on
$\Sigma \times S^1$
with the genus g Riemann surface Σ

The resulting effective susy gauge theory in three dimensions will have 8 supercharges (with the appropriate twist along Σ)

The resulting effective susy gauge theory in three dimensions will have the Hitchin moduli space as the moduli space of vacua.
The gauge group in Hitchin's equations will be the group of the same A,D,E type
as in the definition of the (0,2) theory.

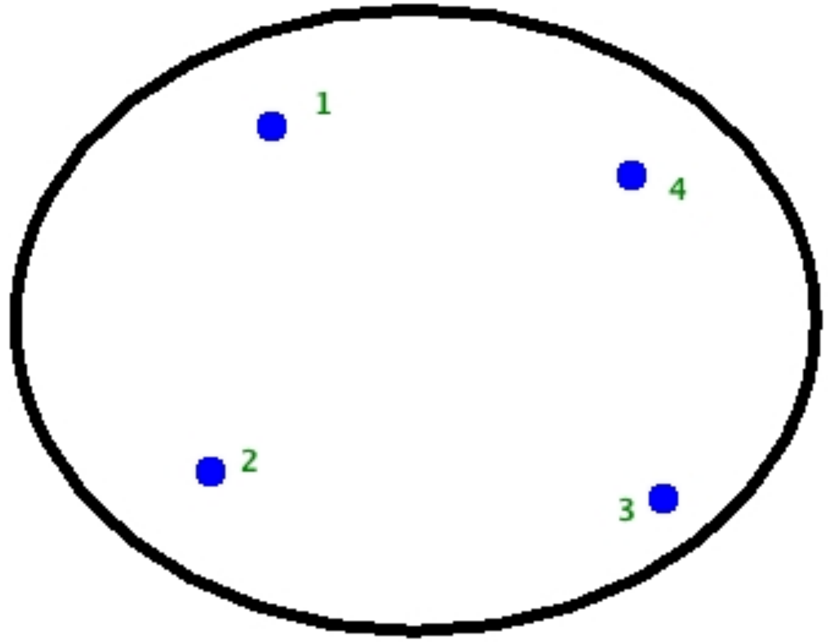
The Hitchin moduli space is the Higgs branch of the 5d gauge theory compactified on Σ

The mirror theory, for which the
Hitchin moduli space
is the Coulomb branch,
is conjectured *Gaiotto*, in the A_1 case,
to be the $SU(2)^{3g-3}$
gauge theory in 4d,
compactified on a circle,
with some matter hypermultiplets in
the
tri-fundamental and/or adjoint
representations

One can allow the Riemann surface
with **n** punctures,
with some local parameters
associated with the punctures.
The gauge group is then **$SU(2)^{3g-3+n}$**
with matter hypermultiplets in the
fundamental,
bi-fundamental,
tri-fundamental representations,
and, sometimes, in the adjoint.

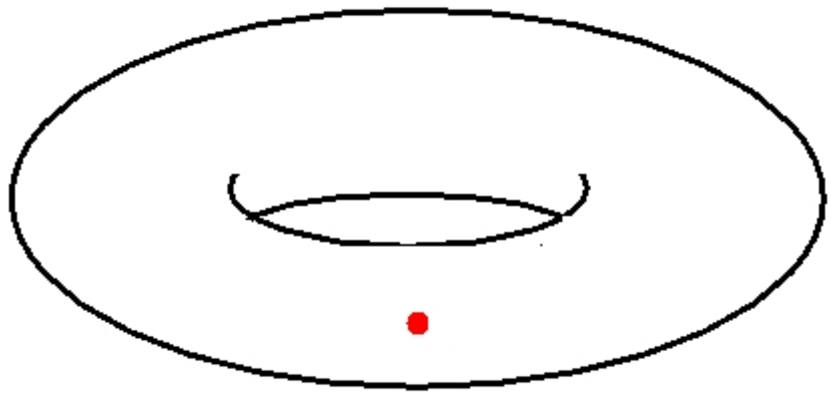
For example,
the SU(2) with N_f=4

Corresponds to the
Riemann surface
of genus zero
with 4 punctures.
The local data at the punctures
determines the masses



For example,
the N=2* SU(2) theory

Corresponds to the
Riemann surface
of genus one
with 1 punctures.
The local data at the puncture
determines the mass of the adjoint



From now on we shall be discussing these « generalized quiver theories »

- The integrable system corresponding to the moduli space of vacua of the 4d theory is the SU(2) Hitchin system on

  The punctured Riemann surface Σ

Hitchin system

Gauge theory on a Riemann surface

The gauge field A_μ and the twisted Higgsfield Φ_μ in the adjoint representation are required to obey:

Hitchin equations

$$\bar{\partial}_{\bar{z}}\Phi_z + [A_{\bar{z}}, \Phi_z] = 0$$

$$\partial_z\Phi_{\bar{z}} + [A_z, \Phi_{\bar{z}}] = 0$$

$$F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0$$

Hitchin system

Modulo gauge transformations:

$$(A_\mu, \Phi_\mu) \longrightarrow (g^{-1} A_\mu g + g^{-1} \partial_\mu g,\, g^{-1} \Phi_\mu g)$$

We get the moduli space $\mathcal{M}_H$

Hyperkahler structure of $\mathcal{M}_H$

- Three complex structures: I,J,K
- Three Kahler forms: ω_I , ω_J , ω_K
- Three holomorphic symplectic forms:

  Ω_I , Ω_J , Ω_K

Hyperkahler structure of $\mathcal{M}_H$

Three Kahler forms: ω_I , ω_J , ω_K

Three holomorphic symplectic forms:

$$\Omega_I = \omega_J + i\,\omega_K,$$
$$\Omega_J = \omega_K + i\,\omega_I,$$
$$\Omega_K = \omega_I + i\,\omega_J$$

Hyperkahler structure of $\mathcal{M}_H$

$$\omega_I = \int_\Sigma \mathrm{Tr}\,(\delta A \wedge \delta A + \delta\Phi \wedge \delta\Phi)$$

$$\omega_J = \int_\Sigma \mathrm{Tr}\,(\delta A \wedge *\delta\Phi)$$

$$\omega_K = \int_\Sigma \mathrm{Tr}\,(\delta A \wedge \delta\Phi)$$

$$\Omega_I = \int_\Sigma \mathrm{Tr}\,(\delta\Phi_z \wedge \delta A_{\bar{z}})\ \mathrm{d}^2 z$$

$$\Omega_J = \int_\Sigma \mathrm{Tr}\,(\delta\mathcal{A} \wedge \delta\mathcal{A})$$

$\mathcal{A} = A + i\,\Phi$

Hyperkahler structure

The linear combinations, parametrized by the points on a twistor two-sphere S^2

$a\,I + b\,J + c\,K$, where

$a^2+b^2+c^2=1$

The integrable structure

In the complex structure I,

the holomorphic functions are: for each Beltrami differential $\mu^{(i)}$, $i=3g-3+n$

$$H_i = \int_\Sigma \mu^{(i)} \mathrm{Tr}\Phi_z^2$$

The integrable structure

These functions Poisson-commute w.r.t. Ω_I

$$\{ H_i, H_j \} = 0$$

The integrable structure

The generalization to other groups is known,

e.g. for G=SU(N)

$$H_{p,i} = \int_\Sigma \nu_{[p]}^{(i)} \mathrm{Tr}\Phi_z^p$$

$$\nu_{[p]}^{(i)} \in H^1(\Sigma, K_\Sigma^{\otimes(1-p)}),$$

$$i = 1, \ldots, (2p-1)(g-1), \qquad p = 2, \ldots, N$$

The integrable structure

The action-angle variables:

Fix <u>the level of the integrals of motion</u>,

ie fix the values of all H_i's

<u>Equivalently</u>:

fix the (spectral)

curve C inside $T^*\Sigma$

$\mathrm{Det}(\lambda - \Phi_z) = 0$

Its Jacobian is the Liouville torus, and

The periods of λdz give the special coordinates

$a_i,\ a_D^i$

The quantum integrable structure

The naïve quantization, using that
in the complex structure I

$\mathcal{M}_H$ is almost = $T^*\mathcal{M}$

Where $\mathcal{M}$=Bun_G

Φ_z becomes the derivative
H_i become the differential operators.
More precisely, one gets the space of twisted (by $K^{1/2}_{\mathcal{M}}$)
differential operators on $\mathcal{M}$=Bun_G

Thinking about the Ω - deformation
of the four dimensional gauge theory,
leads to the conclusion that
the quantum Hitchin system
Is governed by a Yang-Yang function,
The effective twisted superpotential

$$\widetilde{W}(a_1, \ldots, a_{3g-3+n}; m_1, \ldots, m_n, \tau_1, \ldots, \tau_{3g-3+n}; \varepsilon)$$

Here comes the experimental fact

The effective twisted superpotential,

the YY function of

the quantum Hitchin system:

In fact has a

classical mechanical meaning!

$\mathcal{M}_H$ as the moduli space of G_C flat connections

In the complex structure J the holomorphic variables are:

$$\mathcal{A}_\mu = A_\mu + i\,\Phi_\mu$$

which obey (modulo complexified gauge transformations):

$$\mathcal{F} = d\mathcal{A} + [\mathcal{A},\mathcal{A}] = 0$$

$\mathcal{M}_H$ as the moduli space of G_C flat connections

In this complex structure $\mathcal{M}_H$

is defined without a reference to the

complex structure of Σ

$$\mathcal{M}_H = \mathrm{Hom}\,(\,\pi_1(\Sigma)\,,\,G_C\,)_{/}\,G_C$$

$\mathcal{M}_H$ as the moduli space of G_C flat connections

However $\mathcal{M}_H$

Contains interesting complex Lagrangian submanifolds which do depend on the complex structure of Σ

$\mathcal{L}_\Sigma$ = the variety of G-opers

Beilinson, Drinfeld
Drinfeld, Sokolov

$\mathcal{M}_H$ as the moduli space of G_C flat connections

$\mathcal{L}_\Sigma$ = the variety of G-opers

$$\mathcal{A}_z = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \qquad \mathcal{A}_{\bar{z}} = \begin{pmatrix} -\frac{1}{2}\partial\mu & \mu \\ \mu T - \frac{1}{2}\partial^2\mu & \frac{1}{2}\partial\mu \end{pmatrix}$$

The Beltrami differential **μ** is fixed, the projective structure **T** is arbitrary, provided

Beilinson, Drinfeld
Drinfeld, Sokolov

$\mathcal{M}_H$ as the moduli space of G_C flat connections

$\mathcal{L}_\Sigma$ = the variety of G-opers

$$\mathcal{A}_z = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \qquad \mathcal{A}_{\bar{z}} = \begin{pmatrix} -\frac{1}{2}\partial\mu & \mu \\ \mu T - \frac{1}{2}\partial^2\mu & \frac{1}{2}\partial\mu \end{pmatrix}$$

The Beltrami differential **μ** is fixed, the projective structure **T** is arbitrary, provided it is compatible with the complex structure defined by

$$\bar{\partial} - \mu\partial$$

$\mathcal{M}_H$ as the moduli space of G_C flat connections

$\mathcal{L}_\Sigma$:
$$\mathcal{A}_z = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \qquad \mathcal{A}_{\bar z} = \begin{pmatrix} -\frac{1}{2}\partial\mu & \mu \\ \mu T - \frac{1}{2}\partial^2\mu & \frac{1}{2}\partial\mu \end{pmatrix}$$

i.e.

$$\left(\bar\partial - \mu\partial - 2\partial\mu\right) T = -\frac{1}{2}\partial^3\mu$$

Opers on a sphere

For example, on a two-sphere with n punctures, these conditions translate to the following definition of the space of opers with regular singularities: we are studying the space of differential operators of second order, of the form

$$-\partial^2 + T$$

$$T = \sum_{i=1}^{n} \frac{\Delta_i}{(z - z_i)^2} + \frac{\varepsilon_i}{z - z_i}$$

Opers on a sphere

Where Δ_i are fixed, $\Delta_i = \nu_i(\nu_i - 1)$

while the accessory parameters ε_i obey

$$\sum_{i=1}^{n} \varepsilon_i = 0$$

$$\sum_{i=1}^{n} z_i\varepsilon_i + \Delta_i = 0$$

$$\sum_{i=1}^{n} z_i^2\varepsilon_i + z_i\Delta_i = 0$$

Opers on a sphere

All in all we get a (n-3)-dimensional subvariety in the

2(n-3) dimensional moduli space of flat connections on the n-punctured sphere with fixed conjugacy classes of the monodromies around the punctures:

$$m_i = \mathrm{Tr}\,(g_i)$$

$$m_i = 2\cos(2\pi\nu_i)$$

The main conjecture

The YY function is the generating function of the variety of opers

The variety of opers is Lagrangian with respect to Ω_J

We shall now construct a system of Darboux coordinates on M_H

α_i, β_i

$$\Omega_J = \sum_{i=1}^{3g-3+n} d\alpha_i \wedge d\beta_i$$

So that

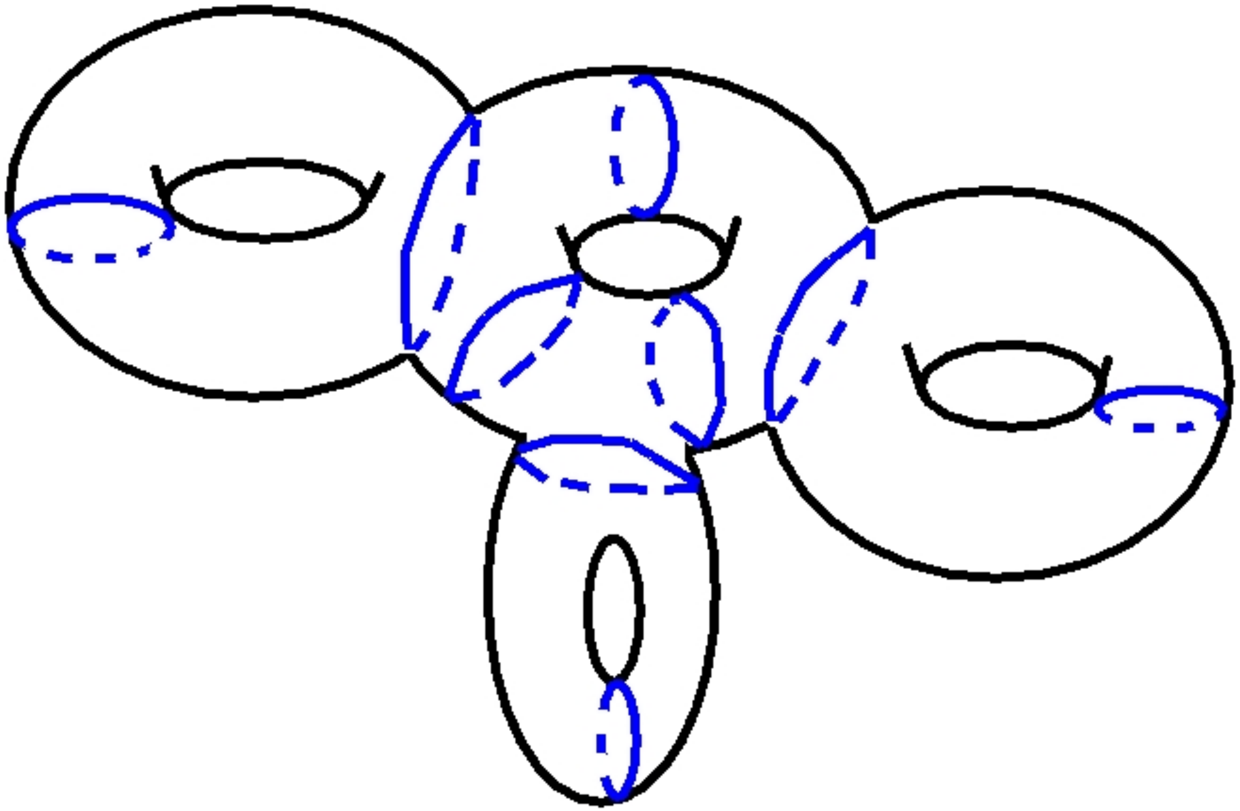
$\mathfrak{L}_{\Sigma}$ = the variety of G-opers,

is described by the equations

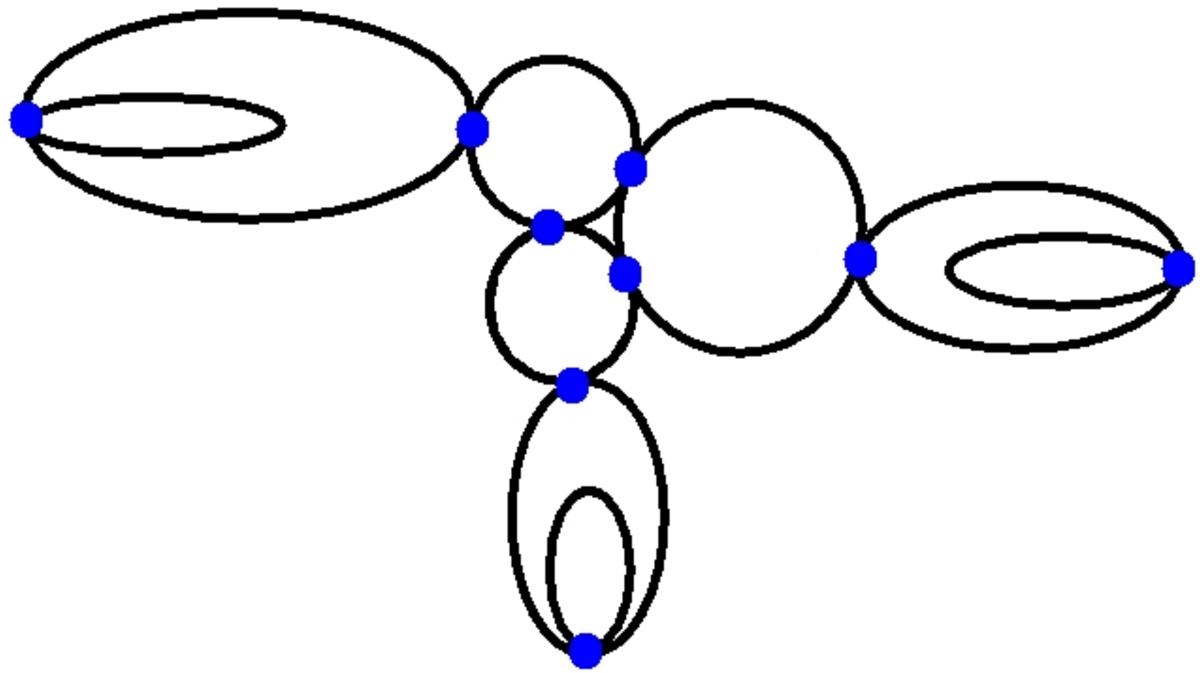
$$\beta_i = \frac{1}{\varepsilon} \frac{\partial \widetilde{W}}{\partial \alpha_i}$$

$$a_i = \varepsilon\, \alpha_i$$

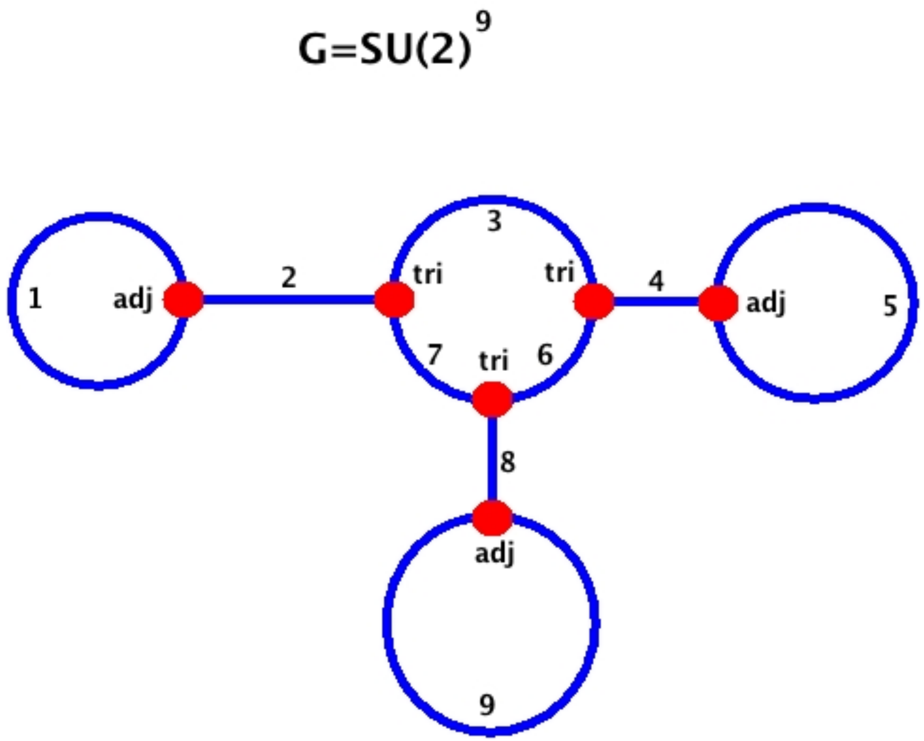
The moduli space is going to be covered by a multitude of Darboux coordinate charts, one per every pair-of-pants decomposition (and some additional discrete choice)



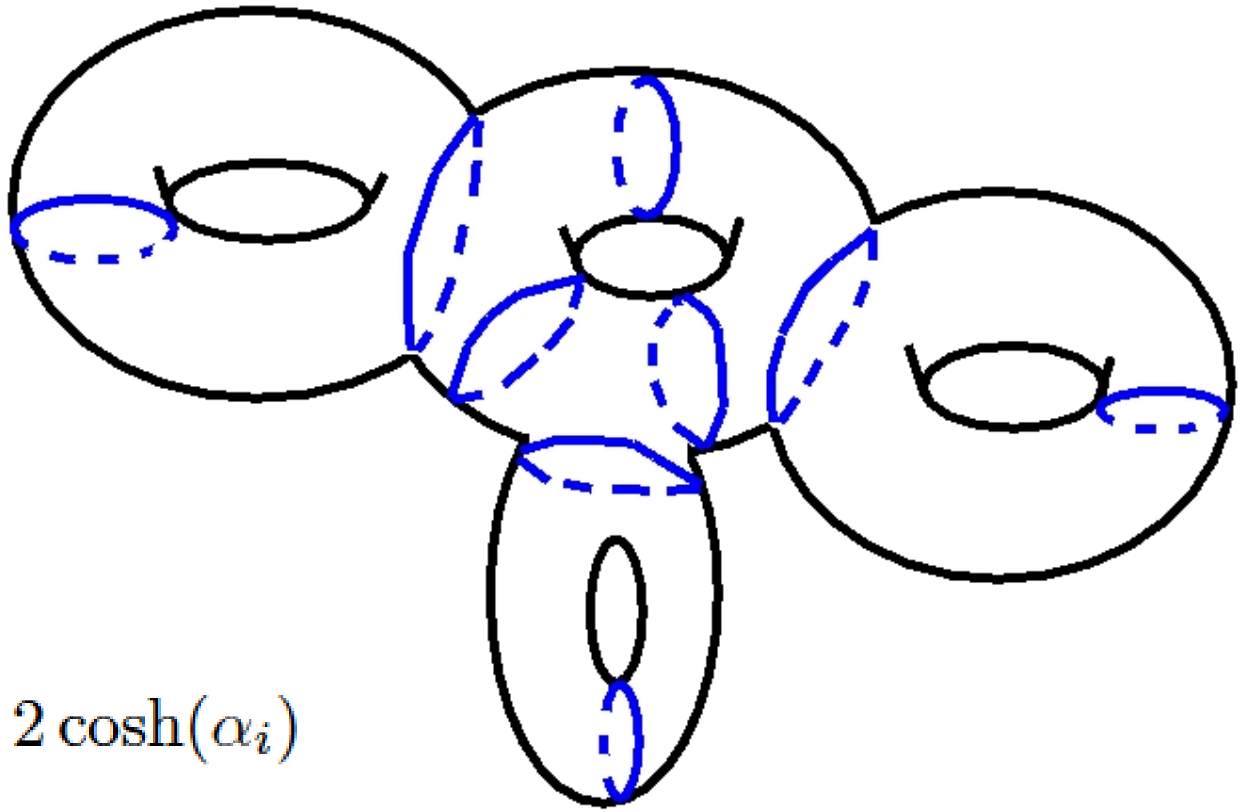
Equivalently,
a coordinate chart
U_Γ
per maximal degeneration
Γ of the complex structure on Σ



The maximal complex structure degenerations = The weakly coupled gauge theory descriptions of Gaiotto theories, e.g. for the previous example



The α_i coordinates are nothing but the logarithms of the eigenvalues of the monodromies around the blue cycles:



$$\mathrm{Tr} P \exp \oint_{\mathcal{C}_i} \mathcal{A} = 2\cosh(\alpha_i)$$

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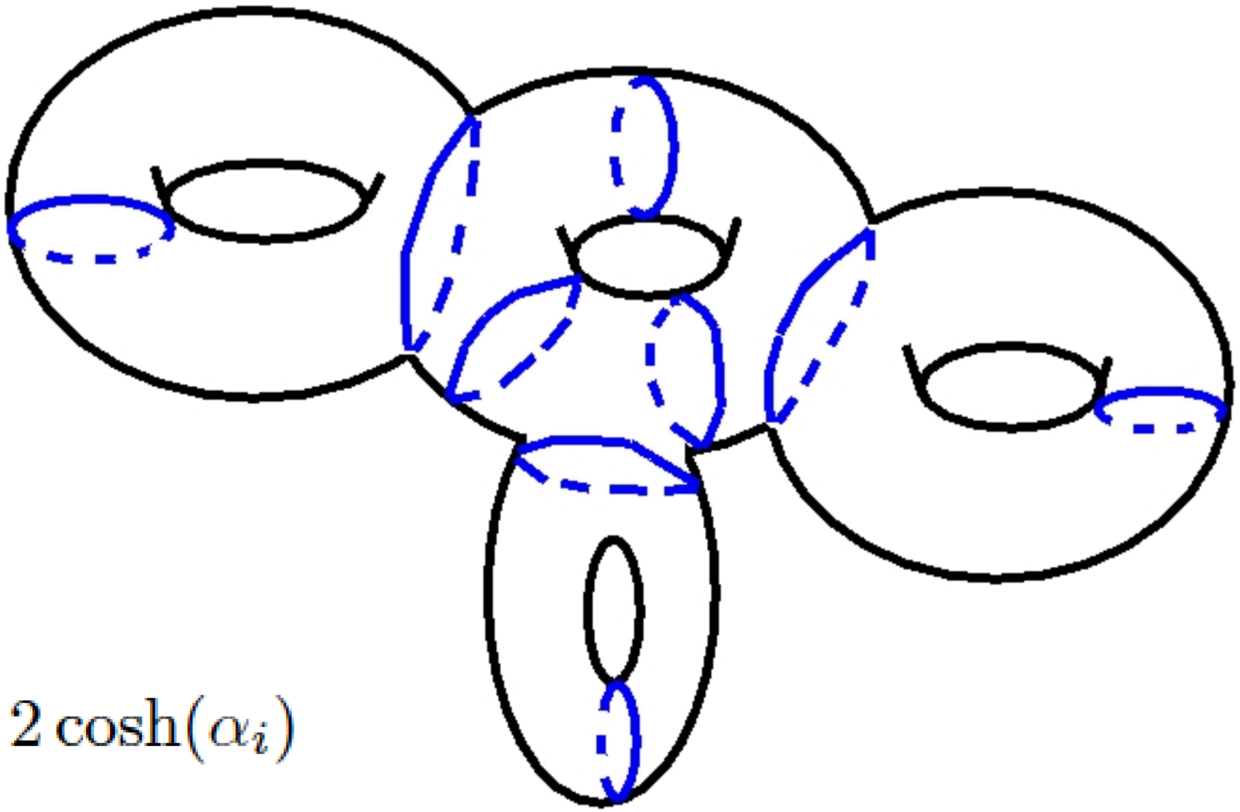
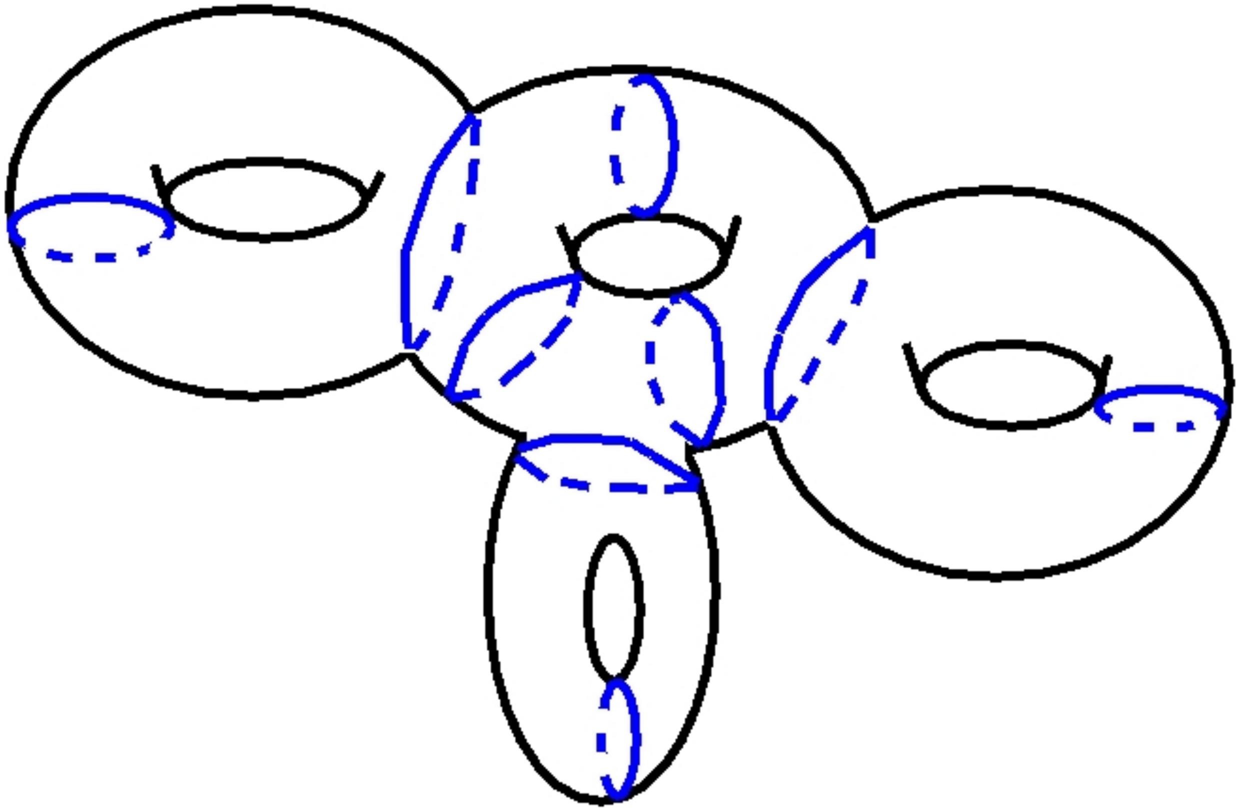


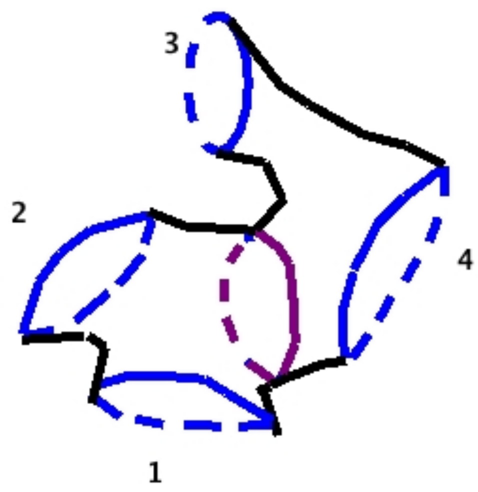
$$\mathrm{Tr} P \exp \oint_{\mathcal{C}_i} \mathcal{A} = 2\cosh(\alpha_i)$$

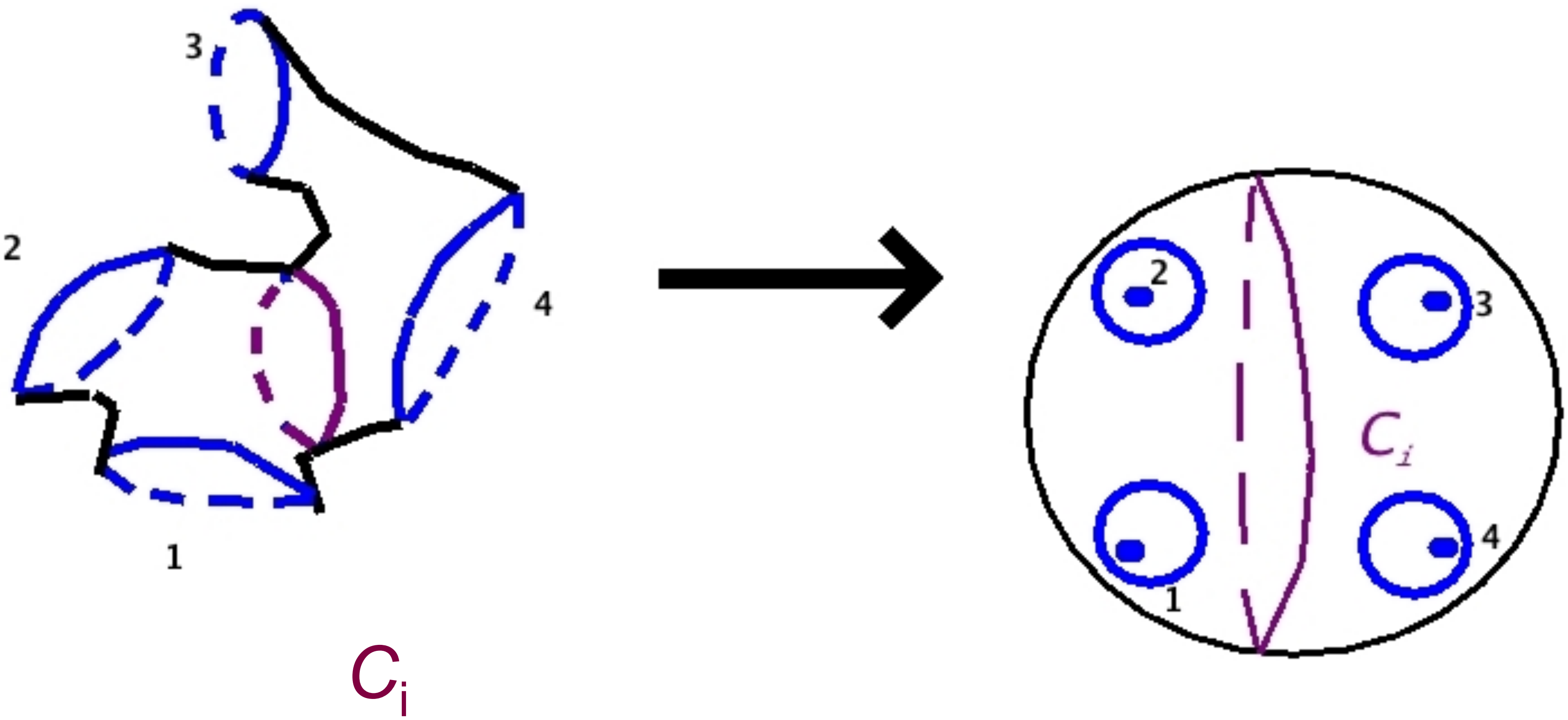
cf. *Drukker, Gomis, Okuda, Teschner; Verlinde; Verlinde*

The $\color{red}{\beta_i}$ coordinates are defined from the local data involving the cycle C_i and its four neighboring cycles (or one, if the blue cycle belongs to a genus one component) :



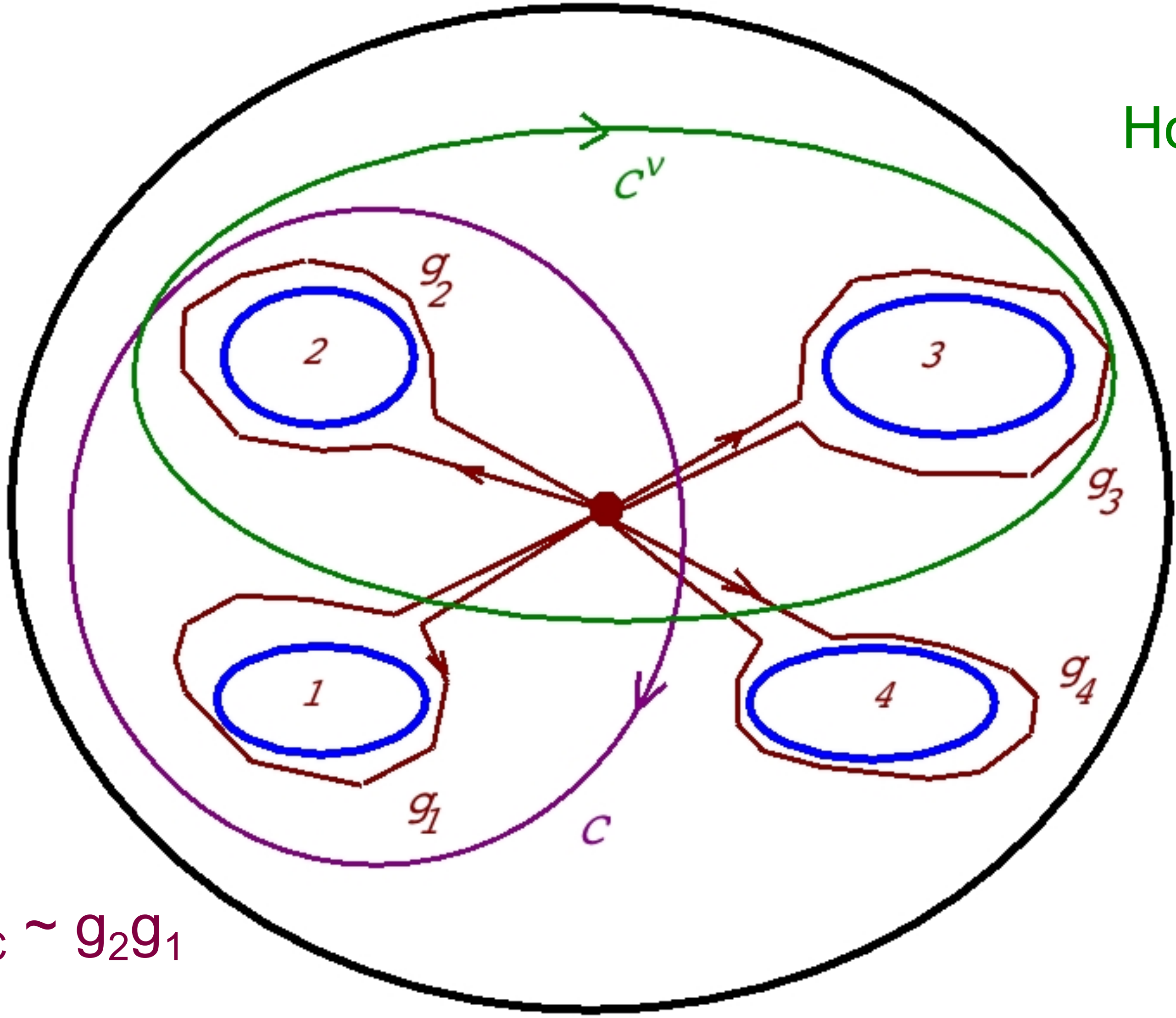
The local data involving the cycle C_i and its four neighboring cycles :





3
2
4
1
C_i
2
3
C_i
4
1

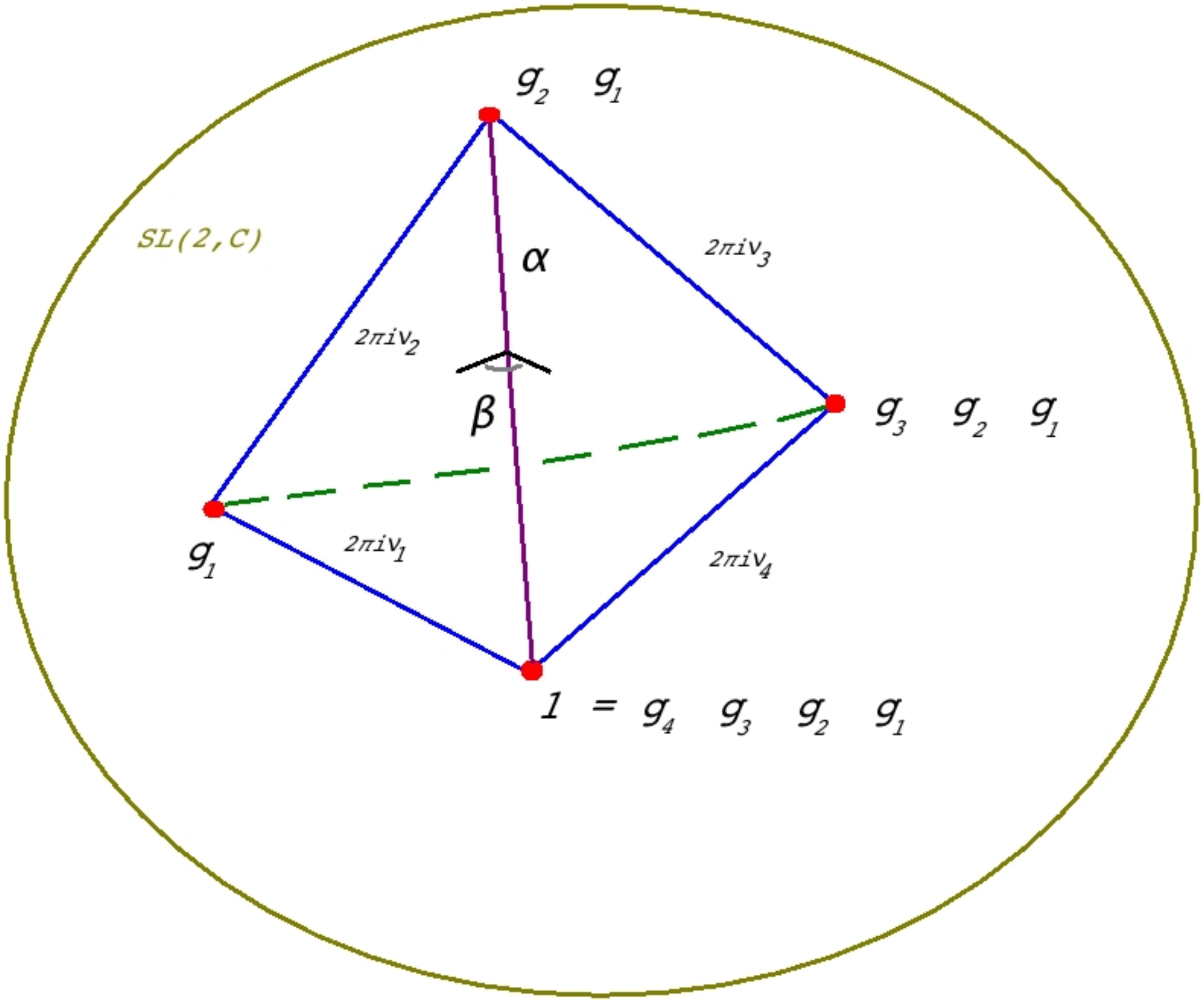
The local picture:
four holes, two interesting holonomies



$Hol_{C^{\vee}} \sim g_3 g_2$

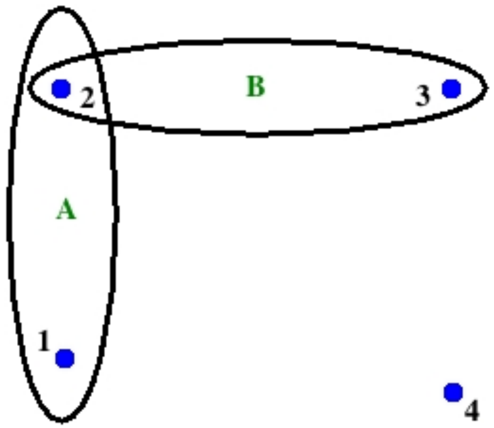
$Hol_C \sim g_2 g_1$

Complexified hyperbolic geometry:



The coordinates α_i ,β_i can be

thus explicitly expressed in terms of the traces of the monodromies:



$B = \mathrm{Tr}\ (\mathrm{Hol}_C{}^{v} \sim g_3 g_2\ )$

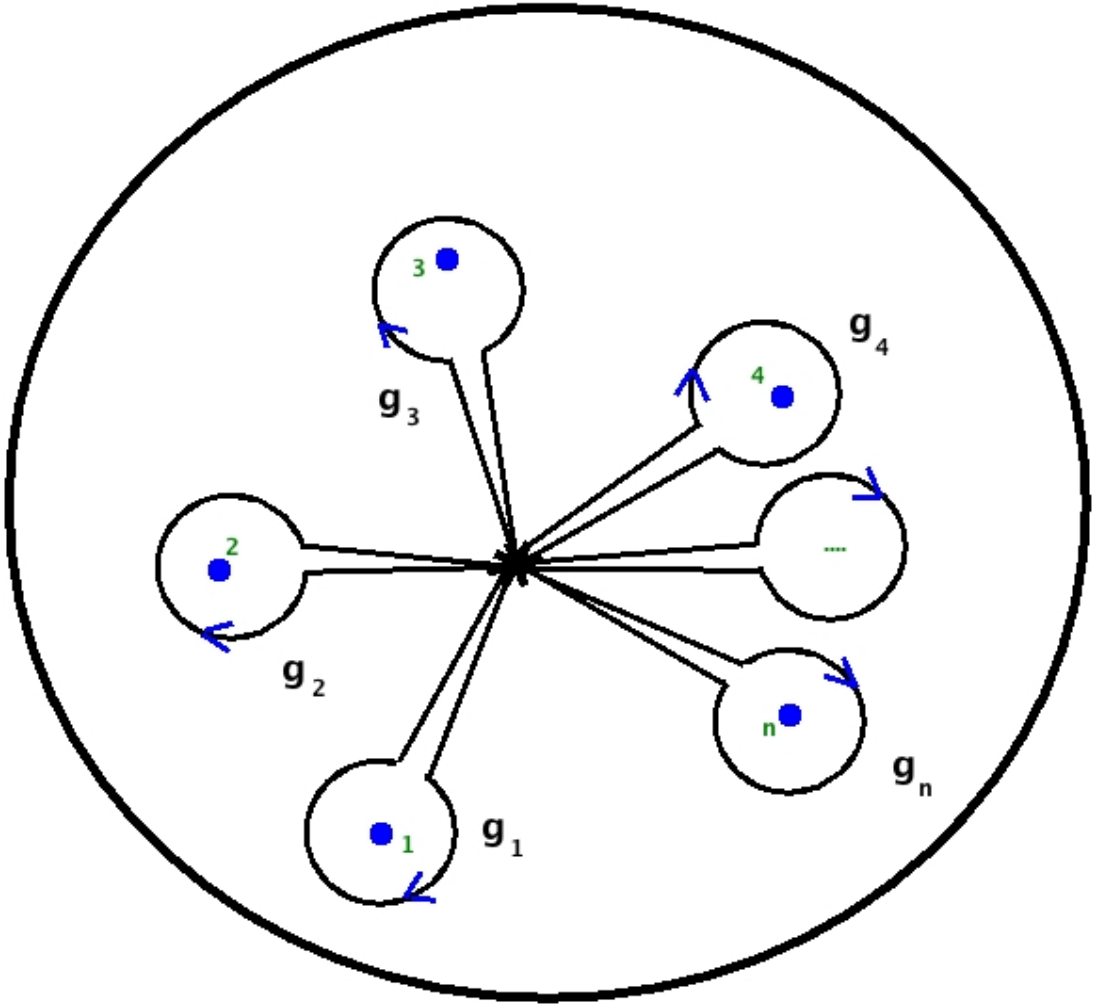
$$m_i = 2\cos(2\pi\nu_i) = \mathrm{Tr}\ g_i$$

$$A = 2\cosh(\alpha) = \mathrm{Tr}\ (\mathrm{Hol}_C \sim g_2 g_1\ )$$

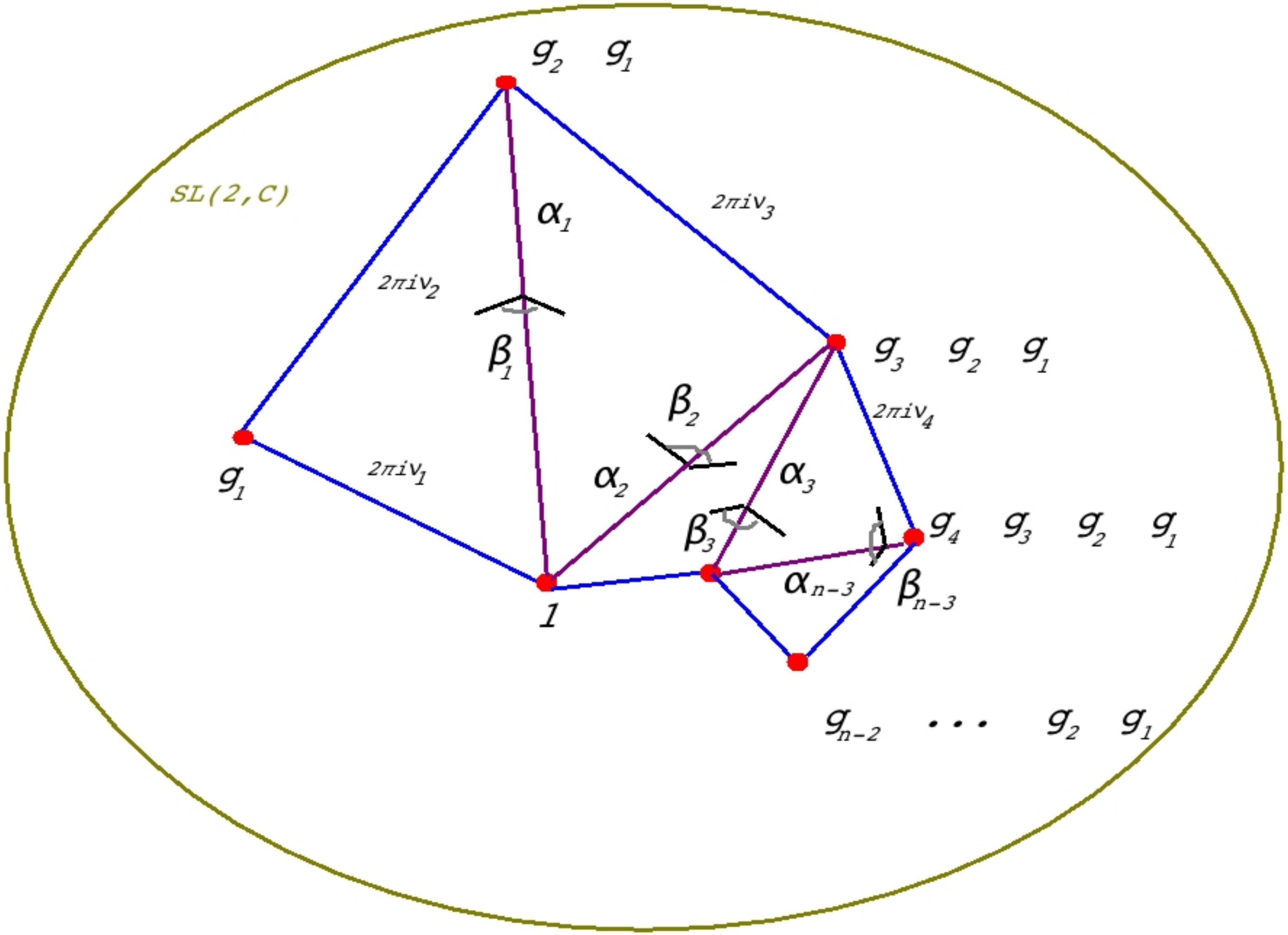
$$\frac{B(A^2-4)+2(m_2m_3+m_1m_4)-A(m_1m_3+m_2m_4)}{\sqrt{c_{12}(A)c_{34}(A)}} = 2\cosh(\beta)$$

$$c_{ij}(A) = A^2 + m_i^2 + m_j^2 - Am_im_j - 4$$

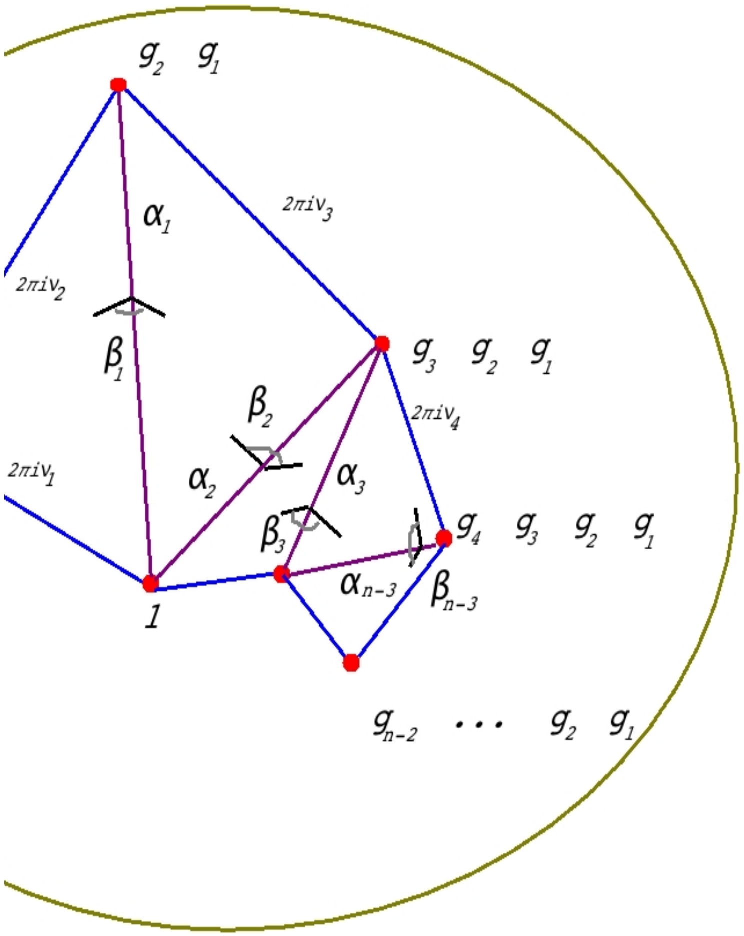
The construction of the hyperbolic polygon generalizes to the case of n punctures:



The construction of the hyperbolic polygon generalizes to the case of n punctures:



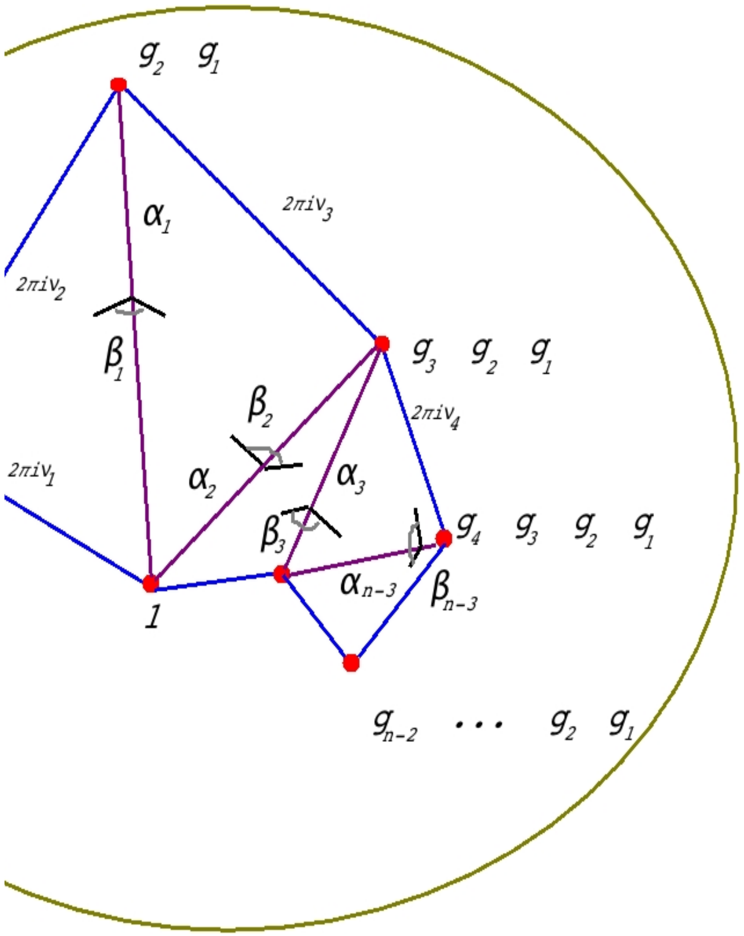
For g_i obeying some reality conditions, e.g. SU(2), SL(2,R), SU(1,1), SO(1,2), or, R^3



We get the real polygons in
S^3, H^3, $R^{2,1}$, E^3
our coordinates reduce to the ones studied by
Klyachko,
Kapovich, Millson
Kirwan, Foth,

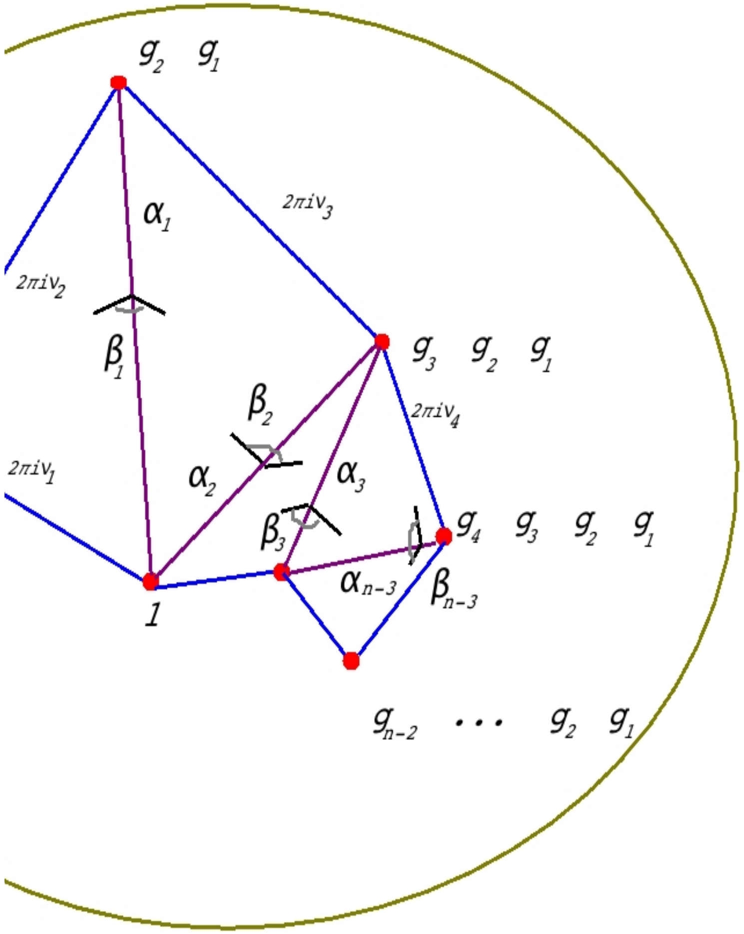
NB: The Loop quantum gravity community (Baez, Charles, Rovelli, Roberts, Freidel, Krasnov, Livin, ….) uses different coordinates

Our polygons sit in the group manifold



<u>An interesting problem:</u>
Relate our coordinates to the coordinates
Introduced by *Fock and Goncharov*,
Based on triangulations of the Riemann surface with punctures.

Our polygons sit in the group manifold



<u>An interesting problem:</u>

Relate our coordinates to the coordinates

Introduced by *Fock and Goncharov*,

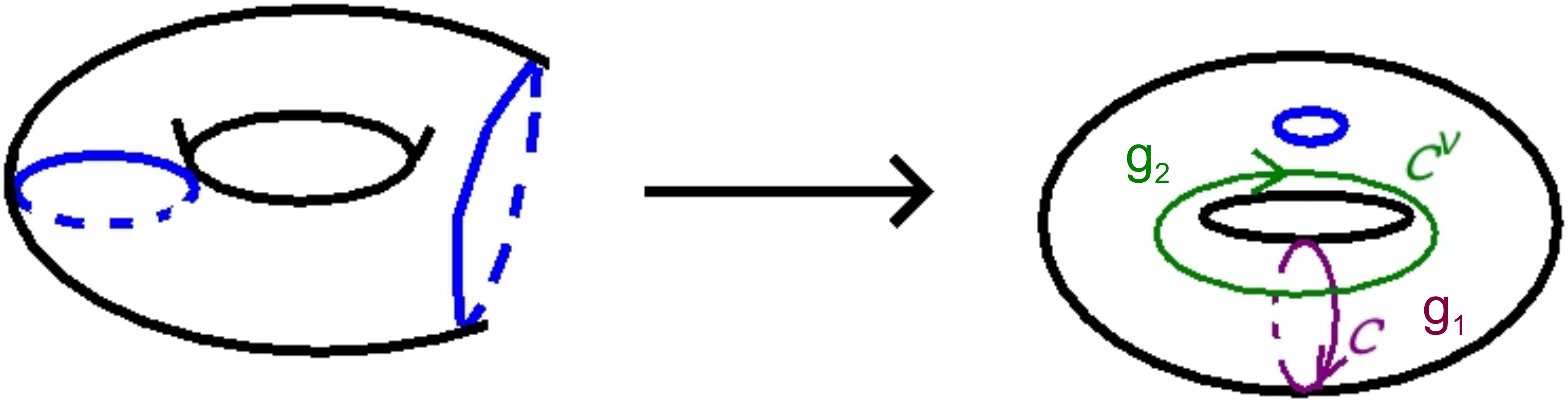
Based on triangulations of the Riemann surface with punctures.

The FG coordinates are the basis of the *Gaiotto-Moore-Neitzke* work on the hyperkahler metric on

$$\mathcal{M}_H$$

The local data involving the cycle C_i on the genus one component

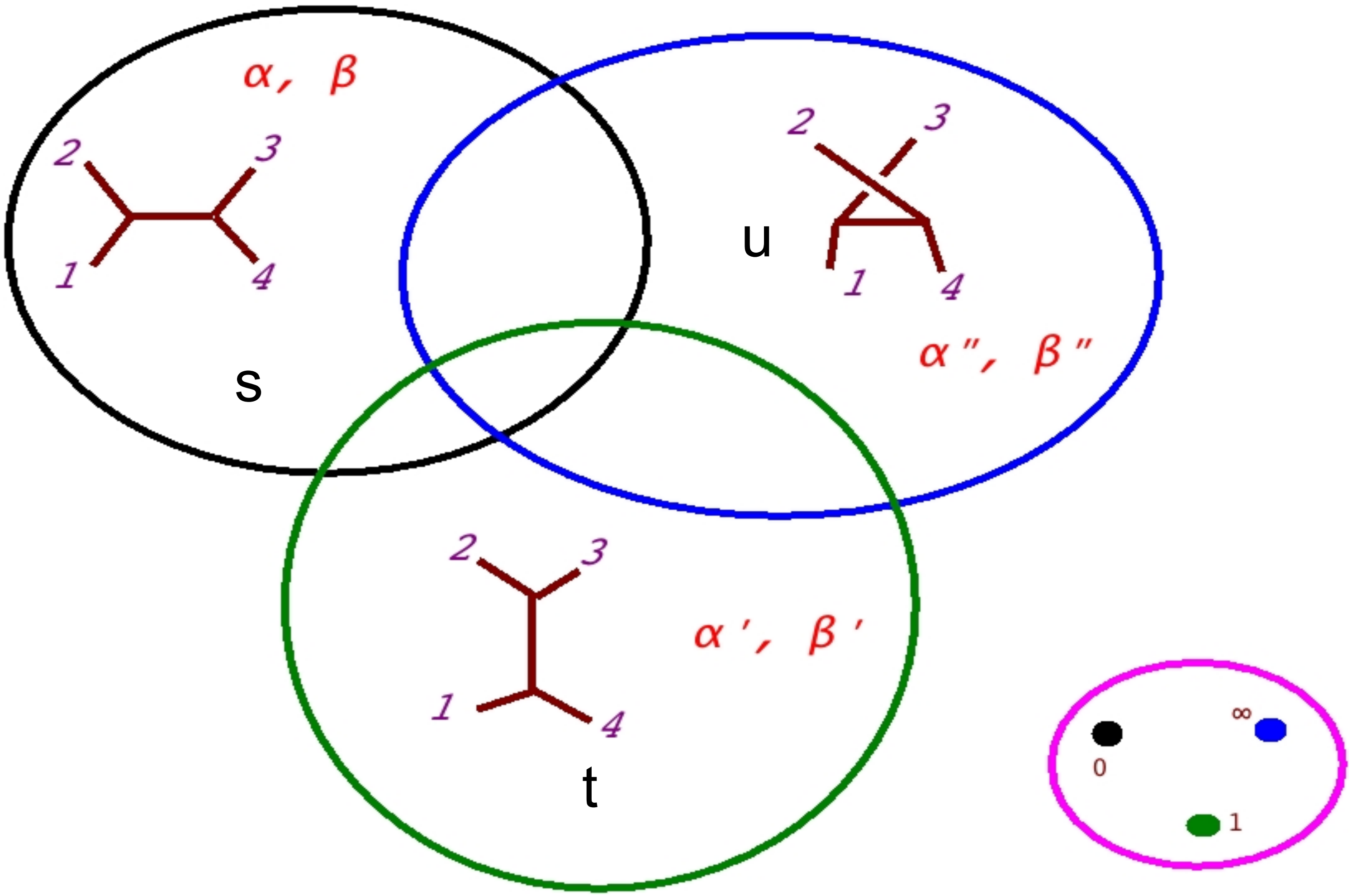
$$\mathrm{tr}\,\left(g_1 g_2 g_1^{-1} g_2^{-1}\right) = m$$



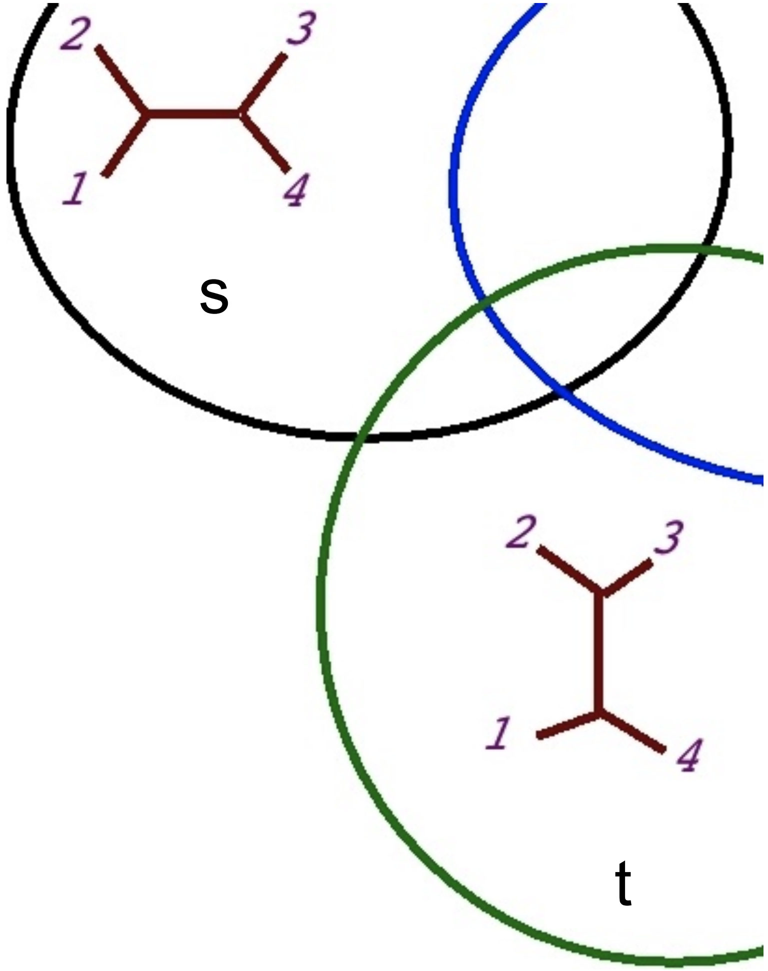
$$A = \mathrm{tr}\,(g_1) = 2\cosh(\alpha)\,,$$

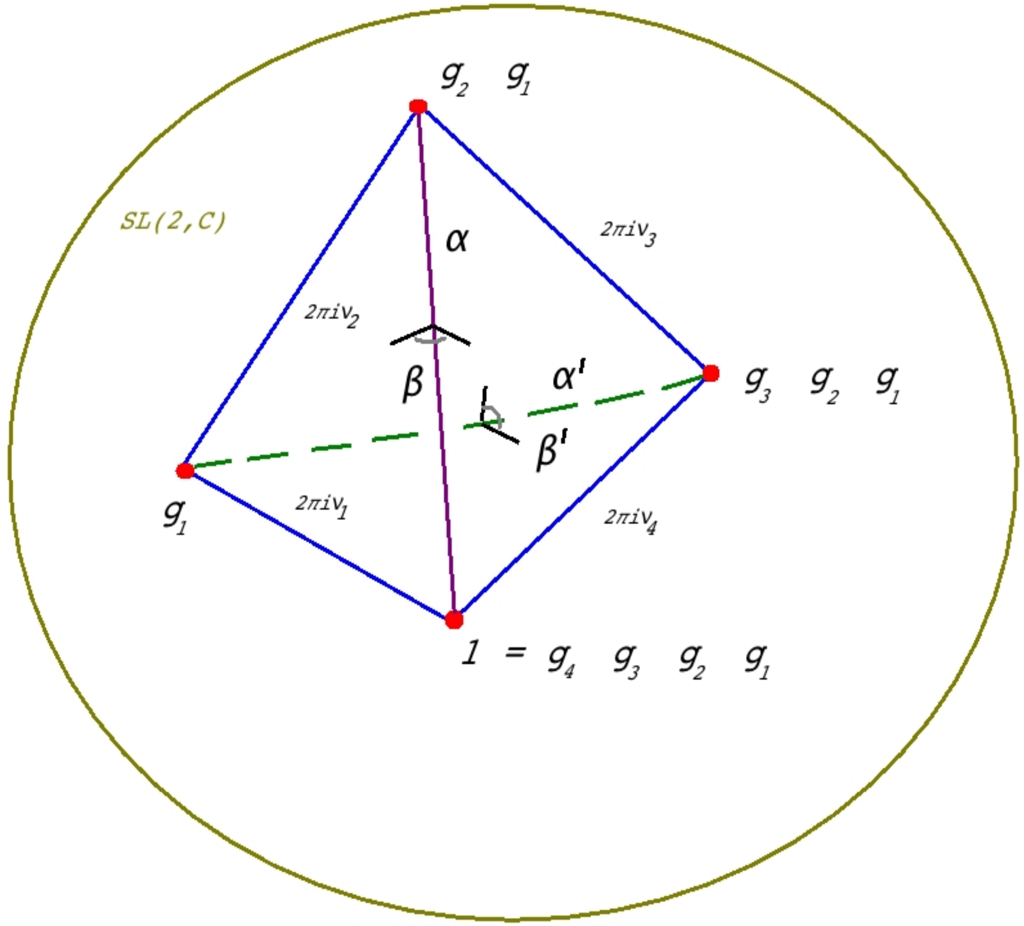
$$B = \mathrm{tr}\,(g_2) = \left(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\right)\sqrt{\frac{A^2 - m - 2}{A^2 - 4}}$$

The canonical transformations (the patching of the coordinates)



The s-t channel flop: the generating function is the hyperbolic volume

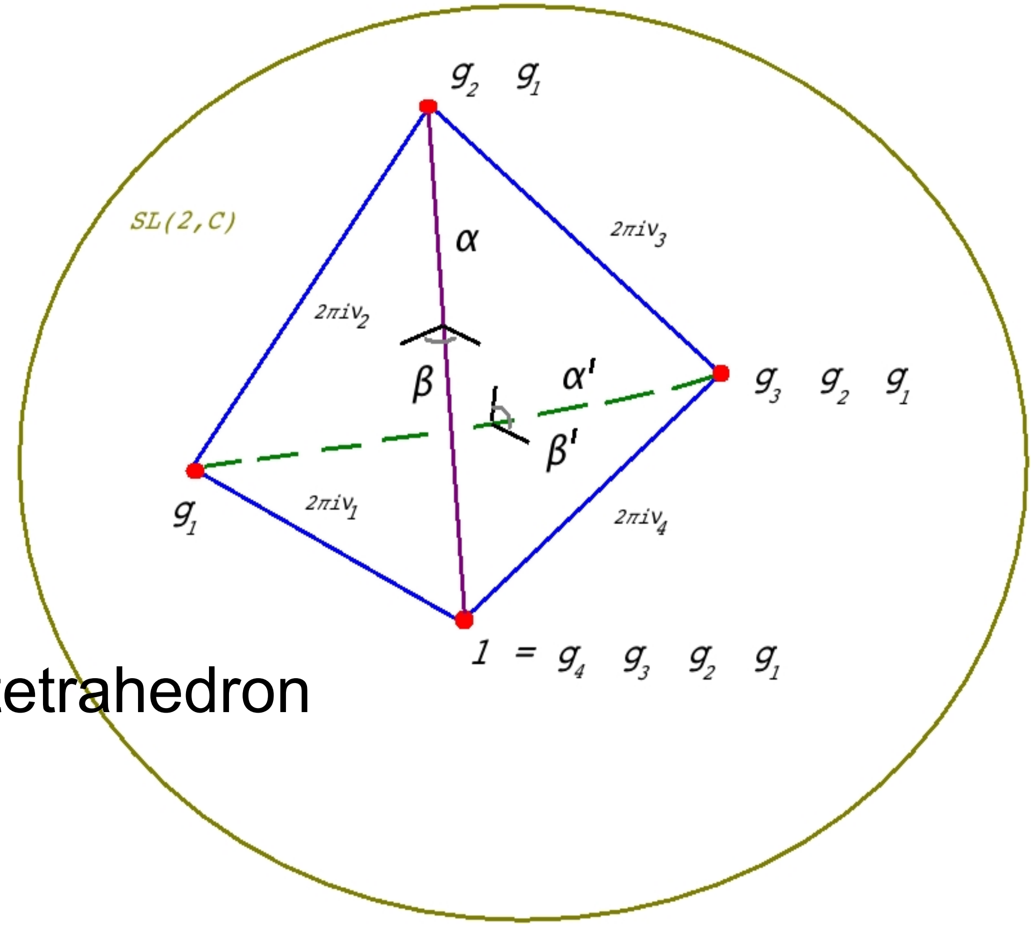




The s-t channel flop: the generating function is the hyperbolic volume

$$\frac{\partial V^{\vee}(\alpha, \alpha', \nu_1, \nu_2, \nu_3, \nu_4)}{\partial \alpha} = \beta$$

$$\frac{\partial V^{\vee}(\alpha, \alpha', \nu_1, \nu_2, \nu_3, \nu_4)}{\partial \alpha'} = \beta'$$



V^v is the volume of the dual tetrahedron

The s-u flop = composition of the 1-2 exchange (a braid group action) and the flop

The 1-2 braiding acts as:

(α,β) goes to $(\alpha, \beta \pm \alpha + \pi i)$

The theory

Why did the twisted superpotential turn into a generating function?
Why did the variety of opers showed up?
What is the meaning of Bethe equations for
quantum Hitchin in terms
of this classical symplectic geometry?
Why did these hyperbolic coordinates
(which generalize the *Fenchel-Nielsen* coordinates on
Teichmuller space and *Goldman* coordinates on the moduli
of SU(2) flat connections) become the special coordinates in the
two dimensional N=2 gauge theory?
What is the relevance of the geometry of hyperbolic polygons for
the M5 brane theory?
For the three dimensional gravity?
For the loop quantum gravity?

The theory

Why did the twisted superpotential turn into a generating function, and why did the variety of opers showed up?

This can be understood by viewing the 4d gauge theory as a 2d theory with an infinite number of fields in two different ways (*NN+EW*)

The theory

What is the meaning of Bethe equations for
quantum Hitchin in terms
of this classical symplectic geometry?

They seem to describe an intersection of the brane of opers with another (A,B,A) brane, a more conventional Lagrangian brane.
The key seems to be in the Sklyanin's separation of variables (NSR)
The full YY function is the difference of
the generating function of the variety of opers and
the generating function of the topological Lagrangian brane
(independent of the complex structure of Σ)

The theory

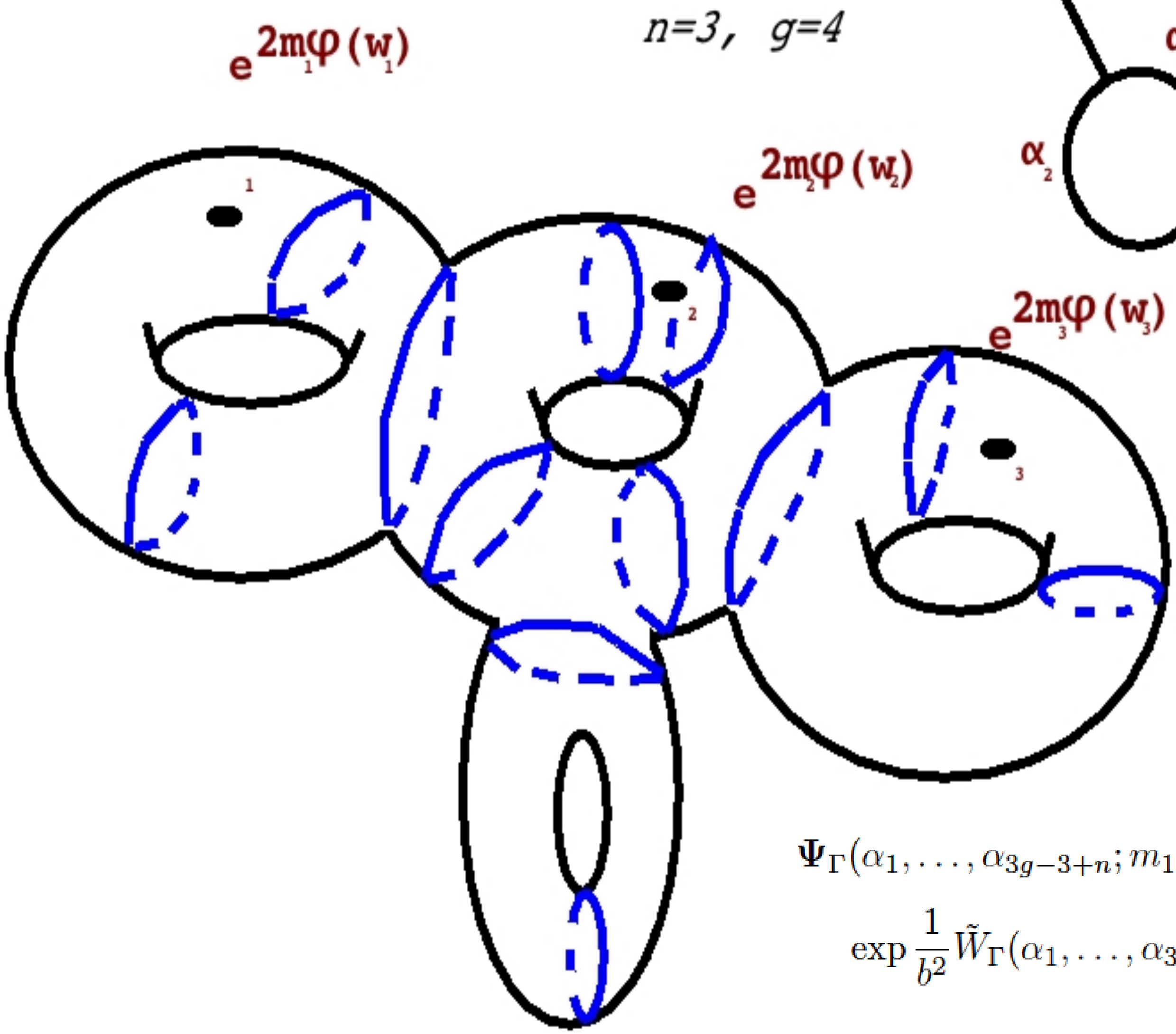
Why did these hyperbolic coordinates
(which generalize the *Fenchel-Nielsen* coordinates on
Teichmuller space and *Goldman* coordinates on the moduli
of SU(2) flat connections) become the special coordinates in the
two dimensional N=2 gauge theory?

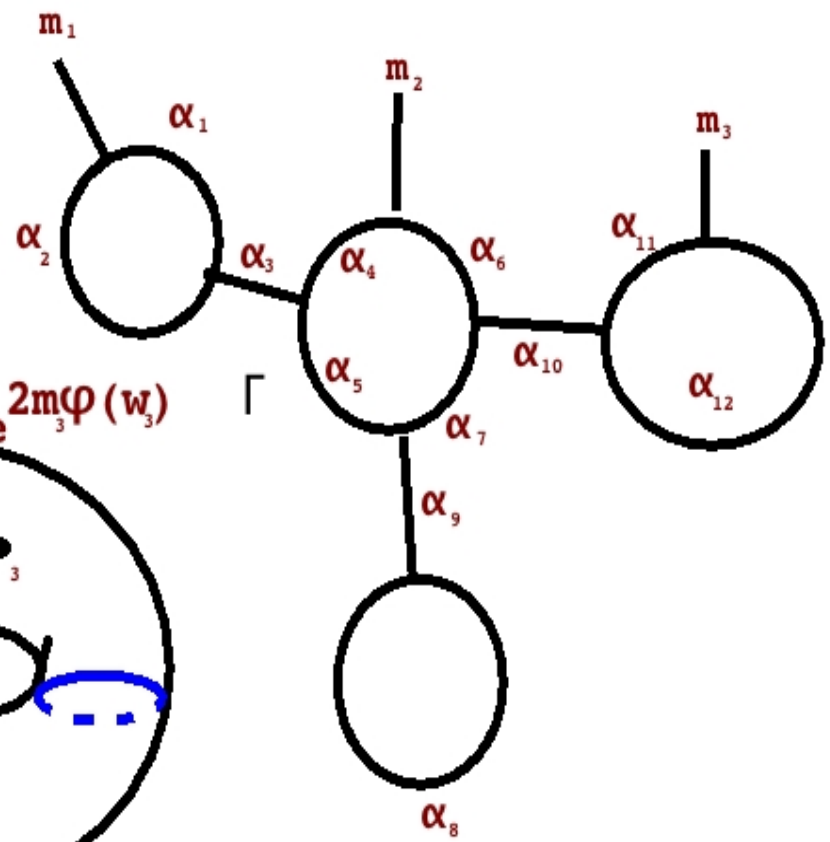
The key seems to be in the relation to
the Liouville theory and the SL(2,**C**) Chern-Simons theory.
A concrete prediction of our formalism is the quasiclassical limit of
the Liouville conformal blocks:

$$\Psi_\Gamma(\alpha_1, \ldots, \alpha_{3g-3+n}; m_1, \ldots, m_n; q_1, \ldots, q_{3g-3+n}; b) \sim$$

$$\exp \frac{1}{b^2} \tilde{W}_\Gamma(\alpha_1, \ldots, \alpha_{3g-3+n}; m_1, \ldots, m_n; q_1, \ldots, q_{3g-3+n})$$

The quasiclassical limit of the Liouville conformal blocks (motivated by the AGT conjecture, but it is independent of the validity of the AGT):

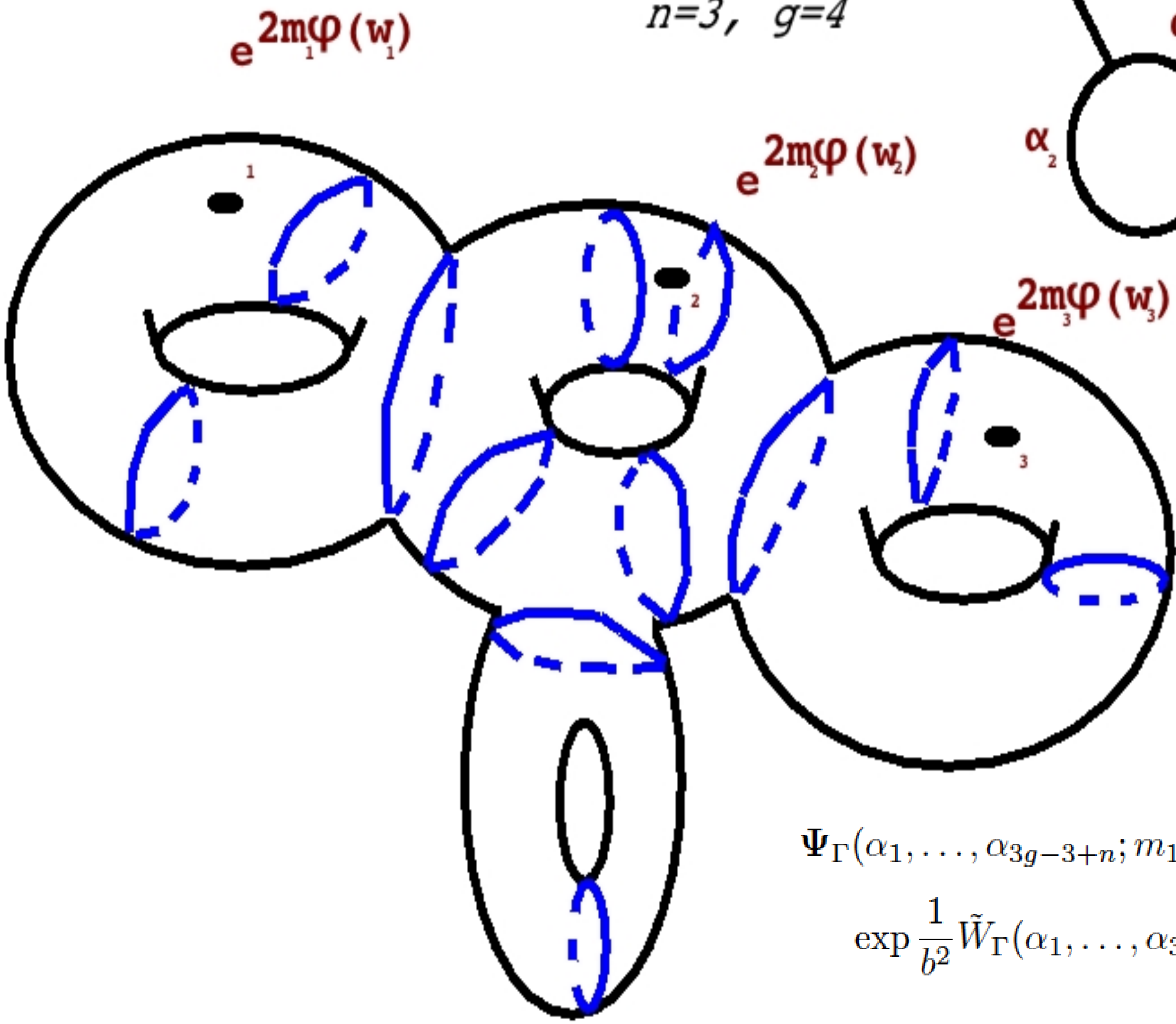


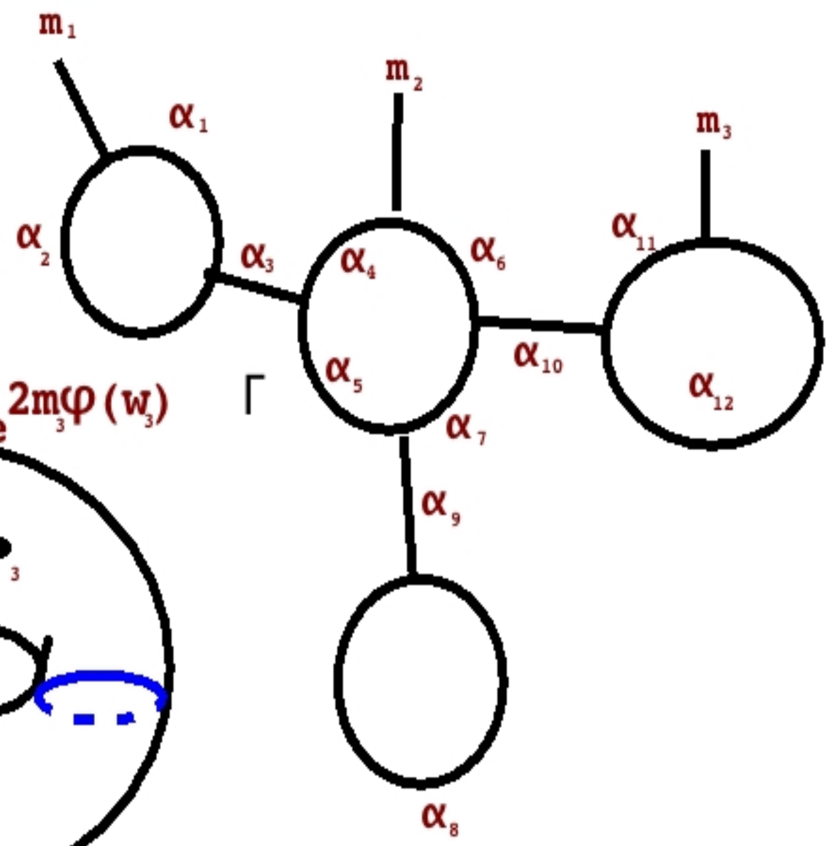


$$\Psi_\Gamma(\alpha_1,\dots,\alpha_{3g-3+n};m_1,\dots,m_n;q_1,\dots,q_{3g-3+n};b)\sim$$

$$\exp\frac{1}{b^2}\tilde{W}_\Gamma(\alpha_1,\dots,\alpha_{3g-3+n};m_1,\dots,m_n;q_1,\dots,q_{3g-3+n})$$

In the genus zero case it should imply the Polyakov's conjecture
(proven for Fuchsian m's by Takhtajan and Zograf);
can be compared with the results of Zamolodchikov,Zamolodchikov; Dorn-Otto





$$\Psi_\Gamma(\alpha_1, \ldots, \alpha_{3g-3+n}; m_1, \ldots, m_n; q_1, \ldots, q_{3g-3+n}; b) \sim$$
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The theory vs experiment

The conjecture in gauge theory
has been tested to a few orders in
instanton expansion for simplest theories (g=0,1),
and at the perturbative level of gauge theory for all theories.
What is lacking is a good understanding of the theories with tri-fundamental hypermultiplets (in progress, *NN+V.Pestun*)

The prediction of the theory

The conjecture implies that the
Twisted superpotential transforms under
the S-duality in the following way:

$$\tilde{W}(\tilde{\alpha}; \mu_1 \pm \mu_4, \mu_2 \pm \mu_3; 1-q) =$$

$$\mathrm{Crit}_{\alpha}\left(\tilde{W}(\alpha; \mu_1 \pm \mu_2, \mu_3 \pm \mu_4; q) + V^{\vee}(\alpha, \tilde{\alpha}; \mu_1, \mu_2, \mu_3, \mu_4)\right)$$

a generalization of the
four dimensional electric-magnetic transformation of the
prepotential

For the N-body Elliptic Calogero-Moser system:

The ingredients of the previous story:
the YY function,
the 4d gauge theory calculation,
the variety of opers are all known.
What is missing is the analogue of our
(α,β) Darboux coordinates

To be continued….

For the rest of the puzzles
there remains much to be said,
Hopefully in the near future.

Thank you!