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In honor of Igor Krichever

Homepage www.mi.ras.ru/$\tilde{}snovikov$ (click publications, items 177, 178, 179),

collaborators: P.Grinevich, A.Mironov

On the Nonrelativistic Purely Magnetic Supersymmetric Pauli Operator (for the 2D particles with spin 1/2) :

Introduction: Completely Integrable PDE Dynamical Systems and Linear Operators:

The typical 1D Case: KdV is equivalent to the Lax Pair
$u_t = 6uu_x - u_{xxx} \rightarrow dL/dt = [A, L]$,
$L = -\partial_x^2 + u(x,t)$
$A = \partial_x^3 + 3/2(u\partial_x + \partial_x u)$
There are also (3/2)D Cases like KP where Krichever's contribution is enormous
$L = \sigma\partial_y + \partial_x^2 + u(x, y, t)$

The right 2D analog of Lax Pair corresponds to one energy level:
$dL/dt = [A, L] + fL$
It was found and used since 1976 Manakov, Dubrovin-Krichever-N.:
$L = -\Delta + U\partial_x + V\partial_y + V(x, y)$
The operator A was chosen of the 3d order in 1980s to get physically interesting Nonlinear Systems. We choose it now of the order two for the second kind applications (below).

At least 3 different ways to use this presentation are known:

I. From the spectral theory of the operator L to the Solutions of Nonlinear System (The Inverse

Scattering Transform for the Solitons and Algebro-Geometric Solutions for the Periodic Problem).

II. It became immediately clear (as Shafarevich pointed out to me in 1974—see our joint Doklady note with Dubrovin) that AG solutions for the Completely Integrable Systems lead to the results in Algebraic Geometry. For example, they give an effective isomorphism of the total family of hyperelliptic Jacobian Varieties with unirational variety. The use of Θ-functional formulas of this Theory for the solution of the Classical Riemann–Shottki Problem was started by Dubrovin and myself in 1980 as an " Effectivization Program of Θ-functional formulas". Taimanov started to apply it to the Prym function (algebraic geometers who did not know right 2D analog of Lax Pairs above mistakably claimed that it is impossible). Igor Krichever with collaborators obtained fundamental results in this Program recently.

III. For the Improvement of Spectral Theory treating Nonlinear System as a Spectral Symmetry Group for the Operator L:The Algebro-Geometric Theory of Finite-Gap Periodic 1D Schrodinger Operators and 2D scalar Schrodinger Operators with One Selected Energy Level ($\epsilon = 0$)was constructed.

Following Conjecture was formulated (N.,1984): The total 2D family of Bloch-Floquet complex eigenfunctions M_L^2 for the Periodic Self-Adjoint Operator $L = -\Delta + V(x,y)$ can be a (Quasi)-Projective Algebraic Variety in the trivial case only $V = u(x) + v(y)$. It was proved several years later by Knorrer and Trubowitz. There are famous counterexamples in the nonself-adjoint case.

An Extended Conjecture follows from our Analysis: Let $L = -\Delta + U\partial_x + V\partial_y + W(x,y)$ be a periodic self-adjoint operator, and $\Gamma \subset M_L^2$ be an algebraic curve. Then either the operator can be reduced to 1 variable or Γ has a DKN form $\epsilon = \epsilon_0$,, and such ''algebraic'' level ϵ_0 is unique.

A number of counterexamples are known in the nonselfadjoint case.

Our Goal now is to extend these ideas to the Purely Magnetic Non-relativistic Pauli Operator L^P. Let me start with History of this Problem:

History: In 1979-1980 three groups of authors studied the ground level of this operator written in the Lorenz gauge: Avron-Seiler[AS], Aharonov-Casher[AC], Dubrovin and myself [DN]

Let $L^P = L^+ \oplus L^-$,and $-L^\pm =$ $= (\partial_x + i\Phi_y)^2 + (\partial_y - i\Phi_x)^2 \pm B$, acting on the space of vector-function $\Psi = (\Psi^+, \Psi^-)$ consisting of 2 spin-sectors $\pm$.

The Scalar Operators $L^\pm$ are Strongly Factorized $L^+ = QQ^+$, $L^- = Q^+Q$, $\partial_z = \partial = \partial_x - i\partial_y, Q =$ $\partial_z - \Phi_z$, and magnetic field $B =$ $\Delta\Phi$, Electric Field is zero.

The most interesting classes of magnetic fields are [AC] and [DN].

1.AC: Rapidly decreasing fields, $|[B]| = |\int_{R^2} B dx dy| < \infty$. Ground states form a finite-dimensional space of dimension $m \in Z$,
$m \leq (1/2\pi)[B] < m + 1$

2.DN: Arbitrary periodic fields with integer flux through the elementary cell $0 \neq (1/2\pi) \int_{cell} B dx dy$. The ground states form an infinite dimensional subspace in the Hilbert Space $L_2(R^2)$ isomorphic to the Landau level.
In both these cases Magnetic Field is Topologically Nontrivial because $(1/2\pi)[B]$ is a Chern Number of the line bundle over CP^1 [AC] and 2-torus [DN] (if integer).

We developed this subject using ideas of Transversality borrowed from Differential Topology. The "Generic" operators and their Topology in the space of Quasimomenta were studied as a byproduct of this work in 1980-81. In particular Chern numbers of the transversal dispersion relations appeared in our works with A.Lyskova. It was partly rediscovered by physicists of the Thouless group few years later after the experimental discovery of the "Integral Quantum Hall Fenomenon".

In the cases AC and DN all ground states are the Instantons belonging to one spin-sector only:

a. They satisfy to the 1st order equations $Q^{+}\psi = 0$ for the case

$[B] > 0$ and $Q\psi = 0$ for the case $[B] < 0$. It is a simple prototype of the self-duality equation.

b. They belong to the Hilbert Space $L_2(R^2)$

The operator $S : (\Psi^+, \Psi^-) \to (0, Q^+\Psi^+)$ is called a "Super-Symmetry" for L^P. The "adjoint" supersymmetry operator is $S^* : (\Psi^+, \Psi^-) \to (Q\Psi^-, 0)$

Both operators S, S^* commute with L^P and $SS^* + S^*S = L^P$. It implies that all higher levels are 2-degenerate (the ground level is ∞-degenerate).

Remark: In the work of the present author with A.Veselov (1997) non-trivial periodic cases were found such that some higher levels are also infinitely degenerate similar to the ground level $\epsilon = 0$. Our technic was associated with the "Laplace Transformations".

Question: Is this Theory related to the Algebro-Geometric (AG) Theory of the scalar 2D Schrodinger Operators based on the Selected

Energy Level $\epsilon = 0$ and 2D Soliton Theory? In the AG case Magnetic Field is always Topologically Trivial $\int_{cell} B dxdy = 0$

The Reduction Problem Some important time-invariant Reductions were actively studied in 1980s. Several authors found them either for Nonlinear Systems or for Inverse Spectral (Scattering) Data (or for both). Solution of this problem for the Inverse Data is more difficult: it implies in particular the description of all reduced hierarchy.

However, the existence of time-invariant reduction is much easier to see for the Nonlinear Equation. This how we use Nonlinear Systems here.

Our Main Goal here is Quantum Mechanics and Spectral Theory.

1.The Data leading to the Self-Adjoint Periodic Operators were found by Cherednik in 1980 [Ch].
2. The Data leading to the traditional operators $L = -\Delta + U$

with zero magnetic were found by Veselov and myself in 1984. Prym Varieties appear here.

Extension of these results to the Rapidly Decreasing Potentials was studied in the works made by Manakov, Grinevich, R.Novikov and myself in the late 1980s.

Krichever proved that every 2D smooth periodic potential can be approximated by the AG ones.

The Problem solved now: Calculate AG Data for the Reduction leading to the Factorized Operators and to the Pauli Operators as a by-product.

Consider a simplest "Manakov-type" System $L_t = [H, L] + fL$ where H is a second order operator: $L = \partial_x \partial_y + G\partial_y + S, H = \Delta + F\partial_y + A$.

It was pointed out in 1988 that the reduction $S = 0$ is time-invariant and looks like the 2D analog of the famous Burgers system (Konopelchenko). How to describe it in terms of the Inverse Spectral Data?

Making replacement $x, y \longrightarrow z, \bar{z}$ we are coming to elliptic operators most interesting for us.

The description of Periodic AG Inverse Data for the Nonlinear System above and the whole hierarchy was obtained in 1976 (DKN) but the "2D Burgers" reduction $S = 0$ never has been studied. Our recent result describes corresponding Inverse Problem Data:

Take Riemann Surface (the Complex Fermi Curve) splitted into nonsingular pieces $\Gamma = \Gamma' \cup \Gamma''$ with genuses g', g''. They cross each other $P_j = Q_j$, $P_j \in \Gamma'', Q_j \in \Gamma'$, $j = 0, ..., l$.

Take 2 points $\infty_1 \in \Gamma', \infty_2 \in \Gamma''$ with local parameters k'^{-1}, k''^{-1}. Construct function $\psi = (\psi', \psi'')$ with asymptotic $\psi' \sim c(x,y)e^{k'\bar{z}}(1+O(k'^{-1}))$, $\psi'' = e^{k''z}(1+O(k''^{-1}))$ and divisors of poles D', D'' of degree $g'+l, g''$ not crossing infinities and intersection points. No problem to include time dynamics in this Data.

Our Theorem claims that such Data generate a function $\psi = (\psi', \psi'')$ and scalar operator $L' = \Delta + G\partial_{\bar{z}}$ with $S = 0$ such that $L'\psi' = L'\psi'' = 0$. To get self-adjoint operator we need to add the degenerate Cherednik-type restriction and to make a gauge transformation $L' \to L = (1/\sqrt{c})L'\sqrt{c}$, $\psi \to \psi/\sqrt{c}$ in Γ', Γ'', This Data generate a Factorized Operator $L = QQ^+$.

Taking $L^+ = L$ and $L^- = Q^+Q$,

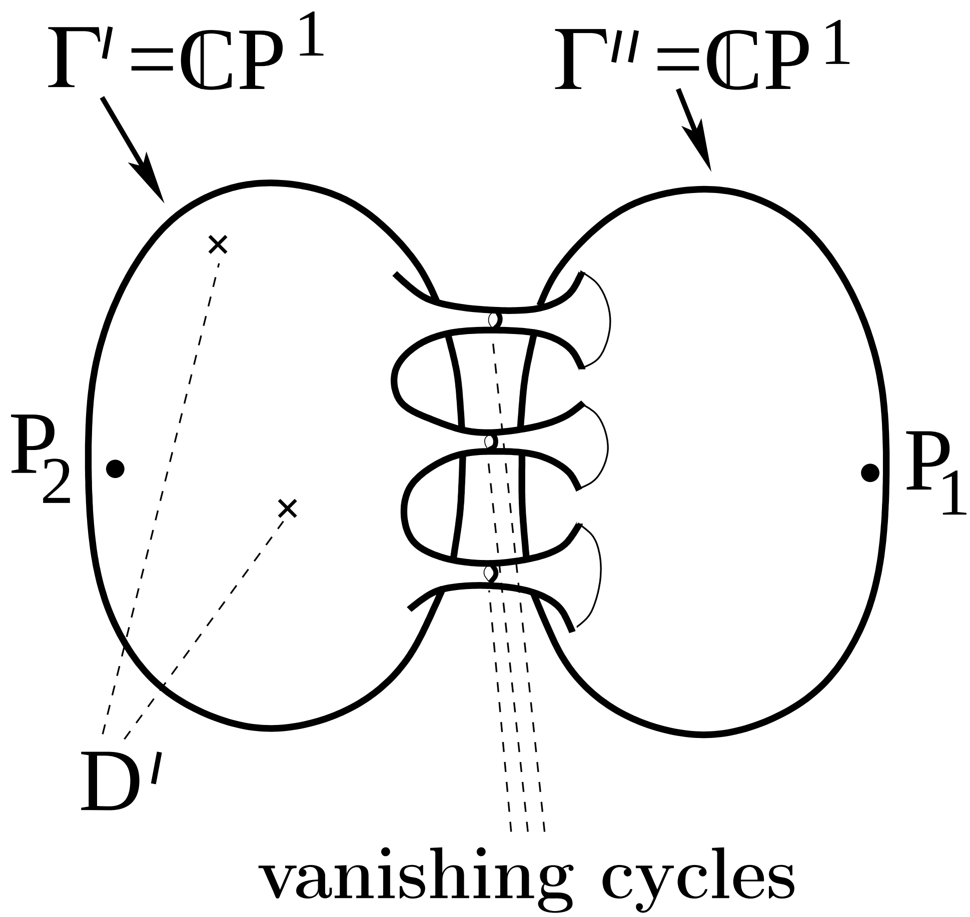
we construct a Purely Magnetic Pauli Operator $L^P = QQ^+ \oplus Q^+Q$. The Magnetic Field is real $B = 1/2\Delta \log c$, periodic or quasiperiodic and **Topologically Trivial**. It is nonsingular if $c \neq 0$, so the operator is self-adjoint in this case.

To find ground states, we take $\psi_0 = (c^{1/2}, 0)$ and $\phi_0 = (0, c^{-1/2})$.

In the case of periodic $c \neq 0$ we have only two periodic ground state functions. They present the bottom of the CONTINUOUS SPECTRUM. Full description of all complex nonsingular Bloch functions of the ground level see below. For the slowly decreasing fields $B = 1/2\Delta \log c$ we will use property that $c' = ce^{az+b\bar{z}}$ lead to the same field B.

The Case of Genus zero (Fig 1)

Fig 1



We take $l+1$ intersection points presented as $k' = k_s$ and $k'' = p_s$ in Γ', Γ'', and divisor $D' = (a_1, ..., a_l)$ of degree l in Γ'. We have $\Psi = e^{k'\bar{z}} \frac{w_0 k'^l + ... + w_l}{(k'-a_1)...(k'-a_l)}, \Psi|_{k'=k_s} = e^{p_s z}$. As we can see, $c = w_0$.

So $c = \sum_{s=0}^{l} \kappa_s e^{W_s(z,\bar{z})}$, where W_s is a linear form. All complex coefficients are possible.

$W_s = \alpha_s x + \beta_s y, (\alpha_s, \beta_s) \in C^2_W$.

Transformation $c \to c' = c e^{\gamma + \alpha x + \beta y}$ leads to the gauge equivalent operator (the same magnetic field)

There exist 3 types of Real Non-singular Solutions:

1.Purely Exponential Positive Case (The Lump-type fields")

$\kappa_s > 0, (\alpha_s, \beta_s) \in R$.

2.Periodic Trigonometric Real Case. It will be considered below jointly with the case $g = 1$

3.The Mixed case. It can be realized only if its "dominating part" belongs to the case 1. So we will not discuss it.

The case 1. Let "the Tropical Sum" of the forms in the set $\{W\}$ is nonnegative $I'_{\{W\}}(\phi) = \max_s(\alpha_s \cos\phi + \beta_s \sin\phi) \geq 0$.

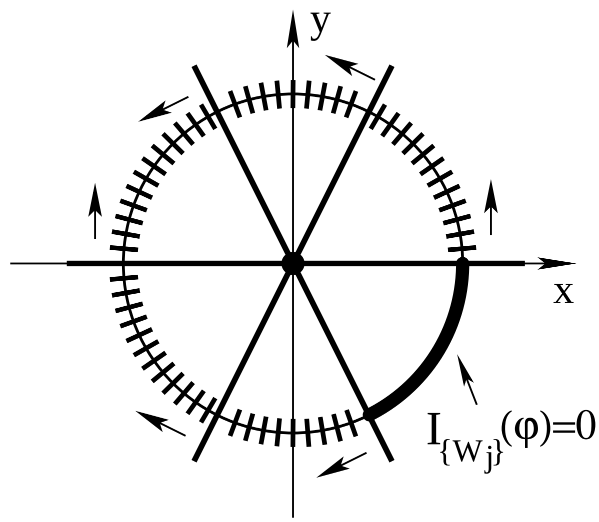
Then $c^{-1/2}$ is bounded in R^2

For the angles $I'_{\{W\}}(\phi) > 0$ we have a rapid decay

$c^{-1/2} \to 0, R \to \infty$, Let $I(\phi) = \max\{I'(\phi), 0\}$

Fig 2a

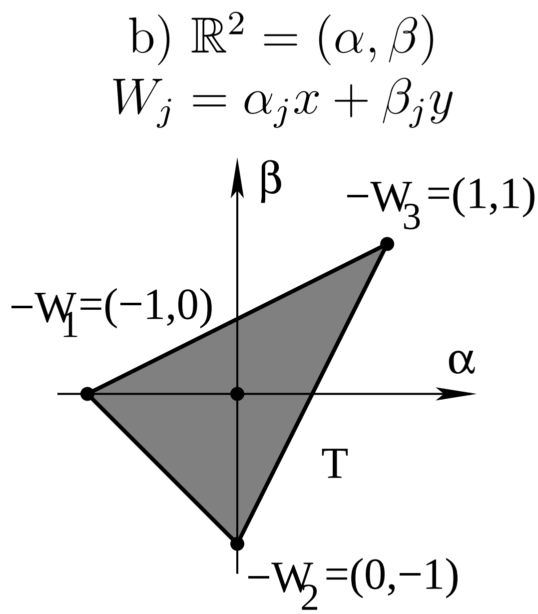
$c = e^{y} + e^{y-2x} + e^{-y-2x}$



In every class $c' \in ce^{W}, W' \in R^2_W$, the set of representatives c' with nonnegative $I = I'_{\{W'\}}(\phi) \geq 0$ forms a convex polytop $\bar{T}_c$. Its inner part $T_c \subset \bar{T}_c$ consists of all

c' such that $I_{\{W'\}} > 0$. Open part T_c is always nonempty for $l > 2$. $\bar{T}_c$ is nonempty for $l > 1$. (see Fig 2b for $l = 3$)

Fig 2b



Here $e^y + e^x + e^{-y-x} = c$

Magnetic field is decaying for $R \to \infty$ except some selected angles, it is a Lump Type Field analogous to the KP "Lump Potentials". A linear sum under the $1/2\Delta\log()$ reflects linearization of the Burgers Hierarchy in the variable c.

$[B] = \int\int_{D_R^2} B dx dy =$

$= -1/2 R \oint_{S1} I_{\{W\}}(\phi) d\phi + O(R^{-1})$

All points in T_c define ground states in the Hilbert Space $L_2(R^2)$. The boundary points define the bottom of continuous spectrum.

The Periodic Problem. Let lattice in R^2 be rectangular and $z = x + iy$. For every real periodic function c we can define a whole family of "possible" meromorphic Bloch functions

$\psi''_{ext,\pm} = f(z)(\sqrt{c})^{\pm}e^{uz-\zeta(p)z}\sigma(z+p+R)/\sigma(z+R)$ where $f(z)$ is an arbitrary elliptic function. We have $Q^+\psi''_{ext,-} = 0$ for $L = L^+ = QQ^+$. For anti-holomorphic functions we get $Q\psi''_{ext,+} = 0$ for $L^- = Q^+Q$.

Let $c \neq 0$. We need only nonsingular functions, so our manifold is $u \in CP^1 = \Gamma''$ and $\psi''_+ = e^{uz}\sqrt{c}$ (or $e^{u\bar{z}}\sqrt{c}$).

Let c has an isotropic zero. Magnetic field became singular. We have larger family of the admissible Bloch functions Ψ'' in the sector $-$ because $\sqrt{c}/\sigma(z+R)$is weakly singular now. So the full Bloch manifold is $M^2 = CP^1 \times \Gamma$ where Γ is an elliptic curve. To calculate ψ' for the sector – we need to deal with all real periodic smooth functions c. Now start the case of genus 1.

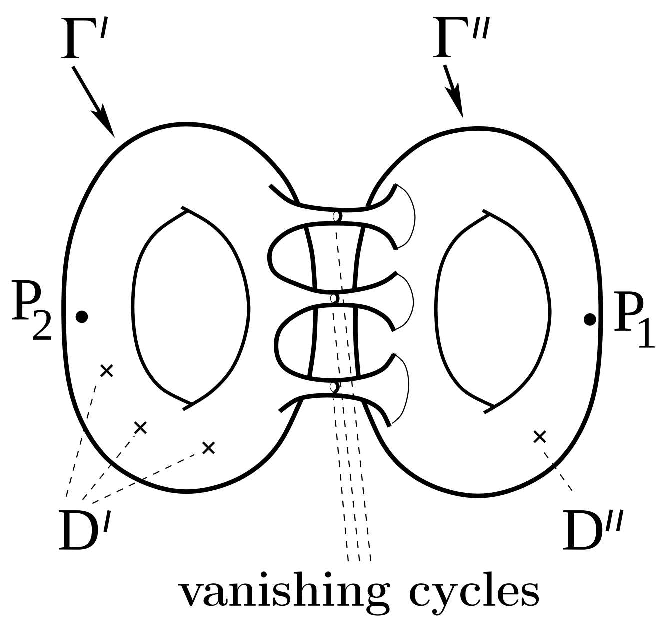


Fig 3

We take elliptic curve $\Gamma' = \Gamma'' = C/\Lambda$ with euclidean local parameters k, p (the point 0 is "infinity"), periods $1, 2i\omega \in iR$, n intersection points $Q_0, Q_1, ..., Q_n \in \Gamma'$ and $R_0, ..., R_n \in \Gamma''$. Divisors $D' = (P_1, ..., P_n), D'' = P$ have degree $n+1, 1$ correspondingly. We have $\psi' = e^{-\bar{z}\zeta(k)\frac{\prod_s \sigma(k-Q_s)}{\prod_l \sigma(k+P_l)}} \times$ $\times (\sum_j w_j \frac{\sigma(k+\bar{z}+\tilde{P}+\tilde{Q}-Q_j)}{\sigma(k-Q_j)})$. Here $\tilde{P} = P_1 + ... + P_n, \tilde{Q} = Q_0 + ... + Q_n$, sum as in C

$\psi'' = e^{-z\zeta(p)}\sigma(p+z+P)/(\sigma(z+P)\sigma(p+P))$, $\psi'(Q_s) = \psi''(R_s)$.

All singularity of the quantity c disappear after multiplication $\tilde{c} = c\sigma(\bar{z}+\tilde{Q}+\tilde{P})\sigma(z+P)$. Take $n = 1, Q_0 = -Q_1, R_0 = Q_1, R_1 = Q_0$ and solution to the equation $\omega\zeta(Q_0) = \eta_1 Q_0$ We have $P = \tilde{Q}+\tilde{P}$, so $-1/2\Delta|\sigma|^2 = -2\pi\delta(z)$.

So our Conclusion based on the case $g = 1$ is:

The magnetic field $\tilde{B} = -1/2\Delta\tilde{c}$

is periodic nonsingular with magnetic flux equal to ONE QUANTUM UNIT. The magnetic field $B = 1/2\Delta c$ is always singular for $g = 1$; it has magnetic flux equal to zero through the elementary cell and δ-singularity in the point P. So this field corresponds to the "Aharonov-Bohm" (AB) situation with quantized flux.

For $g > 1$ number of quantized δ-functions is equal to $k > 1$. Both pieces of the original Riemann surface $\Gamma = \Gamma'' \cup \Gamma'$

are presented in the form of k-sheeted branching covering over elliptic curve $\Gamma'' \to \Gamma_0$ as it was in the works of Krichever dedicated to the elliptic KP. Comparison with [DN] shows that the Aharonov-Bohm terms with quantized δ-flux do not affect spectrum. This question was discussed in the physics literature.

The complex Bloch-Floquet manifolds (consisting of nonsingular Bloch functions) for the level $\epsilon =$

0 and genus $g = 1$ is $M = M^2 \cup \Gamma'$ with functions ψ' and $\psi''_{ext,-} = (1/\sqrt{c})[e^{uz} \times e^{-\zeta(p)z}\sigma(z+p+R)/\sigma(z+R)]$, $L^+\psi''_{ext} = L^+\psi' = 0$.

We did not proved yet that ψ' cannot be extended to the higher dimensional component at the same level, but it is highly probable.

Reconsider now the case $g = 0$ comparing it with $g = 1$.

For $c \neq 0$ and $g = 0$ the Bloch

manifold is equal to the union $\Gamma'' \cup \Gamma'$, and both are CP^1; Let c has an isolated zero (minimum) which is isotropic. Magnetic field became singular, with δ-term. The extended Bloch function can be defined for the operator on the manifold $M^2 = CP^1 \times \Gamma_0$ where Γ_0 is an elliptic curve, $\psi''_{ext,+} = (const(u))e^{p\bar{z}-\zeta(u)\bar{z}}\sigma(\bar{z}+u)\sqrt{c}/\sigma(\bar{z})$.

Our Conclusion is that the periodic case $g = 1$ gives the same result as the special case $g = 0$ where c has an isolated isotropic zero, interchanging sectors $\pm$. The higher number $k \geq 1$ of isotropic zeroes for $g = 0$ leads to the "higher rank" family of nonsingular Bloch functions $M^{k+1} \cup \Gamma'$. Removing δ-singularities by the singular gauge transformations we get smooth periodic magnetic field like in DN with higher flux.

The algebro-geometric $g = 0$ case simply corresponds to the case of trigonometric polynomials. We take rectangular lattice in the plane x, y. Following relation is true $Q^{+}\psi' = M(k)\sqrt{c}e^{\bar{z}k}$. Renormalizing ψ' such that $M(k) = 1$, we extend this construction to the infinite trigonometric series.

The identity $S(\psi', 0) = (0, e^{\bar{z}k})$ is true for "the Sypersymmetry Operator" S.

Let us extend our theory to the "infinite" trigonometric series

We use for that the formula

$\psi' =$

$= k\sum_j[\kappa_j e^{p_j z - k_j\bar{z}}/(k-k_j)]e^{k\bar{z}}$ for this new normalization where k_j are the lattice points.

Here $\sum_j \kappa_j e^{p_j z - k_j\bar{z}} = c$

Apply this result to the function $c' = 1/c$ which is an infinite trigonometric series. It gives us a function ψ' for the second component L^- of the Pauli operator.

Problem: The component Γ' of the Bloch manifold does not affect the ordinary spectrum in the Hilbert space of functions in the whole plane R^2. How to use it for solving physically meaningful self-adjoint boundary problems?

New results dedicated to this problem will be published soon by the authors.