# Integrable kernels and Markov dynamics

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# Preface



• My talk is about a model of continuous time Markov dynamics in infinite dimensions. Its construction exploits some connections between representation theory of so called big groups and probability.

• Existing directions:
(1) Interacting particle systems. local interaction
(2) Some continuous-space analogs [Kondratiev et al]. short range interaction
(3) Models arising from Random Matrix Theory. The basic example is Dyson's Brownian motion. Here particles represent eigenvalues of very large matrices, and dynamics of eigenvalues arises from diffusion processes on matrices. long range interaction

• The model I will speak about is close to (3).

## Laguerre diffusion on $\mathbb{R}_+$



$$D = x\frac{d^2}{dx^2} + (b-x)\frac{d}{dx}, \qquad b > 0.$$

$$Dx^n = -nx^n + \text{lower terms}$$

**Laguerre polynomials:**

$$DL_n = -nL_n, \qquad L_n(x) = x^n + \text{lower terms}$$

$$L_n(x) = (b)_n \sum_{0 \le m \le n} (-1)^{n-m} \frac{\binom{n}{m}}{(b)_m} x^m,$$

where $(z)_n = z(z+1)\dots(z+n-1)$ is the Pochhammer symbol.
**The orthogonality measure** is Gamma distribution:

$$P(dx) = \frac{x^{b-1}}{\Gamma(b)} e^{-x} dx.$$

## Laguerre diffusion on $\mathbb{R}_+$



- Let $\varphi : \mathbb{R}[x] \to \mathbb{R}$ be the **moment functional** defined by $P$:

$$\varphi(F) = \langle F, P \rangle, \qquad F \in \mathbb{R}[x].$$

In terms of $\varphi$, $P$ is characterized as a unique probability measure such that

$$\varphi(1) = 1, \qquad \varphi(DF) = 0 \quad \forall F \in \mathbb{R}[x].$$

- This reflects the fact that $P$ is the **stationary distribution** for the Markov process $X(t)$ generated by $D$.
- The **transition function** of $X(t)$ has the form

$$\mathrm{Prob}\{X(t) \in dy \mid X(0) = x\} = \sum_{n=0}^{\infty} e^{-nt} \frac{L_n(x) L_n(y)}{(L_n, L_n)} P(dy)$$

## Algebraic skeleton-1: Algebra of symmetric functions



• Replace $\mathbb{R}[x]$ by $\mathrm{Sym}$, the **algebra of symmetric functions**.

$$\mathrm{Sym} = \mathbb{R}[e_1, e_2, \dots]$$

$e_1, e_2, \dots$ are the **elementary symmetric functions**.
• Replace monomials $\{x^n\}$ by **Schur symmetric functions** $\{S_\nu\}$.
$\nu = (\nu_1, \nu_2, \dots, 0, 0, \dots)$ is partition=Young diagram.
• Dual Jacobi-Trudi:

$$S_\nu = \det \left[ e_{\nu_i' - i + j} \right]$$

where $\nu'$ is the transposed diagram and the order of determinant is the number of columns in $\nu'$.
• Functions $F \in \mathrm{Sym}$ will be realized as **"polynomial observables"** on the future state space of the Markov process under construction.

## Algebraic skeleton-2: Laguerre symmetric functions



- Analog of the Laguerre polynomials $L_n$: the **Laguerre symmetric functions** $\mathfrak{L}_\nu$. Two complex parameters $z$ and $z'$.

$$\mathfrak{L}_\nu = (z)_\nu (z')_\nu \sum_{\mu \subseteq \nu} (-1)^{|\nu|-|\mu|} \frac{\dim \nu/\mu}{(z)_\mu (z')_\mu \, (|\nu| - |\mu|)!} S_\mu$$

Cf. formula for $L_n$: similarity of structure! Here $|\nu| = \sum \nu_i$ is the number of boxes in $\nu$; $\dim \nu/\mu$ is the number of standard tableaux on $\nu/\mu$; $(z)_\nu$ is an analog of the Pochhammer symbol:

$$(z)_\nu = \prod_{(i,j)\in\nu} (z + j - i)$$

- Cf. Desrosiers and Hallnäs (2011); Sergeev and Veselov (2009).
- $\{\mathfrak{L}_\nu\}$ is an inhomogeneous **basis** in $\mathrm{Sym}_\mathbb{C}$, because

$$\mathfrak{L}_\nu = S_\nu + \text{lower degree terms}.$$

## Algebraic skeleton-3: $\infty$-dim Laguerre differential operator 6

Analog of Laguerre operator $D$ on $\mathbb{R}^*$ is $\mathfrak{D}_{z,z'} : \mathrm{Sym}_{\mathbb{C}} \to \mathrm{Sym}_{\mathbb{C}}$:

$$\mathfrak{D}_{z,z'} \mathfrak{L}_\nu = -|\nu| \mathfrak{L}_\nu, \qquad \forall \nu$$

Equivalently, it can be written as a second order differential operator in formal variables $e_1, e_2, \ldots$:

$$\mathfrak{D}_{z,z'} = \sum_{n \geq 1} \left( \sum_{k=0}^{n-1} (2n-1-2k) e_{2n-1-k} e_k \right) \frac{\partial^2}{\partial e_n^2}$$
$$+ 2 \sum_{m > n \geq 1} \left( \sum_{k=0}^{n-1} (m+n-1-2k) e_{m+n-1-k} e_k \right) \frac{\partial^2}{\partial e_m \partial e_n}$$
$$+ \sum_{n=1}^{\infty} \big( -n e_n + (z-n+1)(z'-n+1) e_{n-1} \big) \frac{\partial}{\partial e_n}$$

## Thoma's cone $\widetilde{\Omega}$ as analog of $\mathbb{R}_+$ 7

$\widetilde{\Omega} \subset \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$, points are triples $\omega = (\alpha, \beta, \delta)$ such that

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0), \quad \delta \geq 0$$

$$\sum \alpha_i + \sum \beta_j \leq \delta.$$

$\widetilde{\Omega}$ is a cone, is closed, $\infty$-dimensional, but locally compact.
Embed $\mathrm{Sym}$ into $\mathrm{Fun}(\widetilde{\Omega})$:

$$1 + \sum_{k=1}^{\infty} e_k(\omega) u^k = e^{\gamma u} \prod_{i=1}^{\infty} \frac{1 + \alpha_i u}{1 - \beta_i u},$$

$$\gamma := \delta - \sum (\alpha_i + \beta_i) \geq 0.$$

Equivalently, in terms of power sums $p_k$,

$$p_k(\omega) = \begin{cases} \sum \alpha_i^k - \sum (-\beta_i)^k, & k \geq 2 \\ \delta, & k = 1. \end{cases}$$

$\mathrm{Sym} \to$ "polynomial functions" on $\widetilde{\Omega}$. Supersymmetry!

## z-Measure $P_{z,z'}$ on $\widetilde{\Omega}$ as analog of $\Gamma$-distribution on $\mathbb{R}_+$



**Formal moment functional** $\varphi_{z,z'} : \mathrm{Sym}_{\mathbb{C}} \to \mathbb{C}$:

$$\varphi_{z,z'}(1) = 1, \qquad \varphi_{z,z'}(\mathfrak{D}_{z,z'} F) = 0 \quad \forall F \in \mathrm{Sym}\,.$$

$(z, z')$ is **admissible** if either $z$ and $z'$ are complex-conjugate, or $m < z, z' < m+1$ with $m \in \mathbb{Z}$. This implies that $(z)_\nu (z')_\nu$ is real and $> 0$, and $\varphi_{z,z'}(S_\nu) > 0$ $\forall \nu$. From now on $(z, z')$ is admissible.

**Theorem**(Borodin-O, 2000). There exists a unique probability measure $P_{z,z'}$ such that all polynomial functions $F(\omega)$ are $P_{z,z'}$-integrable and

$$\varphi_{z,z'}(F) = \langle F, P_{z,z'} \rangle, \qquad F \in \mathrm{Sym}\,.$$

Moreover, the Laguerre symmetric functions form an orthogonal basis in $L^2(\widetilde{\Omega}, P_{z,z'})$.

We call $P_{z,z'}$ the **z-measure**. Its origin: harmonic analysis on $S_\infty$

## The Markov process $X(t)$



• **First Main Theorem** (assume $(z, z')$ admissible).
$\mathfrak{D}_{z,z'} : \mathrm{Sym} \to \mathrm{Sym}$ generates a Feller Markov process $X_{z,z'}(t)$ on $\widetilde{\Omega}$, and $P_{z,z'}$ serves as the stationary (=invariant) distribution for $X_{z,z'}(t)$.

Comments:

• A **Feller Markov process** is determined by a **Feller operator semigroup** $\{T(t)\}_{t\ge 0}$ on the Banach space $C_0(\widetilde{\Omega})$.

• Recall $\widetilde{\Omega}$ is a locally compact. $C_0(\widetilde{\Omega})$ = continuous functions vanishing at infinity.

• A Feller semigroup is a strongly continuous operator semigroup $\{T(t)\}$, $t \ge 0$, where $\|T(t)\| \le 1$, preserves nonnegative functions (+ one more condition).

• Feller semigroup gives rise to a Markov process.

• Any contraction semigroup $\{T(t)\}$ possesses a **generator** $A$, a densely defined dissipative operator. Informally, $T(t) = e^{At}$.

## Difficulties



• The major difficulty: the "pre-generator" $\mathfrak{D}_{z,z'}$ is initially defined on the space $\mathrm{Sym}$ of "polynomial functions" which are unbounded on $\widetilde{\Omega}$ and so are not in the Banach space $C_0(\widetilde{\Omega})$.

• There are two ways to overcome this difficulty.

(1) Define first a semigroup on the Hilbert space $L^2(\widetilde{\Omega}, P_{z,z'})$, then prove that it preserves the space $C_0(\widetilde{\Omega})$ and induces on it a Feller semigroup. After that we see that the transition function of $X(t)$, that is the kernel of $T(t)$, is given by the formula

$$\mathrm{Prob}\{X(t) \in d\omega' \mid X(0) = \omega\} = \sum_{\nu} e^{-|\nu| t} \frac{\mathfrak{L}_\nu(\omega)\mathfrak{L}_\nu(\omega')}{(\mathfrak{L}_\nu, \mathfrak{L}_\nu)} P_{z,z'}(d\omega')$$

(2) Find "natural" extension of $\mathfrak{D}_{z,z'}$ from $\mathrm{Sym}$ to a larger space which has a large enough intersection with $C_0(\widetilde{\Omega})$. Then prove that on this intersection, the operator generates a Feller semigroup. (Based on method of BO 2010 + an idea of B.)

Approximation Meixner → Laguerre

## Random particle ensembles driven by measures on $\widetilde{\Omega}$ 

• $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ – the punctured line. Projection $\widetilde{\Omega} \to \mathrm{Conf}(\mathbb{R}^*)$:

$$\pi : \omega = (\alpha, \beta, \delta) \in \widetilde{\Omega} \longrightarrow \{\alpha_i : \alpha_i \neq 0\} \cup \{-\beta_i : \beta_i \neq 0\}$$

I.e., remove possible zeros and ignore parameter $\delta$. Result: locally finite configurations.

• Every probability measure $P$ gives rise to an **ensemble of random particle configurations** on $\mathbb{R}^*$.

• The language of **correlation functions**:

$$\rho_k(x_1, \dots, x_k), \qquad k = 1, 2, \dots$$

– The $k$th function is the **density** of probability to find a particle at prescribed position $x_i$, for $i = 1, \dots, k$.

• If $\pi \downarrow \operatorname{supp} P$ is injective, then $P$ is determined by $\{\rho_k\}$ uniquely.

## The Whittaker kernel (BO, 2000)



Apply correspondence $\pi$ to $P = P_{z,z'}$ ($(z, z')$ admissible).

- **Theorem**

$$\operatorname{supp} P_{z,z'} \subset \widetilde{\Omega}' := \{\omega : \sum(\alpha_i + \beta_i) = \delta\}.$$

Thus, $P_{z,z'}$ is completely characterized by $\{\rho_k = \rho_k^{(z,z')}\}$

- **Theorem** (BO 2000) The random particle ensemble for $P_{z,z'}$ is **determinantal**:

$$\rho_k^{(z,z')}(x_1, \ldots, x_k) = \det \left[ K_{z,z'}(x_i, x_j) \right]_{i,j=1}^{k}, \qquad k = 1, 2, \ldots$$

where $K_{z,z'}(x, y)$ is a kernel on $\mathbb{R}^*$.

- $K_{z,z'}(x, y)$ is called the **Whittaker kernel**. It is **integrable**.

## Integrable kernels (IIKS 1990, Deift 1998)



• Let $\mathcal{K}$ be an integral operator with kernel $\mathcal{K}(x,y)$. They are **integrable** if $\mathrm{rank}([x,\mathcal{K}]) < \infty$. This means that the kernel has the form

$$\mathcal{K}(x,y) = \frac{\sum A_i(x)B_i(y)}{x-y},$$

where the sum is finite and the numerator vanishes on $x=y$. Notion is due to Its-Izergin-Korepin-Slavnov 1990, further studied by Deift 1998.

• In many examples from Random Matrix Theory,

$$\mathcal{K}(x,y) = \frac{A(x)B(y) - B(x)A(y)}{x-y}.$$

• Notice that $\mathcal{K}(x,y) = \mathcal{K}(y,x)$.

## Christoffel-Darboux, sine kernel, Airy kernel



- Example 1: the **Christoffel-Darboux kernels**:

$$\mathcal{K}_N(x,y) = \sum_{i=0}^{N-1} \frac{f_i(x)f_i(y)}{(f_i,f_i)} = \mathrm{const}_N \, \frac{f_N(x)f_{N-1}(y) - f_{N-1}(x)f_N(y)}{x-y},$$

Here $f_0 = 1, f_2, \ldots$ are orthogonal polynomials, $N = 1, 2, \ldots$.
$\mathrm{rank}\, \mathcal{K}_N = N$

- Example 2: Limit kernels:

(a) sine kernel: $A = \sin$, $B = A' = \cos$
(b) Airy kernel: $A = Ai$, $B = Ai'$.
Here $\mathrm{rank}\, \mathcal{K} = \infty$

- These are **projection kernels**.

## J-symmetry of $K_{z,z'}(x,y)$



• The Whittaker kernel is **J-symmetric**, that is, symmetric with respect to an indefinite inner product in $L^2(\mathbb{R}^*) = L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$:

$$K_{z,z'}(x,y) = \operatorname{sgn}(x)\operatorname{sgn}(y) K_{z,z'}(y,x) \qquad x,\, y \in \mathbb{R}^*.$$

• Informally: $P_{z,z'}$ is a Gibbs measure, but with **long-range** pair interaction between the particles. The pair potential $V$ is of log-gas type:

$$V(x,y) = \pm 2 \log \frac{1}{|x-y|}, \qquad x,\, y \in \mathbb{R}^*, \quad \pm = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

• The picture resembles to RMT: eigenvalues of random matrices in large-$N$ limit.

• Difference: the plus/minus sign. Informal interpretation: the "alpha" and the "beta" particles are oppositely charged.

## Extended Whittaker kernel 

- **Equilibrium version** $X^{\mathrm{eq}}_{z,z'}(t)$: start $X_{z,z'}(t)$ from $P_{z,z'}$. $X^{\mathrm{eq}}_{z,z'}(t)$ is stationary in time.
- f.d. or **multitime distributions**: probability measures on $\mathrm{Conf}(\mathbb{R}^*) \times \cdots \times \mathrm{Conf}(\mathbb{R}^*)$ ($k = 2, 3, \dots$ times).
- Equivalently, the space-time (or dynamical) correlation functions $\rho_k^{(z,z')}(t_1, x_1; \dots; t_k, x_k)$, $k = 1, 2, \dots$: density of probability to find, at prescribed moment $t_i$, a particle at prescribed position $x_i$, for $i = 1, \dots, k$.
- **Second Main Theorem** Again determinantal correlations:

$$\rho_k^{(z,z')}(t_1, x_1; \dots; t_k, x_k) = \det \left[ K_{z,z'}(t_i, x_i \mid t_j, x_j) \right]_{i,j=1}^{k}.$$

for any $k = 1, 2 \dots$, time variables $t_i \in \mathbb{R}$, and space variables $x_i \in \mathbb{R}^*$.
- $K_{z,z'}(s, x \mid t, y)$ is **extended W. kernel**, from BO 2006.

## Problem: survival of Markov property under (scaling) limit 17

• Ext. W. kernel was first derived by a formal scaling limit transition from discrete ("Meixner") models on partitions.

• **General problem** $\{X^{(n)}(t)\}$ a sequence of Markov processes on particle configurations with computable corr. functions.

– Do the (scaling) limit corr. functions correspond to a Markov process, too?

– If yes, how to construct it explicitly?

• Example 1: Extended Ch-D and the corresponding Markov processes are well known.

• Example 2: For the sine or Airy kernels, their extended versions are well known (Tracy-Widom). But Markov processes??? – Spohn, Osada, Katori-Tanemura.

• Comparison: Extended Whittaker vs. Extended sine/Airy

## Analytic continuation in $N$



$N = 1, 2, \dots$; $\mathrm{Sym}(N)$= symmetric polynomials in $N$ variables; $V_N = \prod_{1 \le i,j \le N}(x_i - x_j)$=Vandermonde; $D^{(i)}$ =a copy of $D = x\frac{d^2}{dx^2} + (b-x)\frac{d}{dx}$ acting on the $i$th variable $x_i$, $1 \le i \le N$. The formula

$$D_N = V_N^{-1} \circ \left(D^{(1)} + \cdots + D^{(N)}\right) \circ V_N + \frac{N(N-1)}{2}$$
$$= \sum_{i=1}^{N} x_i \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} \left( b - x_i + \sum_{j:\, j \ne i} \frac{2x_i}{x_i - x_j} \right) \frac{\partial}{\partial x_i},$$

correctly determines a linear operator $\mathrm{Sym}(N) \to \mathrm{Sym}(N)$.

Write $\mathrm{Sym}(N) = \mathbb{R}[e_1, \dots, e_N]$, express $D_N$ as a differential operator in $e_1, \dots, e_N$, and make formal analytic continuation in $b$ and $N$; then rename $z = N$, $z' = N + b - 1$. This gives $\mathfrak{D} : \mathrm{Sym} \to \mathrm{Sym}$.

Rains 2005; Sergeev-Veselov 2009.