FOCK SPACE AS INTEGRAL OVER SPACES ON RANDOM CONFIGURATIONS (PATHS) AND NON-FOCK FACTORIZATIONS.

(PARTIALLY WITH M.I.GRAEV)

A. M. VERSHIK (St.Petersburg)

3 мая 2011 г.

CONFERENCE "The VERSALITY OF INTEGRABILITY"

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Celebrating of Igor Krichever's 60th Birthday
COLUMBIA UNIVERSITY, 4-7 of May 2011, NEW-YORK

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5.**NON-FOCK FACTORIZATIONS - BLACK NOISE.**
0-DIMENSION (VOTING) MODEL; 1-2 DIMENSIONAL EXAMPLES.

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BY DEFINITION this is a continuous tensor product of the Hilbert spaces.

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where right side is L^2 over white noise (gaussian) law ν (for 1-dimensional case — derivative of the brownian motion).
many-particles decomposition, creation and annihilation operators, product-vectors etc.

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Let $L^2(S(M); \mu)$ *where* μ *is a law of Levy process on the space* $S(M)$ *of Schwartz distributions on the manyfold* M *with natural factorization is Fock factorization. The vacuum vectors are multiplicative functionals* $\xi \mapsto \exp\{\int \xi(x) dF(x)\}$

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Here we use only countable tensor product and integration over the space of configurations.

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$\theta > 0$;
For *theta* $= 1$ we called the measure $\mathcal{L}$ *generalized infinite dimensional Lebesgue or stable measure*

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When $r \to 0$ representation $\pi_r \to Id$ - tends to identity representation.

Representation of the current group

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Define the current group of the bounded measurable functions on the manifold X with values in G:

$$G^X = \{x \mapsto g(x) \in G\}$$

with point-wise multiplications.

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Now we correspond to the configuration γ and current $g(.)$ the operator in the space $\bigotimes_s K_{c_s}$:

$$\Pi_\gamma(g(.)) = \bigotimes_s \pi_{c_s)} g(x_s).$$

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STEP 3. Now we can integrate over all configurations γ over generalize Lebesgue measure $\mathcal{L}$:

$$\mathcal{H} = \int_{\gamma \in \mathcal{K}(X)} \bigotimes_{s=1}^{\infty} K_{x_s, c_s} \, d\mathcal{L}(\gamma),$$

IMPORTANT. Measure $\mathcal{L}$ is invariant with respect to multiplication on $r(x)$ iff $\int \ln r(x) = 0$.
We obtain the representation Π of the group G^X.

Theorem

The representation Π is irreducible.

Доказательство.

The group $\mathbb{R}^{*X}$ has ergodic action on $\mathcal{K}(X)$. □

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with multiplication

$$(r_1, u_1 c_1)\,(r_2, u_2, c_2) = (r_1 r_2, u_1 u_2, c_1 + r c_2 u).$$

So this group P is semisimple product

$$P = \mathbb{R}^* \leftthreetimes P_0, \quad \text{where} \quad P_0 = O(n-1) \leftthreetimes \mathbb{R}^{n-1},$$

and elements $r \in \mathbb{R}^*$ acts on P_0 as the automorphisms $(u,c) \mapsto (u,c)^r = (u, rc)$.

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Theorem

Consider $K_r = L^2(B_r)$, where B_r is Euclidean of the radius r with usual representation π_r of the motion group $P_0 = M_{n-1}$. Then the construction above gives the unitary representation of the current group P^X of the bounded measurable functions on the manifold X with values in the parabolic group P, and this representation naturally extends onto current group $O(n,1)^X$.

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A CANTOR set $X = \prod_1^\infty \{0, 1, 2\}$ - the set of the infinite path in the triadic tree $\mathbb{T}_3$ with one root.

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Basis in $\mathbb{C}^2$ — e_1, e_2, in $\mathbb{C}^8$— $e^1 \dots e^8$. Each vertex of cube corresponds to vector of basis:

$$e_1 \sim (0), e_2 \sim (1); e^1 \sim (0,0,0), e^2 \sim (1,0,0), e^3 \sim (0,1,O), e^4 \sim (0,0,1$$

$$e^5 \sim (1,1,0), e^6 \sim (1,0,1), e^7 \sim (0,1,1), e^8 \sim (1,1,1)$$

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Theorem

The factorization of the L^2 by cylindric sets over space of the pathes of triadic tree has no product vectors besides constant and consequently defines a Non-Fock factorization.

Discussion.

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