Gaudin subalgebras and stable rational curves

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Igor Krichever - 60, New York, May 6, 2011

Plan

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Reference

L. Aguirre, G. Felder and A.P. Veselov arXiv:1004.3253. To appear in Compositio Math.

Configuration space Σ_n and cohomology of pure braid group

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Arnold (1969): Cohomology $H^*(\Sigma_n) = H^*(P_n)$ is generated by the elements

$$\omega_{ij} = d\log(z_i - z_j) = \omega_{ji}, \quad , i, j = 1, \ldots, n$$

with the relations

$$\omega_{ij} \wedge \omega_{jk} - \omega_{ik} \wedge \omega_{kj} + \omega_{ij} \wedge \omega_{ik} = 0$$

for all triples $i \neq j \neq k$ (*Arnold's relations*)

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Generalisations: **Brieskorn, Orlik and Solomon**

Kohno-Drinfeld Lie algebra $\mathfrak{t}_n$

is generated by $t_{ij} = t_{ji}$, $1 \leq i \leq j \leq n$ with relations

$$[t_{ij}, t_{kl}] = 0,\ i \neq j \neq k \neq l$$

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3. Universal flat Knizhnik-Zamolodchikov connection (**Drinfeld**):

$$\nabla_i = \partial_i - \kappa \sum_{j \neq i}^{n} \frac{t_{ij}}{z_i - z_j}, \quad [\nabla_i, \nabla_j] = 0$$

Some representations

1. $U\mathfrak{t}_n \to U\mathfrak{G}^{\otimes n}$, $\mathfrak{G}$ is a semisimple Lie algebra:

$$t_{ij} = \sum_{\alpha} J_{\alpha}^{(i)} \otimes J_{\alpha}^{(j)} \in U\mathfrak{G}^{\otimes n}$$

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3. $U\mathfrak{t}_n \to \mathbb{C}[S_n]$:

$$t_{ij} = s_{ij}$$

are abelian subalgebras $\mathfrak{g} \subset \mathfrak{t}_n^1 =< t_{ij} >$ of maximal dimension

Lemma. $\dim \mathfrak{g} = n - 1$.

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Examples.

1. **Gaudin**: Integrable spin chain model

$$\mathfrak{g}_n(z) = \{\sum_{i<j}^{n} \frac{a_i - a_j}{z_i - z_j} t_{ij},\ a \in \mathbb{C}^n, z \in \Sigma_n\} \subset U\mathfrak{G}^{\otimes n}$$

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3. **Jucys, Murphy**: Representation theory of symmetric group S_n

$$JM =< t_{12},\ t_{13} + t_{23},\ t_{14} + t_{24} + t_{34}, \dots, t_{1n} + t_{2n} + \dots + t_{n-1n} >\subset \mathbb{C}[S_n]$$

Moduli space of stable rational curves

Let

$$M_{0,n+1} = \{(z_1, \dots, z_{n+1}), z_i \neq z_j, z_i \in \mathbb{C}P^1\}/PSL_2(\mathbb{C}) = \Sigma_n/Aff$$

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Examples. $\bar{M}_{0,4} = \mathbb{C}P^1$, $\bar{M}_{0,5} = dP_5$ is degree 5 del Pezzo surface.

Witten's picture

E. Witten Nucl. Phys B340 (1990)

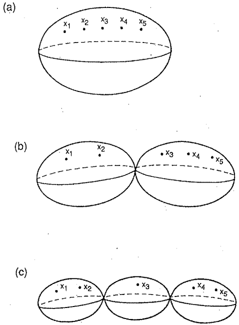

Fig. 1. (a) A generic configuration of distinct points x_i on CP^1. (b, c) To compactify the moduli space, one adds additional configurations in which the underlying Riemann surface breaks up into two or more branches.

Kapranov, Devadoss
$\bar{M}_{0,n+1}(\mathbb{R})$ is smooth and glued from $n!/2$ copies of **Stasheff polytopes** (associahedra) K_n

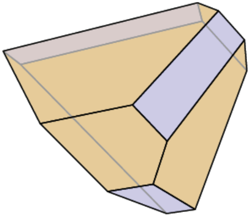

Figure: Stasheff polyhedron K_5

Real version

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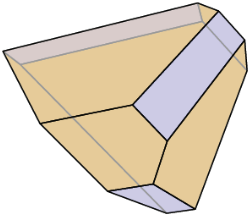

Figure: Stasheff polyhedron K_5

Etingof, Henriques, Kamnitzer, Rains: cohomology ring of $\bar{M}_{0,n+1}(\mathbb{R})$

Main result

Let G_n be the set of all Gaudin subalgebras of Kohno-Drinfeld algebra $\mathfrak{t}_n$. Since every Gaudin subalgebra is a linear subspace in $\mathfrak{t}_n^1 \approx \mathbb{C}^{\frac{n(n-1)}{2}}$ we have a natural imbedding

$$\varphi : G_n \to G(n-1, n(n-1)/2)$$

in the Grassmannian $G(n-1, n(n-1)/2)$.

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Proof is based on results by **Gerritzen, Herrlich and van der Put**.

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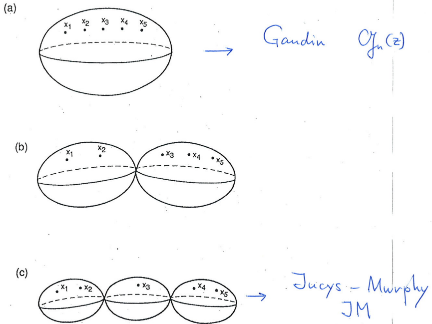

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- **Atiyah et al (2001-2)**: possible links between configuration spaces and flag varieties $F_n = U_n/T_n$

MANY HAPPY RETURNS, IGOR !