Emergent Conformal Symmetry in Dyson-Selberg Integrals

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- Selberg Integral

$$\int_0^1 \prod_{i=1}^{N} \xi_i^{a_1-1}(1-\xi_i)^{a_2-1} \prod_{i>j} |\xi_i - \xi_j|^{2\beta} d\xi_1 \dots d\xi_N$$

- Conformal Field Theory (CFT)

- Comment:

  $\beta = 1/2,\ 1,\ 2$ possesses a determinantal structure and corresponds τ-functions of integrable hierarchies

Outline

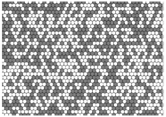
- Dyson-Selberg Integrals (of N variables),
- Conformal Covariance (Symmetry)
- Emergent conformal symmetry at a large N limit.

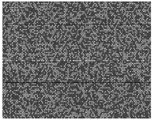
Motivation: Finite Dimensional Reduction of Conformal Field Theory (CFT)

Traditional view: CFT is a limit of 2D lattice statistical mechanics when a mesh tends to zero.

$$N^2 \text{ - variables;}$$

Smirnov: Conformal Symm. is proved for Ising model and Percolation.





Holomorphic nature of CFT suggests only N variables are essential:

$$N^2 \to N \text{ - boundary vs bulk}$$

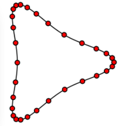
Motivation: Fekete theory - finite dimensional approx. of conformal maps

- Riemann mapping $f(z)$: $\mathbb{C} \setminus \mathscr{D} \to \mathbb{C} \setminus \mathbb{D}$, $f(z) \to z$, $z \to \infty$
  an exterior of a domain $\mathscr{D}$ to the exterior of a disk of a radius r,

$$f(z) = \lim_{N\to\infty} \left(\prod_{i=1}^{N} (z - \xi_i) \right)^{1/N}$$

- A set $\xi_1, \ldots, \xi_N$ are Fekete points, minimizing a Coulomb energy of a conductor

$$\min \sum_{i>j} [-\log|\xi_i - \xi_j|], \quad \xi_i \in \Gamma = \partial D$$



Density of Fekete points and Harmonic measure

- At large N images of Fekete point are uniformly distributed along a circle.
- Density of Fekete points tends to **Harmonic measure**

$$\rho(z)|dz| = \lim_{N\to\infty} \frac{1}{N} \sum_i \delta_\Gamma(z, \xi_i)|dz| = \frac{1}{2\pi}|f'(z)||dz|$$

- Inversed density represents a metric on a boundary $|dz| = \rho^{-1}|df|$.

Dyson-Selberg Integrals

- Selberg Integral:

$$\int_0^1 \prod_{i=1}^N \xi_i^{a_1-1}(1-\xi_i)^{a_2-1} \prod_{i>j} |\xi_i - \xi_j|^{2\beta} d\xi_1 \dots d\xi_N =$$

$$= \prod_{j=0}^{N-1} \frac{\Gamma(a_1+\beta j)\Gamma(a_2+\beta j)\Gamma(1+\beta+\beta j)}{\Gamma(a_1+a_2+(N+j-1)\beta)\Gamma(1+\beta)}$$

- Dyson integral

$$(2\pi i)^{-N} \int_{S^1} \prod_{i=1}^N \xi_i^{\frac{1}{2}a_1-1} |1+\xi_i|^{a_2} \prod_{i>j} |\xi_i - \xi_j|^{2\beta} d\xi_1 \dots d\xi_N =$$

$$= \prod_{j=0}^{N-1} \frac{\Gamma(a_2+\beta j)\Gamma(1+\beta+\beta j)\Gamma(1+a_1+a_2+\beta j)}{\Gamma(a_2-\alpha_1+\beta j)\Gamma(1+\beta)} = \tag{1}$$

$$= \frac{\Gamma(1+N\beta)}{\Gamma^N(1+\beta)}, \quad a_1 = a_2 = 0.$$

Expectation values of operators (Dyson Integral)

$$\langle \mathcal{O} \rangle = Z^{-1} \int_{S^1} \prod_{i=1}^{N} \mathcal{O}(\xi_1, \ldots, \xi_N) \prod_{i>j} |\xi_i - \xi_j|^{2\beta} d\xi_1 \ldots d\xi_N,$$

$$Z = \int_{S^1} \prod_{i>j} |\xi_i - \xi_j|^{2\beta} d\xi_1 \ldots d\xi_N = \frac{\Gamma(1+N\beta)}{\Gamma^N(1+\beta)}$$

- $\mathcal{O}(\xi_1, \ldots, \xi_N)$ is a symmetric function (polynomial) of ξ_i.

$$\mathcal{O} = \prod_{i=1}^{N} \xi_i^{\frac{1}{2}\alpha_1} |1+\xi_i|^{\alpha_2}$$

- More general operators

$$\mathcal{O}(z_1, z_2, \ldots) = \prod_{i=1}^{N} (z_1 - \xi_i)^{a_1} (z_2 - \xi_i)^{a_2} \ldots$$

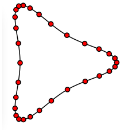
called primaries with charges $\alpha = a_1/\sqrt{\beta}, \alpha_2 = a_2/\sqrt{\beta}, \ldots$.

Primary interest:

- Dyson integrals on

  an arbitrary (not circular) simple closed contour

  at large N.



Digression: Boundary Conformal Field Theory

- A Field Theory defined in a bounded simply-connected domain $\mathscr{D}$ on a plane.
- There is a set of local operators called "**primary** " $\mathcal{O}_\alpha(z)$ which correlation functions are conformally covariant with respect to deformation of the boundary:

$$\langle \mathcal{O}_{h_1}(z_1)\mathcal{O}_{h_2}(z_2)\dots\rangle_{\mathscr{D}} = [f'(z_1)]^{h_1}[f'(z_2)]^{h_2}\dots\langle \mathcal{O}_{h_1}(f(z_1))\mathcal{O}_{h_2}(f(z_2))\dots\rangle_{\mathbb{D}}$$

$$f(z): \quad \mathscr{D} \to \mathbb{D}$$

- A set of h_k is called **dimensions** of primary operators.

Digression: Central Charge and β

Infinitesimal version:

$$f(z) = z + \epsilon(z)$$

$$\langle \delta_{\epsilon(z)} \mathcal{O}_h(z) \dots \rangle \equiv \langle T(z) \mathcal{O}_h(z) \dots \rangle \epsilon = (\epsilon \partial_z + h \partial_z \epsilon) \langle \mathcal{O}_h(z) \dots \rangle$$

- Operator T is called stress energy tensor.
- CFT are characterized by and a central charge.

$$\langle T(z) T(z') \rangle = \frac{c}{2} \frac{1}{(z - z')^4}$$

or parameter β such that

$$c = 1 - 6 \left(\sqrt{\beta} - 1/\sqrt{\beta} \right)^2$$

- One-parametric family β.
- Customary to characterize operators by their charge:

$$h = \alpha(\alpha - \sqrt{\beta} + 1/\sqrt{\beta}), \quad a = \alpha\sqrt{\beta}$$

Digression: Degenerate operators

- Primary operators with special dimension obey differential equations.
- The simplest is an operator on the level 2:

$$\Psi_{12}(z) = \mathcal{O}_{h_{12}}(z), \quad a_{12} = 1,\ a_{21} = -\beta$$

$$D = 2\beta\partial_z^2 - \sum_k \frac{2}{z - z_k}\partial_{z_k} - \sum_k \frac{2h_k}{z - z_k}$$

$$D\,\langle \mathcal{O}_{12}(z)\mathcal{O}_{h_1}(z_1)\dots\rangle_{\mathbb{D}} = 0$$

Main result: Large N limit of Dyson's integral on an arbitrary contour

- Define

$$\mathcal{O}_h(z) = (f(z))^{-aN} \prod_i (z - \xi_i)^a, \quad z \notin \mathcal{D}$$

$$h = \alpha(\alpha - \sqrt{\beta} + 1/\sqrt{\beta}), \quad \alpha = a/\sqrt{\beta}$$

- Then

$$\langle \mathcal{O}(z) \rangle = \left(f'(z)\right)^{h/2}$$

$$\langle \mathcal{O}_{h_1}(z_1)\mathcal{O}_{h_2}(z_2)\ldots\rangle_{\mathcal{D}} \approx (f'(z_1))^{h_1}(f'(z_2))^{h_2}\ldots\left(\frac{f(z_k)-f(z_l)}{z_k - z_l}\right)^{\alpha_k\alpha_l} =$$

$$= (f'(z_1))^{h_1}(f'(z_2))^{h_2}\ldots\langle \mathcal{O}_{h_1}(f(z)_1)\mathcal{O}_{h_2}(f(z)_2)\ldots\rangle_{\mathbb{D}}$$

Screening operators

- Find a general form of Dyson-Selberg density which preserve conformal covariance of the primary operators"?

$$\mathcal{O}_h(z) \propto \prod_i (z-\xi_i)^a, \quad z \notin \mathcal{D}$$

$$\langle \mathcal{O}(z) \rangle_{\mathcal{D}} = \left(r f'(z) \right)^{h/2} \langle \mathcal{O}(f(z)) \rangle_{\mathbb{D}}$$

- Modification of the Dyson-Selberg density a la Dotsenko-Fateev:

$$\prod_{i \le N} (z-\xi_i)^a \prod_{N \ge i > j} |\xi_i - \xi_j|^{2\beta} \times$$
$$\left[\prod_{k=1}^{r} (z-t_i)^{-a/\beta} \frac{\prod_{r \ge k > l} |t_k - t_l|^{-2/\beta}}{\prod_{k \le r, i \le N} (\xi_i - t_k)^2} \right] dt_1 \dots dt_r$$

- There is a one parametric family of Dyson's type integrals reveling conformal symmetry

Degenerate operators as Dyson's integrals

- If one of the charges is chosen to be

$$a = 1, \text{ or} - \beta$$

  then the operator

$$D = 2\beta \partial_z^2 - \sum_k \frac{2}{z - z_k} \partial_{z_k} - \sum_k \frac{2h_k}{z - z_k}$$

  nulls the integral

$$D \, \langle [\prod_{i \leq N} (z - \xi_i)] \, \mathcal{O}_{h_1}(z_1) \ldots \rangle_{\mathbb{D}} = 0.$$

- No large N is necessary (proof through an integration by parts).

Other CFT operators in terms of Dyson's integral

- Bose Field

$$\varphi_-(z) = -\frac{1}{N}\sum_i \beta \log|z-\xi_i|^2$$

- Current

$$\partial\varphi_-(z) = -\frac{1}{N}\sum_i \frac{\beta}{z-\xi_i}$$

- Holomorphic component of s.e. tensor

$$T(z) = \left(\partial\varphi(z)\right)^2 + \frac{1}{N}(1-\beta)\,\partial^2\varphi(z)$$

- Boundary components of s.e. tensor

$$2T_{sn} = \mathrm{Im}(\nu^2 T), \quad z\in\Gamma$$

$$2T_{nn} = \mathrm{Re}(\nu^2 T), \quad z\in\Gamma$$

ν is a normal vector to the boundary

Conformal Boundary Conditions and Ward Identity

At large N CFT conditions emerge

- Conformal Boundary conditions: sn-component continuous through the boundary

$$\operatorname{disc}\langle [T_{sn}(z)]_\Gamma \rangle = 0$$

- Conformal Ward Id: nn component generates a deformation of the boundary

$$\boxed{N^{-2}\delta \log Z_N = -\frac{1}{2\pi\beta}\oint_\Gamma \langle T_{nn}(z)\rangle \delta n(z)\, |dz|}$$

- A "quantum" version of Hadamard formula for variation of conformal maps.

Bose Field and Current

Conformal Boundary conditions for the Bose Field

$$\boxed{\langle \varphi(z) \rangle = -2\beta \log |f(z)| + \frac{2}{N}(\beta - 1) \log |f'|}$$

$$\beta^{-1} N^2 \langle \varphi(z) \varphi(\zeta) \rangle_c = G(z, \zeta) - \log |z - \zeta|$$

$$\langle (\partial \varphi(z))^2 \rangle_c = \frac{\beta}{6} \{f, z\}$$

Conclusion: Finite dimensional approximation of CFT

- Large N Dyson integrals represent objects of Boundary CFT.
- A sort of quantum Fekete theory:

  Finite dimensional approximation of conformal maps $\rightarrow$ Finite dimensional approximation of CFT.