Spaces of diagonal curvature and n-orthogonal coordinate systems

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Plan

1. Spaces of diagonal curvature - elementary theory
2. n - orthogonal coordinate system - new reductions
3. Inverse scattering approach
4. $\bar{\partial}$ - approach
5. Spaces of flat normal bundle and Ferapontov conjecture
6. Can we hope to find new solutions of the Einstein equations?

1. Spaces of diagonal curvature – elementary theory

G^n – n-dim Riemannian space admitting a diagonal metric

$$ds^2 = \sum_{\alpha=1}^{n} H_\alpha^2 \, du_\alpha^2$$

$H_\alpha(u_1, \dots, u_n)$ – Lamé coefficients

Q_{ij} are rotation coefficients defined as follow

$$Q_{ij} = \frac{1}{H_j} \frac{\partial H_i}{\partial u_j} \quad \text{or} \quad \frac{\partial H_i}{\partial u_j} = Q_{ij} H_j$$

(standart notations $\beta_{ij} = Q_{ji}$)

The Riemann curvature tensor

$$R_{ij,kl} = 0 \qquad i \neq j \neq k \neq l$$

$$R_{ij,ik} = H_i H_j \left(\frac{\partial Q_{ij}}{\partial u_k} - Q_{ik} Q_{kj} \right) = H_i H_k \left(\frac{\partial Q_{ik}}{\partial u_j} - Q_{ij} Q_{jk} \right)$$

$$i \neq j \neq k$$

g^n is the space of diagonal curvature if

$R_{ij,kl} = 0$ for $i \neq j \neq k$

Then

$$\frac{\partial Q_{ij}}{\partial u_k} = Q_{ik} Q_{kj}$$

ψ_i – adjoint Lamé coefficients

$$\frac{\partial \psi_i}{\partial u_j} = Q_{ji} \psi_j$$

$$\psi_i H_i = \frac{\partial h}{\partial u_i} \qquad h - \text{potential}$$

$$\frac{\partial^2 h}{\partial u_k \partial u_l} = \Gamma^k_{kl} \frac{\partial h}{\partial u_k} + \Gamma^l_{lk} \frac{\partial h}{\partial u_l}$$

$$\Gamma^i_{jk} = 0 \qquad i \neq j \neq k$$

$$\Gamma^i_{ik} = \frac{1}{H_i} \frac{\partial H_i}{\partial u_k} \qquad \Gamma^i_{kk} = -\frac{H_k}{H_i^2} \frac{\partial H_k}{\partial u_i}$$

First rank solitonic solution

$$Q_{ij} = \frac{A_i(u_i)\, B_j(u_j)}{\Delta}$$

$$\Delta = \Delta_0 - \sum_{i=1}^{n} \int_0^{u_i} B_i(\xi_i)\, A_i(\xi_i)\, d\xi_i$$

Δ_0 - arbitrary constant

$$\frac{\partial \Delta}{\partial u_i} = -A_i(u_i)\, B_i(u_i)$$

$A_i(u_i)$, $B_i(u_i)$ 2n arbitrary functions of one variable

Then

$$H_i = f_i(u_i) + \frac{A_i(u_i)}{\Delta} K \qquad K = k_0 + \sum_{i=1}^{n} \int_0^{u_i} f_i(\xi_i)\, B_i(\xi_i)\, d\xi_i$$

$$\Psi_i = g_i(u_i) + \frac{M}{\Delta} B_i(u_i) \qquad M = M_0 + \sum_{i=1}^{n} \int_0^{u_i} g_i(\xi)\, A_i(\xi)\, d\xi$$

$$h = h_0 + \sum_{i=1}^{n} \int_0^{u_i} f_i(\xi)\, g_i(\xi) + \frac{KM}{\Delta}$$

k_0, M_0, h_0 - arbitrary constants

By a ~~...~~ proper change of variables

$u_i \to u_i(v_i)$

one make obtain $A_i = \begin{cases} 1 \\ 0 \end{cases}$

Solutions of rank K are defined as follow

L^k — k-dim auxiliary space

$A_i(u_i)$ $\quad B_i(u_i)$ — vectors in L^k

$$A_i = A_{ip}(u_i) \qquad 1 < p < k \qquad 1 < i < n$$

$$B_i = B_{ip}(u_i) \qquad 1 < p < k \qquad 1 < i < n$$

Δ — operator in L^k

$$\boxed{\Delta_{pq} = \Delta^0_{pq} - \sum_{i=1}^{n} \int_0^{u_i} B_{ip}(\xi)\, A_{iq}(\xi)\, d\xi}$$

Δ^0_{pq} — arbitrary constant matrix

$$\Delta_{pq} = \Delta^0_{pq} + \sum_{i=1}^{n} \Delta^i_{pq}(u_i)$$

$$\frac{\partial \Delta^i_{pq}}{\partial u_i} = - B_{ip} A_{iq}$$

Then

$$Q_{ij} = A_{ip}(u_i)\left(\Delta^{-1}\right)_{pq} B_{jq}(u_i)$$ — summation over p, q

Then solution of rank k.

$$Q_{ij} = \tilde{A}_{ip} B_{jp}(u_j)$$

$$\tilde{A}_i = A_i \Delta^{-1} \qquad \tilde{A}_{ip} = A_{iq}\left(\Delta^{-1}\right)_{qp}$$ – summation

In solutions of infinite rank $-\infty < p < \infty$ – continuous parameter. $\tilde{A}_{ip}$ is a solution of the integral equation

$$A_{iq} = \int_{-\infty}^{\infty} \tilde{A}_{iq} \Delta_{qp}\, dq$$

Then

$$A_i = f_i(u_i) + \tilde{A}_i(p) k_p$$

$$\Psi_i = g_i(u_i) + M_p\left(\Delta^{-1}\right)_{pq} B_{qi} = g_i(u_i) + M_p \tilde{B}_{pi}$$

$$K_p = K_{op} + \sum_{i=1}^{n} \int_0^{u_i} f_i(\xi) B_{ip}(\xi) d\xi$$

$$M_p = M_{op} + \sum_{i=1}^{n} \int_0^{u_i} g_i(\xi) A_{ip}(\xi) d\xi$$

K_{op}, M_{op} arbitrary constant vectors

$$h = h_0 + \sum_{i=1}^{n} \int_0^{u_i} f_i(\xi) g_i(\xi) d\xi + K_p (\Delta^{-1})_{pq} M_q$$

– summation

In the continuous case we set integration

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_p (\Delta^{-1})_{pq} M_q \, dp \, dq$$

In a general case L^k could be an arbitrary linear space with measure

Notice that $\hat{A}_{ip}$ are combescure equivalent Lamé coefficients at any P

2. n-orthogonal coordinate system – elementary theory

Let $x^1, \ldots x^n$ are carthesian coordinates in R^n

A curvilinear coordinate system

$$u^i = u^i(x^1, \ldots, x^n) \qquad \det \left\| \frac{\partial u^i}{\partial x^j} \right\| \neq 0$$

is orthogonal if

$$\boxed{\sum_{k=1}^{n} \frac{\partial u^i}{\partial x^k} \frac{\partial u^j}{\partial x^k} = h_i^2 \delta_j^i} \tag{2.1}$$

In new coordinates R^n is the Riemannian space with the metrics $H_i^2 = \frac{1}{h_i^2}$

$$ds^2 = H_i^2 \, du_i^2$$

This space is flat, it's Riemann ~~curvature~~ curvature tensor is identically equal to zero. This is a special case of the space of diagonal curvature.

Diagonal elements of the Riemann tensor are

$$R_{ij,ij} = H_i H_j E_{ij} \tag{1.2}$$

Thus equation $E_{ij} = 0$ must be solved together wit (1.4)

$$E_{ij} = \frac{\partial Q_{ij}}{\partial u_j} + \frac{\partial Q_{ji}}{\partial u_i} + \sum_{k \neq i,j} Q_{ik} Q_{jk} \tag{2.3}$$

To find explicit form of curvilinear coordinates one should remember that the straight lines

$x^i(s) = x^i(u^1 .. u^{l-1}, s, u^{l+1} ... u^n)$ for any l. Then are geodesics in R^n

$$\frac{\partial^2 x_i}{\partial u_k \partial u_i} = \Gamma^k_{kl} \frac{\partial x^i}{\partial u_k} + \Gamma^l_{lk} \frac{\partial x^i}{\partial u_l} \tag{2.5}$$

Moreover all x^i are "potential".

$$\frac{\partial^2 x^i}{\partial u_l^2} = \sum_k \Gamma^k_{ll} \frac{\partial x^i}{\partial u_k} \tag{2.6}$$

For a solitonic solution of arbitrary rank

$$E_{ij} = \tilde{A}_{ip} B'_{jp} + \tilde{A}_{jp} B'_{ip} + \tilde{A}_{ip} \tilde{A}_{jq} u_{pq}$$

$$u_{pq} = \sum_{k=1}^{n} B_{kp} B_{kq}$$

If A_{ip}, B_{ip} are connected by following relation

$$A_{ip} = \varphi_{pq} B'_{iq}, \quad \text{where}$$

$$\Delta_{opq} \varphi_{pq} + \Delta_{oqp} \varphi_{qp} = 0$$

$$\downarrow$$

$$E_{ij} = 0$$

The case $\varphi_{pq} + \varphi_{qp} = 0$

$$\Delta_{opq} = \delta_{pq}$$

is described by differential reduction in framework of the IST method

Another example different the IST prescription is presented by the spherical coordinate system in R^3. In this case

$$A_1 = 0 \qquad A_2 = \frac{1}{\sin x_2} \qquad A_3 = 1$$

$$B_1 = 1 \qquad B_2 = \frac{\cos x_2}{\sin x_2} \qquad B_3 = 0$$

now

$$A_2 B_2 = -\frac{\partial}{\partial x_2} \frac{1}{\sin x_2}$$

$$\Delta = \frac{1}{\sin x_2}$$

$$Q_{12} = Q_{13} = Q_{23} = 0 \qquad Q_{12} = \cos x_2$$

$$Q_{31} = \sin x_2$$

$$Q_{21} = 1 \qquad X_3 = \varphi$$

$$x_1 = r \qquad x_2 = \vartheta$$

To accomplish solution of construction of n- orthogonal coordinate one take into account that now potentials x^i must satisfy to additional equation

$$\frac{\partial^2 h}{\partial u_i^2} = \sum \Gamma^k_{\ell\ell} \frac{\partial h}{\partial u^k} \qquad \frac{\partial h}{\partial x_i} = \Psi_i H_i$$

This equation is equivalent to condition

$$\frac{\partial \Psi_i}{\partial x_i} = -\sum_{k \neq i} Q_{ik} \Psi_k$$

This condition is satisfied if all $g_i(u_i)$ are constants. At arbitrary choise of g_i coordinates x^i are affine but not orthogonal. They must be orthogonalized by a proper linear transformation

③ Inverse scattering approach (Dressing method)

let $f_i(s)$ $i = 1, \dots n$ fuctions on R^n,

$1 + F$ is a linear operator

$$[(1+F)f]_i = f_i(s) + \int_{-\infty}^{\infty} \sum_{k=1}^{n} F_{ik}(s,s') f_k(s') ds'$$

We consider two factorization

$$1 + F = (1+K^+)^{-1}(1+K^-)$$

$$1 + F = (1+M^+)(1+M^-)^{-1}$$

$K^+(s,s') = 0$ at $s' < s$

$M^+(s,s') = 0$

$K^-(s,s') = 0$ at $s' > s$

$M^-(s,s') = 0$

$$1 + M^+ = (1+K^+)^{-1}$$

K^+ and M^+ satisfy to Marchenko integral equations

$$K^+(s,s') + F(s,s') + \int_s^\infty K^+(s,s'')F(s'',s')\,ds'' = 0$$

$$M^-(s,s') + F(s,s') + \int_{s'}^\infty F(s,s'')M^-(s'',s')\,ds'' = 0$$

moreover

$$K^+(s,s') + M^+(s,s') + \int_s^{s'} K^+(s,s'')M^+(s'',s')\,ds'' = 0$$

Let F satisfy differential equations

$$D_i F = 0 \qquad D_i F = \frac{\partial F}{\partial u_i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i$$

$$I_i I_j = I_i \delta_{ij} \quad \text{are projectors}$$

$$I_i = \mathrm{diag}\,(0 \ldots \underset{i}{1} \ldots 0)$$

$$F_{ij}(s,s',u) = F_{ij}(s-u_i, s'-u_j) \qquad F_{ii} = 0$$

$N(N-1)$ functions of two variables

Apparentely $[D_i, D_j] = 0$

also

$$\hat{D}_i K^+ = D_i K^+ + [I_i, Q] K^+ = 0$$

$$\hat{D}_i M^- = D_i M^- + M^- [I_i, Q] = 0$$

here $Q = K^+(s, s, u) = M^-(s, s, u)$

and

$$\frac{\partial}{\partial u_k} I_i Q I_j = I_i Q I_k Q I_j$$

or

$$\frac{\partial Q_{ij}}{\partial u_k} = Q_{ik} Q_{kj}$$

Solitonic solutions of rank K are obtained

if

$$F_{ij} = \sum_k F_i^k(s - u_i) G_j^k(s' - u_j)$$

$$\frac{\partial Q_{ij}}{\partial s} + \sum_k \frac{\partial Q_{ij}}{\partial u_k} = 0$$

Then

$$H_i = f_i(s-u_i) + \sum_k \int_s^\infty K^+_{ik}(s,s',u)\,\varphi_k(s'-u_k)\,ds'$$

$$\Psi_i = g_i(s-k_i) + \sum_k \int_s^\infty g_k(s'-u_k)\,M^-_{ki}(s',s,u)\,ds'$$

Suppose

$$\frac{\partial F(s,s')}{\partial s'} + \frac{\partial F^{+z}(s',s)}{\partial s} = 0$$

Then

$$\left(\frac{\partial K^+(s',s)}{\partial s}\right)^{tz} + \frac{\partial M^-(s,s')}{\partial s'} + M^-(s,s')\left[Q(s) - Q^{tz}(s')\right] = 0$$

$$\downarrow$$

$$\left(\frac{\partial K^+(s,s')}{\partial s'} + \frac{\partial K^{+\,tz}(s',s)}{\partial s}\right)\Bigg|_{s'=s} = -Q(s)\,Q^{tz}(s)$$

$$\downarrow$$

$$E_{ij} = 0$$

$$E_{ij} = \frac{\partial Q_{ij}}{\partial u^j} + \frac{\partial Q_{ji}}{\partial u^i} + \sum_{k \neq i,j} Q_{ik} Q_{jk} = 0$$

The solitonic solution of rank K appear if

$$F = \sum_{p,q} \frac{\partial f_p(s-u_p)}{\partial s} \Lambda_{pq} f_q(s'-u)$$

$$\Lambda_{pq} = -\Lambda_{qp} \quad \text{— skew-symmetric}$$

constant matrix

To provide satisfaction of relation

$$\frac{\partial \Psi_i}{\partial x_i} = -\sum_{k \neq i} Q_{ik} \Psi_k$$

One has to put $g_i(s-u_i) = c_i$ are constants

4. $\bar\partial$ approach

$$\chi = \chi_{ij}(\lambda, \bar\lambda) \qquad i, j = 1, \dots, n$$

– quasianalytic matrix function on $\mathbb{C}^1$

$$\frac{\partial \chi}{\partial \bar\lambda} = -\pi i \int \chi(\xi)\, f(\xi, \lambda)\, d\xi$$

$$\frac{\partial \tilde\chi}{\partial \bar\lambda} = \pi i \int \tilde f(\lambda, \xi)\, \tilde\chi(\xi)\, d\xi$$

$$\tilde\chi(\xi) = \chi^{tr}(-\xi) \qquad \tilde f(\lambda, \xi) = f^{tr}(-\xi, -\lambda)$$

$$f(\lambda, \xi) = e^{+i\lambda\varphi}\, f^0(\lambda, \xi)\, e^{-i\xi\varphi}$$

$$\varphi = \sum k_i I_i \qquad I_i I_j = I_i \delta_{ij}$$

– projector operators

$$I_i = \mathrm{diag}\,(\underbrace{0, \dots 1}_{i}, \dots 0\ 0)$$

$$f_{em}(\lambda, \xi) = f^0_{em}\, e^{i(+\lambda k_m - \xi k_e)}$$

f^0_{em} – constant matrix

at $\lambda \to \infty$

$$\chi \to 1 + \frac{Q}{i\lambda} + \frac{P}{(i\lambda)^2}$$

$$\tilde{\chi} \to 1 - \frac{Q^{+z}}{i\lambda} + \frac{P^{+z}}{(i\lambda)^2}$$

$$Q = \int f(\xi, \bar{\xi}, \mu, \bar{\mu})\, \chi(\xi, \bar{\xi})\, d\xi\, d\bar{\xi}\, d\mu\, d\bar{\mu}$$

$$\varphi = \lambda \chi \tilde{\chi}$$

$$\varphi \to \lambda + \frac{B}{\lambda} + \cdots \quad \lambda +$$

$$B = -P \cdot P^{+z} + Q Q^{+z}$$

The condition

$$\int \frac{\partial}{\partial \bar{\lambda}} \varphi\, d\lambda\, d\bar{\lambda} = 0 \quad \longrightarrow \quad B = 0$$

$$D_p \gamma = \frac{\partial \gamma}{\partial u_p} + i\lambda \gamma_p I_p$$

$$D_p \gamma \to i\lambda I_p + Q I_p \qquad \lambda \to \infty$$

$$I_q D_p \gamma \to I_q Q I_p$$

$$L_{qp} \gamma = I_q D_p \gamma - I_q Q I_p \gamma \equiv 0$$ – Lax representation

$$I_q \left(\frac{\partial \gamma}{\partial u_p} + i\lambda \gamma I_p \right) - I_q Q I_p \gamma = 0$$

$$I_q \frac{\partial Q}{\partial u_p} + I_q P I_p - I_q Q I_p Q = 0$$

$$\downarrow$$

$$\frac{\partial Q_{q\ell}}{\partial u_p} = Q_{qp} Q_{p\ell}$$

$$\frac{\partial Q_{qq}}{\partial u_p} = Q_{qp} Q_{pq}$$

$$P_{pq} = Q_{qp} Q_{pp} - \frac{\partial Q_{qp}}{\partial u_p}$$

$$\varphi = \lambda y \hat{x}^2 \rightarrow \lambda + \frac{B}{\lambda} + \cdots$$

$$B_{pq} = \frac{\partial Q_{pq}}{\partial u_q} + \frac{\partial Q_{qp}}{\partial u_p} + \sum_{k \neq p,q} Q_{pk} Q_{qk}$$

Let us denote

$$H(u_1 \ldots u_n, \xi, \bar{\xi}) = f(u_1, \ldots, u_n, \xi, \bar{\xi})\, e^{-i\xi\varphi}$$

Then

$$I_q \frac{\partial H}{\partial u_p} = I_q Q\, I_p H$$

$$H_q = I_q H$$

$$\frac{\partial H_q}{\partial u_p} = Q_{qp} H_p$$

$H_q(u_1, \ldots, u, \xi, \bar{\xi})$ compose a set of combescure equivalent Lamé coefficients, ξ is a „label"

Then

$$B = -i\pi \int \chi(\xi, \bar{\xi})\left(\lambda f(\xi, \lambda) - \xi f^{+2}(-\lambda, -\bar{\xi})\right)\chi^{+}(-\lambda, -\bar{\lambda})\, d\xi\, d\bar{\xi}\, d\lambda\, d\bar{\lambda} =$$

$$= -i\pi \int H(\xi, \bar{\xi})\,(\lambda f_0(\xi, \lambda) - \xi f_0^{+2}(-\lambda, -\xi))\, H^{+2}(-\lambda, -\bar{\lambda})\, d\xi\, d\bar{\xi}\, d\lambda\, d\bar{\xi}$$

The condition

$$\lambda f_0(\xi, \lambda) - \xi f_0^{+2}(-\lambda - \xi)$$

$$\downarrow$$

$$B = 0$$

$$\downarrow$$

$E_{ij} = 0$ – Flat space!

$$-i\pi(\lambda f(\xi, \lambda) - \xi f^{+2}(-\lambda, -\xi)) = R(\xi, -\lambda)$$

$$B = \cancel{i\pi} \int H(\xi, \bar{\xi})\, R(\xi, \lambda)\, H^{+2}(\lambda, \bar{\lambda})\, d\xi\, d\bar{\xi}\, d\lambda\, d\bar{\lambda}$$

5. Spaces of flat normals connection

$$E_{ij} = \int H_{ip}^{(\xi)} H_{jq}^{(\lambda)} R_{pq}(\xi,\lambda)\, d\xi\, d\lambda \qquad i \neq j$$

$$R_{pq}(\xi,\lambda) = R_{qp}(\lambda,\xi)$$

One can check

$$\frac{\partial E_{ij}}{\partial u_k} = Q_{ik} E_{kj} + Q_{jk} E_{ki} \qquad i \neq j \neq k$$

This is the Biachi identity for diagonal curvature tensor

Let $R_{pq}(\xi\ \lambda) = \sum_{\ell=1}^{N} R_p^{\ell}(\xi) R_q^{\ell}(\lambda)\, d^{\ell}$ $\qquad d_\ell = \pm 1$

$$H_i^{\ell} = \sum_p \int H_{ip} R_p^{\ell}(\xi)\, d\xi$$

$$E_{ij} = \sum_{\ell=1}^{N} H_i^{\ell} H_j^{\ell}\, d_\ell$$

This is the metric of a space of flat normal connection (flat normal bundle)

N is arbitrary number

If $N \to \infty$ any $R_{pq}(\xi, \lambda)$ can be obtained. This is a proof of the Ferapontov conjecture for spaces of diagonal curvature solvable by the $\bar{\partial}$ – approach

On elementary level one can ~~put~~ impose the reduction

$$A_{ip} = \varphi_{pq} B_{iq}\,,$$

and put $f_i = 0$

Then

$$E_{ij} = \sum A_{ip} A_{jq} R_{pq}$$

$$R_{pq} = \Delta_{0pq} \varphi_{pq} + \Delta_{0qp} \varphi_{qp}$$

If $R_{pq} = \sum R_p^l R_q^l$ we obtain the space of flat connection

6. Can we find new solution of the Einstein equation?

Not, so far — pity!

Let $n = 4$, then the Einstein tensor C^i_i and Ricci tensor are diagonal.

$$C^i_i = \sum_{\substack{j \neq i \\ k \neq i}} c_{jk}$$

$$C^i_j = \frac{8\pi\varkappa}{c^4} T^i_j + \lambda$$

$$C^i_j = R^i_j - \frac{1}{2} R \delta^i_j$$

$$c_{ij} = \frac{E_{ij}}{H_i H_j}$$

Thus any Einstein space of diagonal curvature give a solution of the Einstein equations with some diagonal energy-momentum tensor. How to get vacuum or dust?

All spherically symmetric solutions of Einstein equations are spaces of diagonal curvature remember that

$$ds^2 = -H_0^2 dx_0^2 + H_1^2 dx_1^2 + H_2^2 dx_2^2 + H_3^2 dx_3^2$$

In the vacuum case

$$C_{01} = C_{23} = \alpha \qquad \alpha + \beta + \gamma = 0$$

$$C_{02} = C_{13} = \beta$$

$$C_{03} = C_{12} = \gamma$$

No new solutions are found.

Let the metric is synchronous $H_0 = 1$

$$Q_{\alpha\beta} = \frac{\partial H_\alpha}{\partial x_\beta} \frac{1}{H_\beta}$$

Offdiagonal Einstein equations are

$$\frac{\partial Q_{\alpha\beta}}{\partial x_\gamma} = Q_{\alpha\gamma} Q_{\gamma\beta}$$ — diagonal curvature condition

$$\frac{1}{H_\alpha}\dot{\overset{\circ}{Q}}_{\alpha\gamma} + \frac{1}{H_\beta}\dot{\overset{\circ}{Q}}_{\beta\gamma} = 0 \qquad \alpha \neq \beta \neq \gamma$$

Metric of system of two black holes moving along a straight line must be inside of this metric!

Suppose $\dot{Q}_{\alpha\beta} = 0$

$F_{\alpha\beta} = E_{\alpha\beta}$ +) are constant in time the Lame' coefficients satisfy to the dynamical system (for evolution of a dust cloud)

$$H_1 \ddot{H}_2 + \dot{H}_1 \dot{H}_2 + \ddot{H}_1 H_2 = F_{12}$$

$$H_1 \ddot{H}_3 + \dot{H}_1 \dot{H}_3 + \ddot{H}_1 H_3 = F_{13}$$

$$H_2 \ddot{H}_3 + \dot{H}_2 \dot{H}_3 + \ddot{H}_2 H_3 = F_{23}$$

This is the Lagrangian system with Lagrangian

$$L = H_1 \dot{H}_2 \dot{H}_3 + H_2 \dot{H}_1 \dot{H}_3 + H_3 \dot{H}_1 \dot{H}_2 + F_{12} H_3 + F_{13} H_2 + F_{23} H_1$$

and Hamiltonian

$$\mathcal{H} = H_1 \dot{H}_2 \dot{H}_3 + H_2 \dot{H}_1 \dot{H}_3 + H_3 \dot{H}_1 \dot{H}_3 - F_{12} H_3 - F_{13} H_2 - F_{23} H_1$$

in the vacuum case

$$\mathcal{H} \equiv 0$$

For the symmetric case $F_{\alpha\beta} = \pm \frac{1}{R_0^2}$

one ~~get~~ can put

$$H_i = a(t) h_i$$

and get the Friedmann equation for the scale factor of Universe

$$2a\ddot{a} + \dot{a}^2 = \underline{\underline{C}}$$

If $H_3 = H_2 \sin x_2$ one set

$$Q_{21} = \frac{\partial H_2}{\partial x_1}\frac{1}{H_1} = \sqrt{1+f(R)} = \text{const}$$

$$Q_{31} = \sin\vartheta \, Q_{21}$$

$$Q_{32} = \cos\vartheta$$

$$H_2 = r$$

$$\dot{r}^2 = f(R) + \frac{F(R)}{r}$$ — Tolman solution for the ~~cat~~ collapse of dust cloud.

Remember, that all equation must be solved ~~better~~ together with constraints

$$\frac{\partial H_\alpha}{\partial x_\beta} = Q_{\alpha\beta} H_\beta$$

Evolution of H_α is the "Combescure flow" in ~~the~~ a ~~sp~~ 3-d space of diagonal curvature with given constant rotation coefficients.