In honor of Igor.

KP equation, flexes, and trisecants to the Kummer variety.

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Theorem

(EA, C. Codogni, G. Pareschi, '20) The base locus of an ample linear system on an abelian variety is reduced.

(Strong form of a conjecture by O.Debarre and C. Hacon, '06)

Corollary

(A-C-P, '20) (X, Θ) *a p.p.a.v.,* $G \subset X$ *a closed algebraic subgroup. Then* $\Sigma(X, \Theta, G) := \bigcap_{g \in G} \Theta_g$ *is a reduced scheme.*

• The Kadomtsev-Petviashvili (KP) equation and the trisecants to the Kummer variety.

• Igor's results ('75–'10)

• An algebro-geometrical approach. (Fay '73, Mumford '83 A.-De Concini '84, Debarre '97, Marini '97, A.-Codogni-Pareschi '20)

Preliminaries:

$$X = \mathbb{C}^g / \Lambda_\tau , \qquad \Lambda = (I_g , \tau)$$

$$\tau \in \mathcal{H}_g , \qquad \tau \in M_{g\times g}(\mathbb{C}) , \quad {}^t\tau = \tau , \quad \mathcal{I}m(\tau) > 0$$

$$\theta(z,\tau) = \sum_{m\in\mathbb{Z}^g} \exp \frac{1}{2}\{{}^t m\tau m + 2{}^t mz\} \qquad z \in \mathbb{C}^g$$

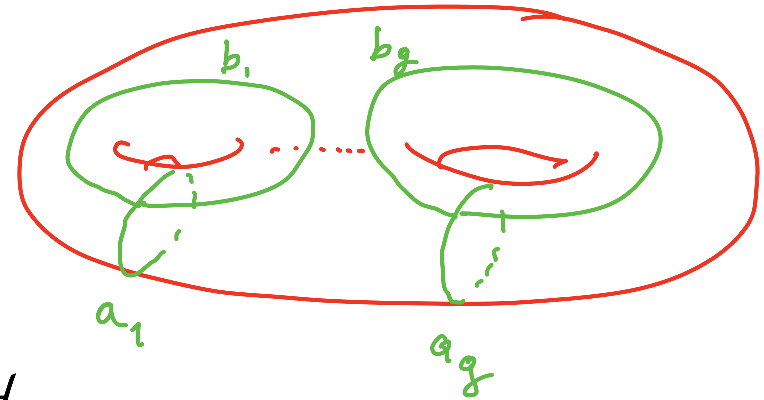
$$\Theta := (\theta)_0 \subset X$$

(X, Θ) is a p.p.a.v

$$\mathcal{H}_g / \operatorname{Sp}(g,\mathbb{Z}) = \mathcal{A}_g = \{\text{moduli space of p.p.a.v.}\}$$

$$\dim \mathcal{A}_g = \frac{g(g+1)}{2}$$

C genus g smooth curve. Riemann:



$\left(\int_{b_i} \omega_j\right)$ = I

$\langle \omega_1, \dots, \omega_g \rangle = H^0(\Omega^1_C)$

$$\left(\int_{b_i} \omega_j\right) = \tau \in \mathcal{H}_g\,,$$

$$\mathcal{J}_g = \text{jacobian locus} \quad \subset \quad \mathcal{H}_g$$

$$\dim \mathcal{J}_g = 3g - 3\,, \qquad g > 1$$

Abel Jacobi

$$\mathcal{J}_g / \operatorname{Sp}(g, \mathbb{Z}) = \mathcal{M}_g \quad \subset \quad \mathcal{A}_g$$

$\gamma : C \longrightarrow \Gamma \subset J(C)$; $\gamma : C_{g-1} \longrightarrow W_{g-1} \subset J(C)$

KP equation:

$$\boxed{\frac{\partial}{\partial x}(2u_{xxx} + 3uu_x - u_t) + 3u_{yy} = 0}, \qquad u = u(x, y, t).$$

Zakharov, Shabat, Faddeev, Matveev, Novikov, Dubrovin, Krichever,....

Gardner, Miura, Lax,...

McKean, Moser,...

Sato, Kashiwara, Jimbo, Miwa, Hirota, Mulase, Shiota,....

Segal-Wilson: *Loop groups and equations of KdV type.*

KdV:

$$2u_{xxx} + 3uu_x - u_t = 0$$

Isospectral deformations of the Schrödinger operator $L = \partial^2 + u$

Eigenvalues of L are "conserved quantities"

$\rightsquigarrow$ Arnold-Liouville

$\rightsquigarrow$ Tori

Burchnall Chaundy

$L\psi = \lambda\psi$

$P\psi = \mu\psi$

$[P, L] = 0$

Back to KP:

$$\boxed{\frac{\partial}{\partial x}\left(2u_{xxx} + 3uu_x - u_t\right) + 3u_{yy} = 0}, \qquad u = u(x, y, t).$$

Theorem

Krichever '76 (Novikov, Dubrovin,....) C a genus-g curve , θ its Riemann theta-function

$$\gamma : C \longrightarrow \Gamma \subset J(C) \qquad (Abel\text{-}Jacobi)$$

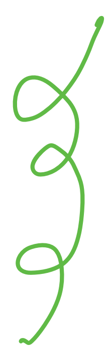
$$\epsilon \mapsto 2D_1\epsilon + 2D_2\epsilon^2 + 2D_3\epsilon^3 + \cdots \qquad D_i \in \mathbb{C}^g$$

Then

$$u(x, y, t; z) = \frac{\partial}{\partial x^2} \log \theta(xD_1 + yD_2 + tD_3 + z)$$

satisfies the KP equation.

Solitons: $\longleftrightarrow$



genus g Castelnuovo curves

Theorem

(Novikov's conjecture) (Shiota '86) An i.p.p.a.v (X, Θ) *is the Jacobian of an algebraic curve* C*, if (and only if) there exist vectors* $D_1,\ D_2,\ D_3 \in \mathbb{C}^g$*, with* $D_1 \neq 0$*, and* $d \in \mathbb{C}$ *s.t. the function*

$$u(x, y, t; z) = \frac{\partial}{\partial x^2} \log \theta(xD_1 + yD_2 + tD_3 + z) + d$$

satisfies the KP equation: $\frac{\partial}{\partial x}(2u_{xxx} + 3uu_x - u_t) + 3u_{yy} = 0.$

Unwinding the KP (Hirota '71):

$$\boxed{\begin{aligned} & D_1^4\theta(z) \cdot \theta(z) - 4D_1^3\theta(z) \cdot D_1\theta(z) + 3(D_1^2\theta(z))^2 - 3(D_2\theta(z))^2 \\ & + 3D_2^2\theta(z) \cdot \theta(z) + 3D_1\theta(z) \cdot D_3\theta(z) - 3D_1D_3\theta(z) \cdot \theta(z) + d\theta^2(z) = 0 \end{aligned}}$$

where $d \in \mathbb{C}$, $z \in \mathbb{C}^g$, and $D \equiv (a_1, \ldots, a_g) \leftrightarrow \sum a_i \frac{\partial}{\partial z_i}$ thought of as constant vector field

Apparently unrelated point of view:

(X, Θ) p.p.a.v. Kummer morphism

$$\phi : X \longrightarrow K(X) \subset |2\Theta)| = \mathbb{P}^{2^g-1} \quad z \mapsto [\vec{\theta}(z)] = [\ldots, \theta[n](z), \ldots] \quad K$$

Trisecant lines to $K(X) \cong X/\pm 1$:
$\ell \subset \mathbb{P}, \quad \ell \cong \mathbb{P}^1$

$$\ell \cdot K(X) = Y$$

a) $Y = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^3$ *flex*,

b) $Y = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2 + \operatorname{Spec} \mathbb{C}$ *degenerate trisecant,*

c) $Y = \operatorname{Spec} \mathbb{C} + \operatorname{Spec} \mathbb{C} + \operatorname{Spec} \mathbb{C}$ *bona fide trisecant*.

Bridging trisecants to $K(X)$ and the KP: Mumford ('77), based on Fay's trisecant formula ('73).

C genus g curve,

$$u : C \xrightarrow{\cong} \Gamma \subset X := J(C) \qquad \text{Abel-Jacobi.}$$

$\alpha, \beta, \gamma \in \Gamma$,

$$\zeta \in \frac{1}{2}(\Gamma - \alpha - \beta - \gamma) \subset X, \qquad \text{then}$$

$$\theta(z-\alpha)\theta(z+\alpha+2\zeta)+c\theta(z-\beta)\theta(z+\beta+2\zeta)+d\theta(z-\gamma)\theta(z+\gamma+2\zeta) = 0 (*)$$

Proof:

$$(*) \iff \Theta_\alpha \cap \Theta_\beta \quad \subset \quad \Theta_\gamma \cup \Theta_{\gamma+2\zeta}$$

(Weil):

$$W_{g-1} = \Theta + \kappa$$

$$(W_{g-1})\cap(W_{g-1}+(p-q)) \quad \subset \quad (W_{g-1}+(p-r))\cup(W_{g-1}+(q-s))$$

Kummer morphism:

$$\overrightarrow{\theta} : X \longrightarrow K(X) \subset \mathbb{P}H^0(X, 2\Theta) = \mathbb{P}^{2^g - 1}$$
$$z \mapsto [\overrightarrow{\theta}(z)] = [\ldots, \theta[n](z), \ldots], \qquad n \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$$

Riemann bilinear rel.

$$\theta(x-y)\theta(x+y) = \overrightarrow{\theta}(x) \cdot \overrightarrow{\theta}(y)$$

Thus, for $\zeta \in \frac{1}{2}(\Gamma - \alpha - \beta - \gamma) \subset X$ and $\alpha, \beta, \gamma \in \Gamma$

$\overrightarrow{\theta}(\zeta + \alpha), \overrightarrow{\theta}(\zeta + \beta), \overrightarrow{\theta}(\zeta + \gamma)$ are collinear in $K(X) = X/\pm$

so $\frac{1}{2}(\Gamma - \alpha - \beta - \gamma) \subset X$ gives a curve of trisecants to $K(X)$:

$$\overrightarrow{\theta}(\zeta + \alpha) \wedge \overrightarrow{\theta}(\zeta + \beta) \wedge \overrightarrow{\theta}(\zeta + \gamma) = 0$$

We'll see:

θ satisfies the KP equation $\iff$ $\exists$ a third order germ of flexes to $K(X)$

Theorem

(Gunning '82, Welters '83) An i.p.p.a.v. (X, Θ) is a Jacobian of an algebraic curve $\iff$ the Kummer variety $K(X)$ has a curve Γ of trisecants.

Matsusaka: $[\Gamma] = \dfrac{\Theta^{g-1}}{(g-1)!}$

Welter's conjecture:

Theorem

(Krichever '06, '10) An i.p.p.a.v. (X, Θ) is a Jacobian of an algebraic curve $\iff$ the Kummer variety $K(X)$ admits a single trisecant, i.e.:

a) one flex,

b) one degenerate trisecant,

c) one bona fide trisecant.

Theorem

(Grushevsky-Krichever '10) X is a Prym variety $\iff$ $K(X)$ admits two quadrisecant planes.

Curve of flexes to $K(X)$, and bridge to KP hierarchy.

$\frac{1}{2}\Gamma$ near $0 \in X$:

$$\zeta(\epsilon) = \epsilon D_1 + \epsilon^2 D_2 + \epsilon^3 D_3 + \cdots$$

Let $\alpha, \beta, \gamma \to 0$, in $\overrightarrow{\theta}(\zeta + \alpha) \wedge \overrightarrow{\theta}(\zeta + \beta) \wedge \overrightarrow{\theta}(\zeta + \gamma) = 0$

$$\overrightarrow{\theta}(\zeta(\epsilon)) \wedge D_1 \overrightarrow{\theta}(\zeta(\epsilon)) \wedge (D_1^2 + D_2)\overrightarrow{\theta}(\zeta(\epsilon)) = 0$$

Taylor series:

$$f(\zeta(\epsilon)) = \sum_s (\Delta_s f_{|\zeta=0})\epsilon^s\,, \quad \Delta_s = \sum_{i+1+2i_2\cdots+si_s=s} \frac{1}{i_1!\cdots i_s!} D_1^{i_1}\cdots D_s^{i_s}$$

Use Riemann's bilinear relation and expand in ϵ. Coefficient of ϵ^s:

$$\left[\Delta_s D_1 - \Delta_{s-1}(D_1^2 + D_2) + \sum_{i=3}^{s} d_{i+1}\Delta_{s-i}\right]\theta(z+\zeta)\cdot\theta(z-\zeta)_{|\zeta=0} = 0$$

Call this $P_s = 0$, $s \geq 3$. It depends on $D_1, \ldots, D_s$, $d_4, \ldots, d_{s+1}$.

via Gunning → $X = J(\Gamma)$

For $s = 3$ get the KP (Mumford):

$$D_1^4\theta(z)\cdot\theta(z) - 4D_1^3\theta(z)\cdot D_1\theta(z) + 3(D_1^2\theta(z))^2 - 3(D_2\theta(z))^2$$
$$+ 3D_2^2\theta(z)\cdot\theta(z) + 3D_1\theta(z)\cdot D_3\theta(z) - 3D_1D_3\theta(z)\cdot\theta(z) + d_4\theta^2(z) = 0$$

EA-DeConcini '83:

$$\{\theta = 0\} \cap \{D_1\theta = 0\} \subset \{(D_1^2 - D_2)\theta = 0\} \cup \{(D_1^2 + D_2)\theta = 0\}$$

Define

$$D_1\Theta := \{\theta = 0\} \cap \{D_1\theta = 0\}$$

via :

$$0 \longrightarrow \mathcal{O}_X(\Theta) \xrightarrow{\cdot\theta} \mathcal{O}_X(2\Theta) \longrightarrow \mathcal{O}_\Theta(2\Theta) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_\Theta(\Theta) \xrightarrow{D_1\theta} \mathcal{O}_\Theta(2\Theta) \longrightarrow \mathcal{O}_{D_1\Theta}(2\Theta) \longrightarrow 0$$

$$P_{3|D_1\Theta} = \left[(D_1^2 - D_2)\theta(D_1^2 + D_2)\theta\right]_{D_1\Theta} = 0$$

$\iff \exists\ D_3, d_4, \quad s.t.\ P_3 = 0$

$\iff \exists$ third order germ of flexes to $K(X)$

$\iff$ KP holds

On the other hand (Gunning-Welters):

$$P_s = 0\,,\ \ s \geq 3 \quad \Longleftrightarrow \quad \exists \text{ a curve of flexes of } K(X)$$
$$\Longleftrightarrow \quad X \text{ is the jacobian of a curve}$$

As in the case $s = 3$:

$$P_{s|D_1\Theta} = 0 \ \Rightarrow \ \exists\ D_s, d_{s+1} \text{ s.t. } P_s = 0$$

To prove Shiota's theorem, suffices to prove, inductively:

$$P_3 = \cdots = P_{s-1} = 0 \quad \Rightarrow \quad P_{s|D_1\Theta} = 0$$

If $P_3 = \cdots = P_{s-1} = 0 \Rightarrow$, then

$$P_{s|D_1\Theta} = 0 \Leftrightarrow \quad \left[\widetilde{\Delta}_{s-1}\theta(z)\cdot(D_1^2 - D_2)\theta(z)\right]_{|D_1\Theta} = 0 \qquad (*)$$

$$\Delta_s = \Delta_s(D_1,\ldots,D_s)\,,\quad \widetilde{\Delta}_s = \Delta_s(2D_1,\ldots,2D_s)$$

$D_1 \Theta = \cup V_i$

Let V an irreducible component of $D_1\Theta$. Assume V reduced.

$$\text{KP} \quad \Rightarrow \quad \left[(D_1^2 + D_2)\theta(z) \cdot (D_1^2 - D_2)\theta(z)\right]_{|V} = 0$$

$\Rightarrow$ either $(D_1^2 - D_2)\theta(z)_{|V} = 0$, or $(D_1^2 + D_2)\theta(z)_{|V} = 0$, and $(*)$ follows easily.

Only difficult case is when a) V non-reduced, and

$$\text{b)} \qquad V \subset \{D_1^n\theta = 0\,,\, n \geq 0\} = \Sigma(X, \Theta, A_{D_1})$$

where $A_{D_1} = \{\text{abelian subvariety generated by } D_1\}$.

But the Corollary of the theorem on base loci says that $\Sigma(X, \Theta, A_{D_1})$ is reduced.

Conclusion of the algebro-geometric approach: An i.p.p.a.v. (X, Θ) is a Jacobian of an algebraic curve $\iff$ the Kummer variety $K(X)$ admits one of the following:

✓ a) A third order germ of flexes $\iff$ the KP equation holds. (Relevant $\Sigma = \Sigma_{D_1}$)

✓ b) One flex: $\ell \cdot K(X) = 3\psi(b)$. (Relevant $\Sigma = \Sigma(X, \Theta, \langle a \rangle)$, $a = 2b$)

Debarre, Marini

Krichever

✓ c) One degenerate trisecant: $\ell \cdot K(X) = 2\psi(u) + \psi(b)$, (Relevant $\Sigma = \Sigma(X, \Theta_u, \langle a \rangle)$, $a = 2b$)

? d) One bona fide trisecant. $\ell \cdot K(X) = \psi(u) + \psi(v) + \psi(w)$.

Thank you Igor!